AP Calculus AB — Name: ______________________________

2.3 – Continuity and Intermediate Value Theorem — Date: _______________ Period: _______

Intermediate Value Theorem

1. Consider the equation $x^2 - \cos \pi x - 1 = 0$. Use the Intermediate Value Theorem to show that the equation has a solution on the interval [0, 1].

We know that $f(x) = x^2 - \cos \pi x - 1$ is continuous because it is the sum of continuous functions. Therefore, the IVT applies and, since

$f(0) = 0^2 - \cos 0\pi - 1 = 0 - 1 - 1 = -2$ and
$f(1) = 1^2 - \cos 1\pi - 1 = 4 - 1 - 1 = 2$, there must be a c in [0, 1] such that $f(c)$ is in [–2, 2].

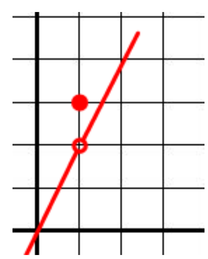
2. Use the Intermediate Value Theorem to show that the equation $x^3 - 3x - 1 = 0$ has a solution.

We know that $f(x) = x^3 - 3x - 1$ is continuous because it is the sum of continuous functions. Therefore, the IVT applies and, since

$f(0) = 0^3 - 3*0 - 1 = -1$ and

$f(2) = 2^3 - 3*2 - 1 = 1$, there must be a c in [0, 2] such that $f(c)$ is in [–1, 1].

Former AP Exam Problems



3. The graph of the function f is shown. The value of $\lim_{x \to 1}(\sin(f(x)))$ is:

A) 0.909 B) 0.841 C) 0.141 D) –0.416 E) Nonexistent

This will be the sin of the limit as x goes to 1 of f. Therefore, it's the sin of 2.
sin 2 = 0.909

4. If the function f is continuous for all real numbers and if when $f(x) = \dfrac{x^2 - 4}{x + 2}$, $x \neq 2$, then $f(-2) =$:

A) –4 B) –2 C) –1 D) 0 E) 2

$f(x) = (x - 2)(x + 2)/(x + 2) = x - 2$ when $x \neq -2$. $f(-2) = -2 - 2 = -4$.

5. Let f be defined as follows, where $a \neq 0$. $f(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a}, x \neq a \\ 0, x = a \end{cases}$

Which of the following are true about f?

I. $\lim_{x \to a} f(x)$ exists. II. $f(a)$ exists. III. $f(x)$ is continuous at $x = a$.

A) None B) I Only C) II Only D) I and II Only E) I, II, and III

I is true because the function $= (x + a)(x - a)/(x - a) = x + a$ when $x \neq a$. So on both sides of a, the function is the same $(x + a)$. II is true because $f(a)$ is defined to = 0. III is false because $x + a = 2a$ (the limit to the left and right), not 0, $f(a)$. So the correct answer is D.

6. If the graph of $y = \frac{ax+b}{x+c}$ has a horizontal asymptote $y = -2$ and a vertical asymptote $x = -3$, then $a + c$ =:

(A) –5 (B) –1 (C) 0 (D) 1 (E) 5

Since the degrees of the numerator and denominator are both 1, we use the leading coefficients to find the horizontal asymptote, which will happen at $y = a/1$. Therefore, $a = -2$.

The vertical asymptote will happen when the denominator = 0, so $x + c = 0$. So $c = 3$ to make $x + 3 = 0$ solve to $x = -3$.

$a + c = -2 + 3 = 1$.

7. Given that f is a function defined by $f(x) = \frac{x^3 - x}{x^3 - 4x}$,

Factoring this, I get $x(x^2 - 1)/(x(x^2 - 4)) = x(x - 1)(x + 1)/(x(x - 2)(x + 2))$. Notice the common factor of x in the numerator and denominator, so there will be a hole at $x = 0$.

A) Find $\lim_{x \to 0} f(x)$.

Crossing out the common factor of x, we get $(x^2 - 1)/(x^2 - 4)$, so at $x = 0$, we have $(0^2 - 1)/(0^2 - 4) = 1/4$

B) Find the zeros of $f(x)$.

They will occur where the numerator = 0 after eliminating the holes: $(x - 1)(x + 1) = 0$, so at $x = 1$, and -1.

C) Write an equation for each horizontal and vertical asymptote to the graph of f.

Since the degrees of the numerator and denominator are the same (3), there will be a horizontal asymptote at y = leading coefficient of numerator/leading coefficient of denominator = 1/1 = 1. $y = 1$.

Vertical asymptotes can be found by setting the denominator = 0 after eliminating the holes. So $(x - 2)(x + 2) = 0$, meaning $x = 2$ and $x = -2$ will be vertical asymptotes.

D) Describe the symmetry of the graph of f.

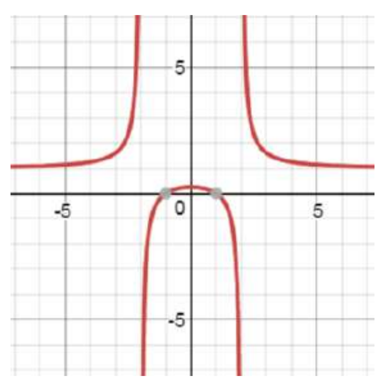
Well, let's find $f(-x)$ to see if the function is odd, even, or neither.

$f(-x) = ((-x)^3 - (-x))/((-x)^3 - 4(-x)) = (-x^3 + x)/(-x^3 + 4x) = -1(x^3 - x)/((-1)(x^3 - 4x)) = (x^3 - x)/(x^3 - 4x)$ $= f(x)$, so the function is even and is symmetric about the y-axis.

E) Using the information found in parts a – d, sketch the graph of f.

Check with your calculator.

(The hole at $x = 0$ doesn't show on my graph here.)



8. Suppose $\begin{cases} f(x)=\dfrac{3x(x-1)}{x^2-3x+2}, x\neq 1,2 \\ f(1)=-3 \\ f(2)=4 \end{cases}$. Then $f(x)$ is continuous:

A) except at $x = 1$ B) except at $x = 2$ C) except at $x = 1$ or 2 D) except at $x = 0$, 1, or 2
(E) at each real number

Factoring, we get $3x(x – 1)/((x – 1)(x – 2))$. There is a hole at $x = 1$ and a vertical asymptote at $x = 2$.

Let's fill the hole first: Eliminating the common factors, $f(x) = 3x/(x – 2)$.
At $x = 1$, that is $3*1/(1 – 2) = –3$. Since the limit at $x = 1$ exists and $= f(1)$, $f(x)$ is continuous at $x = 1$.

Since there is a vertical asymptote at $x = 2$, the function is only discontinuous there.

9. If $\begin{cases} f(x)=\dfrac{x^2-x}{2x}, x\neq 0 \\ f(0)=k \end{cases}$, and if f is continuous at $x = 0$, then k =:

A) –1 B) – ½ C) 0 D) ½ E) 1

Factoring: $x(x – 1)/(2x)$ shows us a common factor of x, showing a hole at $x = 0$. Removing that common factor: we get $(x – 1)/2$ if $x \neq 0$. If $x = 0$, $(x – 1)/2$ is $–1/2$. So f will be continuous if we fill the hole, making $f(0) = k = –1/2$.

More IVT Practice

10. Prove that the function $f(x)=x^2-4x+2$ intersects the x-axis on the interval [0, 2]. Can the same be said for the function: $g(x)=\dfrac{2x-3}{x-1}$?

$f(x)$ is a polynomial, so it is continuous and we can use the IVT. $f(0) = 2$ and $f(2) = 4 – 8 + 2 = –2$. By the IVT, there must be a value c in [0, 2] where $f(c) = 0$ since 0 is between 2 and –2.

$g(x)$ has a vertical asymptote at $x = 1$, so $g(x)$ is not continuous over [0, 2] and therefore, we can't use the IVT. ☹

11. Prove that the equation $x^3 + x – 5 = 0$ has at least one solution $x = a$ such that $1 < a < 2$.

$f(x) = x^3 + x – 5$ is a polynomial function, so it is continuous and we can use the IVT.
$f(1) = 1^3 + 1 – 5 = –3$
$f(2) = 2^3 + 2 – 5 = 5$
So, by the IVT, there must be at least one a in the interval (1, 2) such that $f(a) = 0$ since 0 is in the interval (–3, 5).

12. Given the function $f(x) = x^3 - x^2 + 1$, can it be said that there is at least one point, c, inside the interval [1, 2] which verifies $f(c) = 0$?

$f(1) = 1^3 - 1^2 + 1 = 1$, $f(2) = 2^3 - 2^2 + 1 = 5$. Even though f is continuous, the IVT does not help here because $f(c) = 0$ is not between $f(1)$ and $f(2)$.

13. Prove that the equation $e^{-x} + 2 = x$ has at least one solution.

$e^{-x} + 2 = x$ can be rewritten as $f(x) = e^{-x} - x + 2 = 0$. $f(x)$ is continuous because it's the sum of continuous functions. We need to find two values for x that will have y-values on opposite sides of 0.

$f(0) = e^{-0} - 0 + 2 = 1 + 2 = 3 > 0$.

$f(3) = e^{-3} - 3 + 2 = 1/e^3 - 1 < 0$. So there must be a solution in the interval (0, 3).

14. Prove that there is a real number, x, such that $\sin x = x$.

Let's investigate the equivalent statement: $f(x) = \sin x - x = 0$. It's continuous, so we can use the IVT.
$f(0) = \sin 0 - 0 = 0$. Oops. That's a solution right there. That's not supposed to happen.

Let's try $f(\pi)$. $f(\pi) = \sin(\pi) - \pi = 0 - \pi = -\pi < 0$.

$f(-\pi) = \sin(-\pi) - (-\pi) = 0 + \pi = \pi > 0$ By the IVT, there is a solution between $-\pi$ and π.