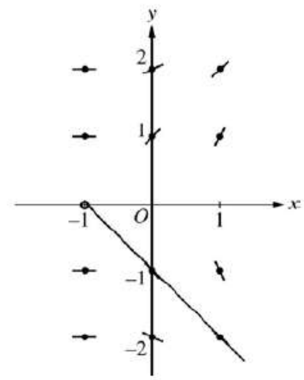
AP Calculus – Free Response **Unit 7 Free Response #2**
Solutions File

#1a (2010 ABB5)

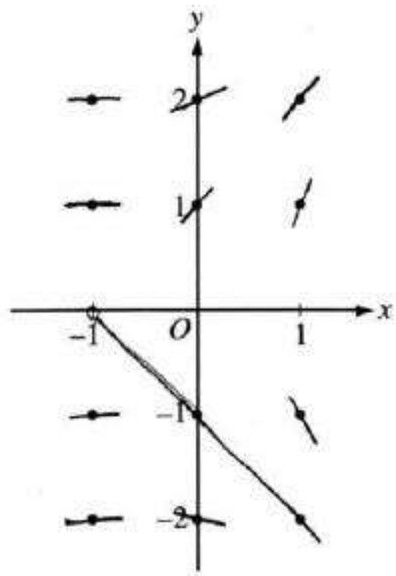
AP Scoring Rubric

$$3 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{nonzero slopes} \\ 1 : \text{solution curve through } (0, -1) \end{cases}$$

(a)



Actual Solution Receiving Full Credit



#1b

AP Scoring Rubric

(b) $-1 = \dfrac{x+1}{y} \Rightarrow y = -x - 1$

$\dfrac{dy}{dx} = -1$ for all (x, y) with $y = -x - 1$ and $y \neq 0$

1 : description

Actual Solution Receiving Full Credit

$$\frac{dy}{dx} = \frac{x+1}{y} = -1 \Rightarrow x+1=-y \Rightarrow x+y=-1$$

except point$(-1, 0)$, points $(x, -1-x)$

are all solution to $\frac{dy}{dx} = -1$.

They are on the line $y = -1-x$

#1c

AP Scoring Rubric

(c) $\int y\,dy = \int (x+1)\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + x + C$

$\frac{(-2)^2}{2} = \frac{0^2}{2} + 0 + C \Rightarrow C = 2$

$y^2 = x^2 + 2x + 4$

Since the solution goes through $(0,-2)$, y must be negative. Therefore $y = -\sqrt{x^2 + 2x + 4}$.

$5: \begin{cases} 1: \text{separates variables} \\ 1: \text{antiderivatives} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition} \\ 1: \text{solves for } y \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

Actual Solution Receiving Full Credit

$$\frac{dy}{dx} = \frac{x+1}{y}$$

$$dy \cdot y = dx \cdot x+1$$

$$\int y \cdot dy = \int x+1\,dx$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + x + c \qquad \text{passes through } f(0) = -2$$

$$\frac{1}{2} \times 4 = c \Rightarrow c = 2$$

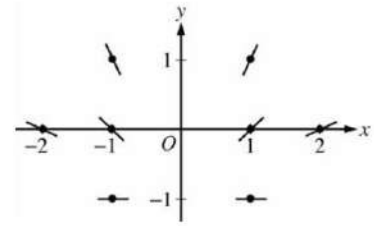
$$y = -\sqrt{x^2+2x+4}$$

#2a (2006 AB5)

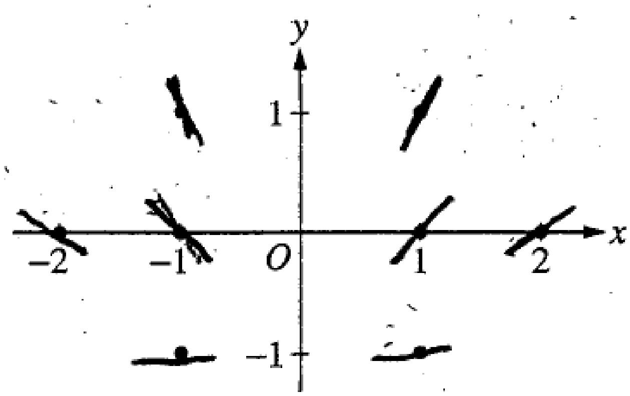
AP Scoring Rubric

2 : sign of slope at each point and relative steepness of slope lines in rows and columns

(a)



Actual Solution Receiving Full Credit



#2b

AP Scoring Rubric

(b) $\dfrac{1}{1+y}\,dy = \dfrac{1}{x}\,dx$

$\ln|1+y| = \ln|x| + K$

$|1+y| = e^{\ln|x|+K}$

$1+y = C|x|$

$2 = C$

$1+y = 2|x|$

$y = 2|x| - 1$ and $x < 0$

or

$y = -2x - 1$ and $x < 0$

7 :
- 6 :
  - 1 : separates variables
  - 2 : antiderivatives
  - 1 : constant of integration
  - 1 : uses initial condition
  - 1 : solves for y
- Note: max $3/6$ [1-2-0-0-0] if no constant of integration
- Note: $0/6$ if no separation of variables
- 1 : domain

Actual Solution Receiving Full Credit

$$\frac{dy}{dx} = \frac{1+y}{x}$$

$$\int \frac{dy}{1+y} = \int \frac{dx}{x}$$

$$e^{\ln|1+y|} = e^{(\ln|x|+C)}$$

$$1+y = Cx \qquad 1+y>0$$

$$y = Cx-1, \qquad y>-1$$

$$f(x) = Cx-1, \qquad f(x)>-1$$

$$f(-1) = C(-1)-1 = 1$$

$$C(-1) = 2$$

$$C = -2$$

$$\boxed{f(x) = -2x-1}, \quad f(x)>-1$$

$$D = \{x \in \mathbb{R} \mid x<0\} \qquad -2x-1>-1$$

or

$$-2x>0$$

$$D = (-\infty, 0)$$

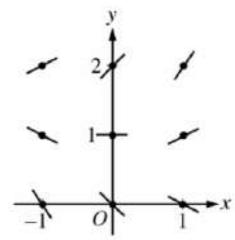
$$2x<0 \qquad x<0$$

#3a (2007 ABB5)

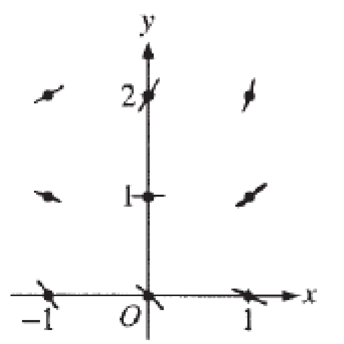
AP Scoring Rubric

2 : sign of slope at each point and relative steepness of slope lines in rows and columns

(a)



Actual Solution Receiving Full Credit



#3b

AP Scoring Rubric

(b) $\frac{d^2y}{dx^2} = \frac{1}{2} + \frac{dy}{dx} = \frac{1}{2}x + y - \frac{1}{2}$

Solution curves will be concave up on the half-plane above the line $y = -\frac{1}{2}x + \frac{1}{2}$.

$3 : \begin{cases} 2 : \frac{d^2y}{dx^2} \\ 1 : \text{description} \end{cases}$

Actual Solution Receiving Full Credit

$$\frac{d^2y}{dx^2} = \frac{1}{2} + \frac{dy}{dx} = \frac{1}{2}x + y - \frac{1}{2}$$

curves are concave up : $\frac{d^2y}{dx^2} > 0$

$\frac{1}{2}x + y - \frac{1}{2} > 0$

$\frac{1}{2}x + y > \frac{1}{2}$

$x + 2y > 1$

$\boxed{x + 2y - 1 > 0}$

when coordinates of the solution curves satisfy this condition, the solution curves are concave up.

$\therefore$ $x \geq 0$ and $y > \frac{1}{2}$

#3c

AP Scoring Rubric

(c) $\left.\frac{dy}{dx}\right|_{(0,1)} = 0 + 1 - 1 = 0$ and $\left.\frac{d^2y}{dx^2}\right|_{(0,1)} = 0 + 1 - \frac{1}{2} > 0$

Thus, f has a relative minimum at $(0, 1)$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

Actual Solution Receiving Full Credit

$f(0) = 1$

$$\frac{dy}{dx} = \frac{1}{2} \cdot 0 + 1 - 1 = 0$$

$$\frac{d^2y}{dx^2} = \frac{1}{2} \cdot 0 + 1 - \frac{1}{2} = \frac{1}{2} > 0$$

f has a relative minimum at $x = 0$

as $\frac{dy}{dx}$ attains zero and change its sign from

negative to positive.

#3d

AP Scoring Rubric

(d) Substituting $y = mx + b$ into the differential equation:

$m = \frac{1}{2}x + (mx + b) - 1 = \left(m + \frac{1}{2}\right)x + (b - 1)$

Then $0 = m + \frac{1}{2}$ and $m = b - 1$: $m = -\frac{1}{2}$ and $b = \frac{1}{2}$.

$2: \begin{cases} 1: \text{value for } m \\ 1: \text{value for } b \end{cases}$

Actual Solution Receiving Full Credit

$y = mx + b$

$\frac{dy}{dx} = m = \underline{\frac{1}{2}x + y} - 1 \rightarrow \quad m = \frac{1}{2} - 1 = -\frac{1}{2}$

$\uparrow$

$\frac{d^2y}{dx^2} = 0 = \underline{\frac{1}{2}x + y} - \frac{1}{2} \rightarrow \quad \underline{\frac{1}{2}x + y} = \frac{1}{2}$

$$\begin{cases} y = -\frac{1}{2}x + b \rightarrow \underline{2y} = -x + 2b \\ \frac{1}{2}x + y = \frac{1}{2} \rightarrow x + \underline{2y} = 1 \rightarrow \cancel{x - x} + 2b = 1 \end{cases}$$

$b = \frac{1}{2}$

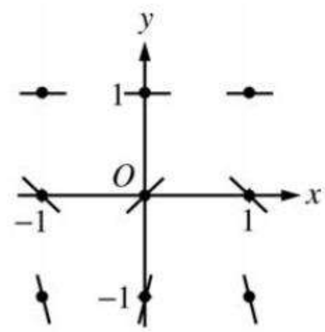
$m = -\frac{1}{2}, \quad b = \frac{1}{2}$

#4a . (2006 ABB5)

AP Scoring Rubric

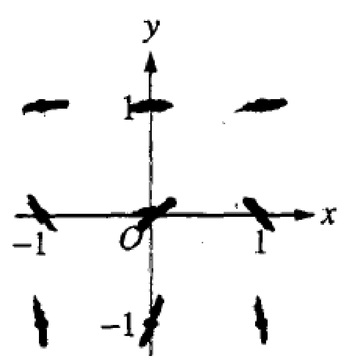
$$2 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{all other slopes} \end{cases}$$

(a)



Actual Solution Receiving Full Credit

$\frac{dy}{dx} = (y-1)^2 \cdot \cos \pi x.$



#4b

AP Scoring Rubric

(b) The line $y = 1$ satisfies the differential equation, so $c = 1$.	$1 : c = 1$

Actual Solution Receiving Full Credit

Work for problem 5(b)

$y = C$

The horizontal line is $y=1$, because where $y=1$ the derivative of the the function is zero (it does not depend of what x we take). And as equation of the linear function is $y=bx+c$ and the slope of it is zero, we may say That $\underline{y=1}$.

#4c

AP Scoring Rubric

(c) $\dfrac{1}{(y-1)^2}\,dy = \cos(\pi x)\,dx$

$-(y-1)^{-1} = \dfrac{1}{\pi}\sin(\pi x) + C$

$\dfrac{1}{1-y} = \dfrac{1}{\pi}\sin(\pi x) + C$

$1 = \dfrac{1}{\pi}\sin(\pi) + C = C$

$\dfrac{1}{1-y} = \dfrac{1}{\pi}\sin(\pi x) + 1$

$\dfrac{\pi}{1-y} = \sin(\pi x) + \pi$

$y = 1 - \dfrac{\pi}{\sin(\pi x) + \pi}$ for $-\infty < x < \infty$

$6 : \begin{cases} 1 : \text{separates variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables

Actual Solution Receiving Full Credit

Work for problem 5(c)

$$\begin{cases} \dfrac{dy}{dx} = (y-1)^2 \cdot \cos(\pi x) \\ y(1) = 0. \end{cases}$$

$$\int \frac{dy}{(y-1)^2} = \int \cos(\pi x)\,dx$$

$$-\frac{1}{y-1} = \frac{\sin(\pi x)}{\pi} + C.$$

$$-\frac{1}{0-1} = \frac{\sin \pi}{\pi} + C.$$

$$1 = 0 + C.$$

$$\underline{C = 1}.$$

$$-\frac{1}{y-1} = \frac{\sin(\pi x)}{\pi} + 1.$$

$$\frac{1}{y-1} = -\frac{\sin(\pi x)}{\pi} - 1.$$

$$y - 1 = -\frac{1}{\frac{\sin(\pi x)}{\pi} + 1}.$$

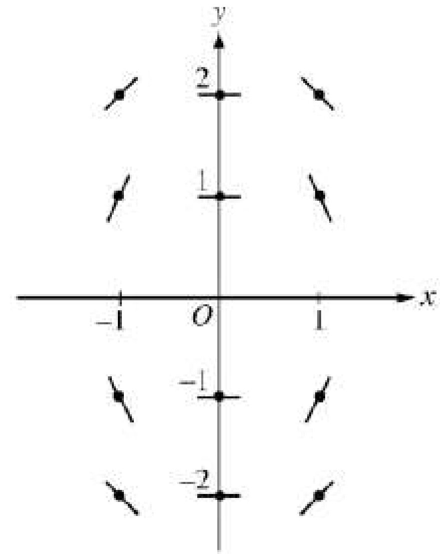
$$\underline{y = -\frac{\pi}{\sin(\pi x) + \pi} + 1}.$$

#5a (2005 AB6)

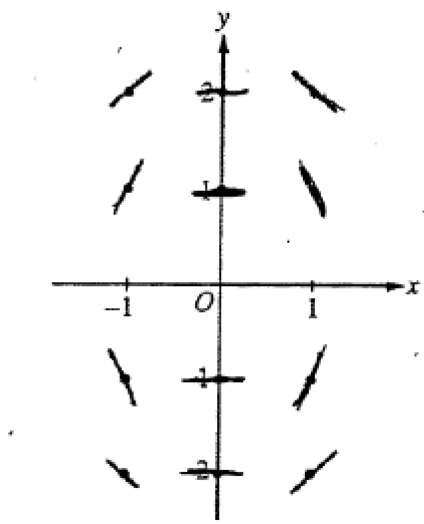
AP Scoring Rubric

$2:\begin{cases}1: \text{zero slopes}\\ 1: \text{nonzero slopes}\end{cases}$

(a)



Actual Solution Receiving Full Credit



#5b

AP Scoring Rubric

(b) The line tangent to f at $(1, -1)$ is $y+1 = 2(x-1)$.
Thus, $f(1.1)$ is approximately -0.8.

$2:\begin{cases}1: \text{equation of the tangent line}\\ 1: \text{approximation for } f(1.1)\end{cases}$

Actual Solution Receiving Full Credit

line tangent: $y+1 = 2(x-1)$

$f(x) = 2(x-1)-1$

$f(1.1) = 2(1.1-1)-1$

$f(1.1) = .2-1 = -.8$

#5c

AP Scoring Rubric

(c) $\dfrac{dy}{dx} = -\dfrac{2x}{y}$

$y\,dy = -2x\,dx$

$\dfrac{y^2}{2} = -x^2 + C$

$\dfrac{1}{2} = -1 + C;\ C = \dfrac{3}{2}$

$y^2 = -2x^2 + 3$

Since the particular solution goes through $(1, -1)$, y must be negative.

Thus the particular solution is $y = -\sqrt{3 - 2x^2}$.

5 :
- 1 : separates variables
- 1 : antiderivatives
- 1 : constant of integration
- 1 : uses initial condition
- 1 : solves for y

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

Actual Solution Receiving Full Credit

$\dfrac{dy}{dx} = \dfrac{-2x}{y}$

$\int y\,dy = \int -2x\,dx$

$\dfrac{y^2}{2} = -x^2 + C$

$y^2 = -2x^2 + C$

$y = \pm\sqrt{-2x^2+C}$

$-1 = -\sqrt{-2(1)^2+C}$

$1 = -2+C$

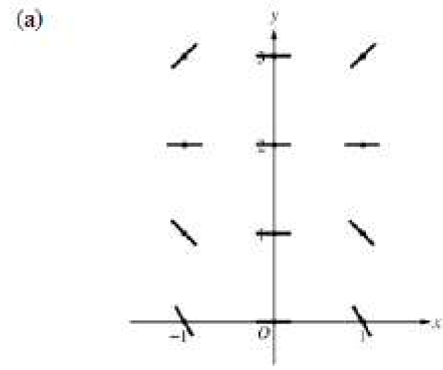
$C = 3$

$y = -\sqrt{-2x^2+3}$

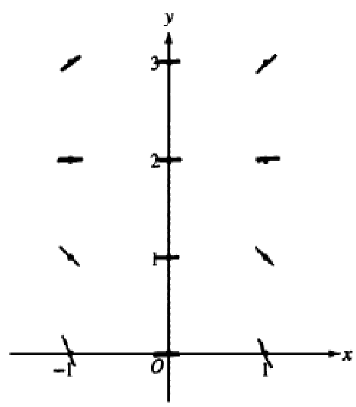
#6a (2004 AB5)

AP Scoring Rubric

2 :
- 1 : zero slope at each point (x, y) where $x = 0$ or $y = 2$
- 1 :
  - positive slope at each point (x, y) where $x \neq 0$ and $y > 2$
  - negative slope at each point (x, y) where $x \neq 0$ and $y < 2$



Actual Solution Receiving Full Credit



#6b

AP Scoring Rubric

(b) Slopes are negative at points (x, y) where $x \neq 0$ and $y < 2$.

1 : description

Actual Solution Receiving Full Credit

x^4 is always positive $\Rightarrow \frac{dy}{dx} < 0$ iff $y < 2$ ^ $x \neq 0$

∴ the negative slope's where $y<2$ and $x \neq 0$.

~~the negative slopes~~ become greater in magnitude as $|x|$ become greater and $|y-2|$ become greater

#6c

AP Scoring Rubric

(c) $\frac{1}{y-2}dy = x^4 dx$

$\ln|y-2| = \frac{1}{5}x^5 + C$

$|y-2| = e^C e^{\frac{1}{5}x^5}$

$y - 2 = Ke^{\frac{1}{5}x^5}$, $K = \pm e^C$

$-2 = Ke^0 = K$

$y = 2 - 2e^{\frac{1}{5}x^5}$

$6: \begin{cases} 1: \text{separates variables} \\ 2: \text{antiderivatives} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition} \\ 1: \text{solves for } y \\ \quad 0/1 \text{ if } y \text{ is not exponential} \end{cases}$

Note: max 3/6 [1-2-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

Actual Solution Receiving Full Credit

$\frac{dy}{dx} = x^4(y-2) \Rightarrow \frac{dy}{(y-2)} = x^4\,dx \Rightarrow \int \frac{dy}{(y-2)} = \int x^4\,dx$

$\Rightarrow \ln|y-2| = \frac{x^5}{5} + c_1 \Rightarrow y-2 = e^{\frac{x^5}{5}+c_1} \Rightarrow y = Ce^{x^5/5} + 2$

$f(0) = 0 \Rightarrow 0 = ce^0 + 2 \Rightarrow c = -2$

$\therefore y = -2e^{\frac{x^5}{5}} + 2$