Course and Exam Description

AP® Calculus AB and AP® Calculus BC

Including the Curriculum Framework

Revised Edition, Effective Fall 2016

CollegeBoard

AP® Calculus AB and AP® Calculus BC

Course and Exam Description

Effective Fall 2016

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CollegeBoard
New York, NY

About the College Board

The College Board is a mission-driven not-for-profit organization that connects students to college success and opportunity. Founded in 1900, the College Board was created to expand access to higher education. Today, the membership association is made up of over 6,000 of the world's leading educational institutions and is dedicated to promoting excellence and equity in education. Each year, the College Board helps more than seven million students prepare for a successful transition to college through programs and services in college readiness and college success — including the SAT® and the Advanced Placement Program®. The organization also serves the education community through research and advocacy on behalf of students, educators, and schools. For further information, visit **www.collegeboard.org**.

AP® Equity and Access Policy

The College Board strongly encourages educators to make equitable access a guiding principle for their AP® programs by giving all willing and academically prepared students the opportunity to participate in AP. We encourage the elimination of barriers that restrict access to AP for students from ethnic, racial, and socioeconomic groups that have been traditionally underrepresented. Schools should make every effort to ensure their AP classes reflect the diversity of their student population. The College Board also believes that all students should have access to academically challenging course work before they enroll in AP classes, which can prepare them for AP success. It is only through a commitment to equitable preparation and access that true equity and excellence can be achieved.

This version of the *AP Calculus AB and AP Calculus BC Course and Exam Description* includes additional information on pages 45-46 on completing the free-response questions.

This version also corrects an error in the sample exam question on page 83. The series now begins at 1 (n=1) instead of zero (n=0).

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About AP®

The College Board's Advanced Placement Program® (AP®) enables students to pursue college-level studies while still in high school. Through more than 30 courses, each culminating in a rigorous exam, AP provides willing and academically prepared students with the opportunity to earn college credit and/or advanced placement. Taking AP courses also demonstrates to college admission officers that students have sought out the most rigorous course work available to them.

Each AP course is modeled upon a comparable college course, and college and university faculty play a vital role in ensuring that AP courses align with college-level standards. Talented and dedicated AP teachers help AP students in classrooms around the world develop and apply the content knowledge and skills they will need later in college.

Each AP course concludes with a college-level assessment developed and scored by college and university faculty, as well as experienced AP teachers. AP Exams are an essential part of the AP experience, enabling students to demonstrate their mastery of college-level course work. Most four-year colleges and universities in the United States and universities in more than 60 countries recognize AP in the admission process and grant students credit, placement, or both on the basis of successful AP Exam scores. Visit **www.collegeboard.org/apcreditpolicy** to view AP credit and placement policies at more than 1,000 colleges and universities.

Performing well on an AP Exam means more than just the successful completion of a course; it is a gateway to success in college. Research consistently shows that students who receive a score of 3 or higher on AP Exams typically experience greater academic success in college and have higher graduation rates than their non-AP peers.[1] Additional AP studies are available at **www.collegeboard.org/research**.

Offering AP Courses and Enrolling Students

This *AP Course and Exam Description* details the essential information required to understand the objectives and expectations of an AP course. The AP Program unequivocally supports the principle that each school implements its own curriculum that will enable students to develop the content knowledge and skills described here.

Schools wishing to offer AP courses must participate in the AP Course Audit, a process through which AP teachers' syllabi are reviewed by college faculty. The AP Course Audit was created at the request of College Board members who sought a means for the College Board to provide teachers and administrators with clear guidelines on curricular and resource requirements for AP courses and to help colleges and universities validate courses marked "AP" on students' transcripts. This process ensures that AP teachers' syllabi meet or exceed the curricular and

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[1] See the following research studies for more details:

Linda Hargrove, Donn Godin, and Barbara Dodd, *College Outcomes Comparisons by AP and Non-AP High School Experiences* (New York: The College Board, 2008).

Chrys Dougherty, Lynn Mellor, and Shuling Jian, *The Relationship Between Advanced Placement and College Graduation* (Austin, Texas: National Center for Educational Accountability, 2006).

resource expectations that college and secondary school faculty have established for college-level courses. For more information on the AP Course Audit, visit **www.collegeboard.org/apcourseaudit**.

The College Board strongly encourages educators to make equitable access a guiding principle for their AP programs by giving all willing and academically prepared students the opportunity to participate in AP. We encourage the elimination of barriers that restrict access to AP for students from ethnic, racial, and socioeconomic groups that have been traditionally underrepresented. Schools should make every effort to ensure their AP classes reflect the diversity of their student population. The College Board also believes that all students should have access to academically challenging course work before they enroll in AP classes, which can prepare them for AP success. It is only through a commitment to equitable preparation and access that true equity and excellence can be achieved.

How AP Courses and Exams Are Developed

AP courses and exams are designed by committees of college faculty and expert AP teachers who ensure that each AP subject reflects and assesses college-level expectations. To find a list of each subject's current AP Development Committee members, please visit **press.collegeboard.org/ap/committees**. AP Development Committees define the scope and expectations of the course, articulating through a curriculum framework what students should know and be able to do upon completion of the AP course. Their work is informed by data collected from a range of colleges and universities to ensure that AP coursework reflects current scholarship and advances in the discipline.

The AP Development Committees are also responsible for drawing clear and well-articulated connections between the AP course and AP Exam — work that includes designing and approving exam specifications and exam questions. The AP Exam development process is a multi-year endeavor; all AP Exams undergo extensive review, revision, piloting, and analysis to ensure that questions are high quality and fair and that there is an appropriate spread of difficulty across the questions.

Throughout AP course and exam development, the College Board gathers feedback from various stakeholders in both secondary schools and higher education institutions. This feedback is carefully considered to ensure that AP courses and exams are able to provide students with a college-level learning experience and the opportunity to demonstrate their qualifications for advanced placement upon college entrance.

How AP Exams Are Scored

The exam scoring process, like the course and exam development process, relies on the expertise of both AP teachers and college faculty. While multiple-choice questions are scored by machine, the free-response questions are scored by thousands of college faculty and expert AP teachers at the annual AP Reading. AP Exam Readers are thoroughly trained, and their work is monitored throughout the Reading for fairness and consistency. In each subject, a highly respected college faculty member fills the role of Chief Reader, who, with the help of AP readers in leadership positions, maintains the accuracy of the scoring standards. Scores on

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the free-response questions are weighted and combined with the results of the computer-scored multiple-choice questions, and this raw score is converted into a composite AP score of 5, 4, 3, 2, or 1.

The score-setting process is both precise and labor intensive, involving numerous psychometric analyses of the results of a specific AP Exam in a specific year and of the particular group of students who took that exam. Additionally, to ensure alignment with college-level standards, part of the score-setting process involves comparing the performance of AP students with the performance of students enrolled in comparable courses in colleges throughout the United States. In general, the AP composite score points are set so that the lowest raw score need to earn an AP score of 5 is equivalent to the average score among college students earning grades of A in the college course. Similarly, AP Exam scores of 4 are equivalent to college grades of A-, B+, and B. AP Exam scores of 3 are equivalent to college grades of B-, C+, and C.

Using and Interpreting AP Scores

College faculty are involved in every aspect of AP, from course and exam development to scoring and standards alignment. These faculty members ensure that the courses and exams meet colleges' expectations for content taught in comparable college courses. Based upon outcomes research and program evaluation, the American Council on Education (ACE) and the Advanced Placement Program recommend that colleges grant credit and/or placement to students with AP Exam scores of 3 and higher. The AP score of 3 is equivalent to grades of B-, C+, and C in the equivalent college course. However, colleges and universities set their own AP credit, advanced standing, and course placement policies based on their unique needs and objectives.

AP Score	Recommendation
5	Extremely well qualified
4	Well qualified
3	Qualified
2	Possibly qualified
1	No recommendation

Additional Resources

Visit **apcentral.collegeboard.org** for more information about the AP Program.

About the AP Calculus AB and AP Calculus BC Courses

Building enduring mathematical understanding requires students to understand the *why* and *how* of mathematics in addition to mastering the necessary procedures and skills. To foster this deeper level of learning, AP® Calculus is designed to develop mathematical knowledge conceptually, guiding students to connect topics and representations throughout each course and to apply strategies and techniques to accurately solve diverse types of problems.

AP Calculus includes two courses, AP Calculus AB and AP Calculus BC, which were developed in collaboration with college faculty. The curriculum for AP Calculus AB is equivalent to that of a first-semester college calculus course, while AP Calculus BC is equivalent to a first-semester college calculus course and the subsequent single-variable calculus course. Calculus BC is an extension of Calculus AB rather than an enhancement; common topics require a similar depth of understanding. Both courses are intended to be challenging and demanding, and each is designed to be taught over a full academic year.

College Course Equivalents

AP Calculus AB is roughly equivalent to a first semester college calculus course devoted to topics in differential and integral calculus. AP Calculus BC is roughly equivalent to both first and second semester college calculus courses; it extends the content learned in AB to different types of equations and introduces the topic of sequences and series.

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Prerequisites

Before studying calculus, all students should complete the equivalent of four years of secondary mathematics designed for college-bound students: courses which should prepare them with a strong foundation in reasoning with algebraic symbols and working with algebraic structures. Prospective calculus students should take courses in which they study algebra, geometry, trigonometry, analytic geometry, and elementary functions. These functions include linear, polynomial, rational, exponential, logarithmic, trigonometric, inverse trigonometric, and piecewise-defined functions. In particular, before studying calculus, students must be familiar with the properties of functions, the composition of functions, the algebra of functions, and the graphs of functions. Students must also understand the language of functions (domain and range, odd and even, periodic, symmetry, zeros, intercepts, and descriptors such as increasing and decreasing). Students should also know how the sine and cosine functions are defined from the unit circle and know the values of the trigonometric functions at the numbers $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ and their multiples.

Students who take AP Calculus BC should have basic familiarity with sequences and series, as well as some exposure to polar equations.

Participating in the AP Course Audit

Schools wishing to offer AP courses must participate in the AP Course Audit. Participation in the AP Course Audit requires the online submission of two documents: the AP Course Audit form and the teacher's syllabus. The AP Course Audit form is submitted by the AP teacher and the school principal (or designated administrator) to confirm awareness and understanding of the curricular and resource requirements. The syllabus, detailing how requirements are met, is submitted by the AP teacher for review by college faculty.

Please visit **http://www.collegeboard.com/html/apcourseaudit/teacher.html** for more information to support syllabus development including:

Annotated Sample Syllabi — Provide examples of how the curricular requirements can be demonstrated within the context of actual syllabi.

Curricular and Resource Requirements — Identify the set of curricular and resource expectations that college faculty nationwide have established for a college-level course.

Example Textbook List — Includes a sample of AP college-level textbooks that meet the content requirements of the AP course.

Syllabus Development Guide — Includes the guidelines reviewers use to evaluate syllabi along with three samples of evidence for each requirement. This guide also specifies the level of detail required in the syllabus to receive course authorization.

Syllabus Development Tutorial — Describes the resources available to support syllabus development and walks through the syllabus development guide requirement by requirement.

AP Calculus AB and AP Calculus BC Curriculum Framework

The *AP Calculus AB and AP Calculus BC Curriculum Framework* specifies the curriculum — what students must know, be able to do, and understand — for both courses. AP Calculus AB is structured around three big ideas: limits, derivatives, and integrals and the Fundamental Theorem of Calculus. AP Calculus BC explores these ideas in additional contexts and also adds the big idea of series. In both courses, the concept of limits is foundational; the understanding of this fundamental tool leads to the development of more advanced tools and concepts that prepare students to grasp the Fundamental Theorem of Calculus, a central idea of AP Calculus.

Overview

Based on the Understanding by Design (Wiggins and McTighe) model, this curriculum framework is intended to provide a clear and detailed description of the course requirements necessary for student success. It presents the development and organization of learning outcomes from general to specific, with focused statements about the content knowledge and understandings students will acquire throughout the course.

The **Mathematical Practices for AP Calculus (MPACs)**, which explicitly articulate the behaviors in which students need to engage in order to achieve conceptual understanding in the AP Calculus courses, are at the core of this curriculum framework. Each concept and topic addressed in the courses can be linked to one or more of the MPACs.

This framework also contains a **concept outline**, which presents the subject matter of the courses in a table format. Subject matter that is included only in the BC course is indicated with blue shading. The components of the concept outline are as follows:

- **Big ideas:** The courses are organized around big ideas, which correspond to foundational concepts of calculus: limits, derivatives, integrals and the Fundamental Theorem of Calculus, and (for AP Calculus BC) series.
- **Enduring understandings:** Within each big idea are enduring understandings. These are the long-term takeaways related to the big ideas that a student should have after exploring the content and skills. These understandings are expressed as generalizations that specify what a student will come to understand about the key concepts in each course. Enduring understandings are labeled to correspond with the appropriate big idea.
- **Learning objectives:** Linked to each enduring understanding are the corresponding learning objectives. The learning objectives convey what a student needs to be able to do in order to develop the enduring understandings. The learning objectives serve as targets of assessment for each course. Learning objectives are labeled to correspond with the appropriate big idea and enduring understanding.

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- **Essential knowledge:** Essential knowledge statements describe the facts and basic concepts that a student should know and be able to recall in order to demonstrate mastery of each learning objective. Essential knowledge statements are labeled to correspond with the appropriate big idea, enduring understanding, and learning objective.

Further clarification regarding the content covered in AP Calculus is provided by **examples** and **exclusion statements**. Examples are provided to address potential inconsistencies among definitions given by various sources. Exclusion statements identify topics that may be covered in a first-year college calculus course but are not assessed on the AP Calculus AB or BC Exam. Although these topics are not assessed, the AP Calculus courses are designed to support teachers who wish to introduce these topics to students.

Mathematical Practices for AP Calculus (MPACs)

The Mathematical Practices for AP Calculus (MPACs) capture important aspects of the work that mathematicians engage in, at the level of competence expected of AP Calculus students. They are drawn from the rich work in the National Council of Teachers of Mathematics (NCTM) Process Standards and the Association of American Colleges and Universities (AAC&U) Quantitative Literacy VALUE Rubric. Embedding these practices in the study of calculus enables students to establish mathematical lines of reasoning and use them to apply mathematical concepts and tools to solve problems. The Mathematical Practices for AP Calculus are not intended to be viewed as discrete items that can be checked off a list; rather, they are highly interrelated tools that should be utilized frequently and in diverse contexts.

The sample items included with this curriculum framework demonstrate various ways in which the learning objectives can be linked with the Mathematical Practices for AP Calculus.

The Mathematical Practices for AP Calculus are given below.

MPAC 1: Reasoning with definitions and theorems

Students can:

a. use definitions and theorems to build arguments, to justify conclusions or answers, and to prove results;

b. confirm that hypotheses have been satisfied in order to apply the conclusion of a theorem;

c. apply definitions and theorems in the process of solving a problem;

d. interpret quantifiers in definitions and theorems (e.g., “for all,” “there exists”);

e. develop conjectures based on exploration with technology; and

f. produce examples and counterexamples to clarify understanding of definitions, to investigate whether converses of theorems are true or false, or to test conjectures.

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MPAC 2: Connecting concepts

Students can:

a. relate the concept of a limit to all aspects of calculus;

b. use the connection between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, antidifferentiation) to solve problems;

c. connect concepts to their visual representations with and without technology; and

d. identify a common underlying structure in problems involving different contextual situations.

MPAC 3: Implementing algebraic/computational processes

Students can:

a. select appropriate mathematical strategies;

b. sequence algebraic/computational procedures logically;

c. complete algebraic/computational processes correctly;

d. apply technology strategically to solve problems;

e. attend to precision graphically, numerically, analytically, and verbally and specify units of measure; and

f. connect the results of algebraic/computational processes to the question asked.

MPAC 4: Connecting multiple representations

Students can:

a. associate tables, graphs, and symbolic representations of functions;

b. develop concepts using graphical, symbolical, verbal, or numerical representations with and without technology;

c. identify how mathematical characteristics of functions are related in different representations;

d. extract and interpret mathematical content from any presentation of a function (e.g., utilize information from a table of values);

e. construct one representational form from another (e.g., a table from a graph or a graph from given information); and

f. consider multiple representations (graphical, numerical, analytical, and verbal) of a function to select or construct a useful representation for solving a problem.

MPAC 5: Building notational fluency

Students can:

a. know and use a variety of notations (e.g., $f'(x), y', \frac{dy}{dx}$);

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b. connect notation to definitions (e.g., relating the notation for the definite integral to that of the limit of a Riemann sum);

c. connect notation to different representations (graphical, numerical, analytical, and verbal); and

d. assign meaning to notation, accurately interpreting the notation in a given problem and across different contexts.

MPAC 6: Communicating

Students can:

a. clearly present methods, reasoning, justifications, and conclusions;

b. use accurate and precise language and notation;

c. explain the meaning of expressions, notation, and results in terms of a context (including units);

d. explain the connections among concepts;

e. critically interpret and accurately report information provided by technology; and

f. analyze, evaluate, and compare the reasoning of others.

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The Concept Outline

Big Idea 1: Limits

Many calculus concepts are developed by first considering a discrete model and then the consequences of a limiting case. Therefore, the idea of limits is essential for discovering and developing important ideas, definitions, formulas, and theorems in calculus. Students must have a solid, intuitive understanding of limits and be able to compute various limits, including one-sided limits, limits at infinity, the limit of a sequence, and infinite limits. They should be able to work with tables and graphs in order to estimate the limit of a function at a point. Students should know the algebraic properties of limits and techniques for finding limits of indeterminate forms, and they should be able to apply limits to understand the behavior of a function near a point. Students must also understand how limits are used to determine continuity, a fundamental property of functions.

Enduring Understandings (Students will understand that …)	**Learning Objectives** (Students will be able to …)	**Essential Knowledge** (Students will know that …)
EU 1.1: The concept of a limit can be used to understand the behavior of functions.	**LO 1.1A(a):** Express limits symbolically using correct notation. **LO 1.1A(b):** Interpret limits expressed symbolically.	**EK 1.1A1:** Given a function f, the limit of $f(x)$ as x approaches c is a real number R if $f(x)$ can be made arbitrarily close to R by taking x sufficiently close to c (but not equal to c). If the limit exists and is a real number, then the common notation is $\lim_{x\to c} f(x)=R$. **EXCLUSION STATEMENT (EK 1.1A1):** *The epsilon-delta definition of a limit is not assessed on the AP Calculus AB or BC Exam. However, teachers may include this topic in the course if time permits.*
		EK 1.1A2: The concept of a limit can be extended to include one-sided limits, limits at infinity, and infinite limits.
		EK 1.1A3: A limit might not exist for some functions at particular values of x. Some ways that the limit might not exist are if the function is unbounded, if the function is oscillating near this value, or if the limit from the left does not equal the limit from the right. **EXAMPLES OF LIMITS THAT DO NOT EXIST:** $\lim_{x\to 0}\frac{1}{x^2}=\infty$ $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ *does not exist* $\lim_{x\to 0}\frac{\lvert x\rvert}{x}$ *does not exist* $\lim_{x\to 0}\frac{1}{x}$ *does not exist*
	LO 1.1B: Estimate limits of functions.	**EK 1.1B1:** Numerical and graphical information can be used to estimate limits.

Note: In the Concept Outline, subject matter that is included only in the BC course is indicated with blue shading.

Enduring Understandings (Students will understand that …)	**Learning Objectives** (Students will be able to …)	**Essential Knowledge** (Students will know that …)
EU 1.1: The concept of a limit can be used to understand the behavior of functions. *(continued)*	**LO 1.1C**: Determine limits of functions.	**EK 1.1C1:** Limits of sums, differences, products, quotients, and composite functions can be found using the basic theorems of limits and algebraic rules.
		EK 1.1C2: The limit of a function may be found by using algebraic manipulation, alternate forms of trigonometric functions, or the squeeze theorem.
		EK 1.1C3: Limits of the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ may be evaluated using L'Hospital's Rule.
	LO 1.1D: Deduce and interpret behavior of functions using limits.	**EK 1.1D1:** Asymptotic and unbounded behavior of functions can be explained and described using limits.
		EK 1.1D2: Relative magnitudes of functions and their rates of change can be compared using limits.
EU 1.2: Continuity is a key property of functions that is defined using limits.	**LO 1.2A**: Analyze functions for intervals of continuity or points of discontinuity.	**EK 1.2A1:** A function f is continuous at $x = c$ provided that $f(c)$ exists, $\lim_{x \to c} f(x)$ exists, and $\lim_{x \to c} f(x) = f(c)$.
		EK 1.2A2: Polynomial, rational, power, exponential, logarithmic, and trigonometric functions are continuous at all points in their domains.
		EK 1.2A3: Types of discontinuities include removable discontinuities, jump discontinuities, and discontinuities due to vertical asymptotes.
	LO 1.2B: Determine the applicability of important calculus theorems using continuity.	**EK 1.2B1:** Continuity is an essential condition for theorems such as the Intermediate Value Theorem, the Extreme Value Theorem, and the Mean Value Theorem.

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Big Idea 2: Derivatives

Using derivatives to describe the rate of change of one variable with respect to another variable allows students to understand change in a variety of contexts. In AP Calculus, students build the derivative using the concept of limits and use the derivative primarily to compute the instantaneous rate of change of a function. Applications of the derivative include finding the slope of a tangent line to a graph at a point, analyzing the graph of a function (for example, determining whether a function is increasing or decreasing and finding concavity and extreme values), and solving problems involving rectilinear motion. Students should be able to use different definitions of the derivative, estimate derivatives from tables and graphs, and apply various derivative rules and properties. In addition, students should be able to solve separable differential equations, understand and be able to apply the Mean Value Theorem, and be familiar with a variety of real-world applications, including related rates, optimization, and growth and decay models.

Enduring Understandings (Students will understand that …)	**Learning Objectives** (Students will be able to …)	**Essential Knowledge** (Students will know that …)
EU 2.1: The derivative of a function is defined as the limit of a difference quotient and can be determined using a variety of strategies.	**LO 2.1A:** Identify the derivative of a function as the limit of a difference quotient.	**EK 2.1A1:** The difference quotients $\frac{f(a+h)-f(a)}{h}$ and $\frac{f(x)-f(a)}{x-a}$ express the average rate of change of a function over an interval.
		EK 2.1A2: The instantaneous rate of change of a function at a point can be expressed by $\lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h}$ or $\lim\limits_{x\to a}\frac{f(x)-f(a)}{x-a}$, provided that the limit exists. These are common forms of the definition of the derivative and are denoted $f'(a)$.
		EK 2.1A3: The derivative of f is the function whose value at x is $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$ provided this limit exists.
		EK 2.1A4: For $y=f(x)$, notations for the derivative include $\frac{dy}{dx}$, $f'(x)$, and y'.
		EK 2.1A5: The derivative can be represented graphically, numerically, analytically, and verbally.
	LO 2.1B: Estimate derivatives.	**EK 2.1B1:** The derivative at a point can be estimated from information given in tables or graphs.

Enduring Understandings (Students will understand that …)	**Learning Objectives** (Students will be able to …)	**Essential Knowledge** (Students will know that …)
EU 2.1: The derivative of a function is defined as the limit of a difference quotient and can be determined using a variety of strategies. *(continued)*	**LO 2.1C:** Calculate derivatives.	**EK 2.1C1:** Direct application of the definition of the derivative can be used to find the derivative for selected functions, including polynomial, power, sine, cosine, exponential, and logarithmic functions.
		EK 2.1C2: Specific rules can be used to calculate derivatives for classes of functions, including polynomial, rational, power, exponential, logarithmic, trigonometric, and inverse trigonometric.
		EK 2.1C3: Sums, differences, products, and quotients of functions can be differentiated using derivative rules.
		EK 2.1C4: The chain rule provides a way to differentiate composite functions.
		EK 2.1C5: The chain rule is the basis for implicit differentiation.
		EK 2.1C6: The chain rule can be used to find the derivative of an inverse function, provided the derivative of that function exists.
		EK 2.1C7: (BC) Methods for calculating derivatives of real-valued functions can be extended to vector-valued functions, parametric functions, and functions in polar coordinates.
	LO 2.1D: Determine higher order derivatives.	**EK 2.1D1:** Differentiating f' produces the second derivative f'', provided the derivative of f' exists; repeating this process produces higher order derivatives of f.
		EK 2.1D2: Higher order derivatives are represented with a variety of notations. For $y = f(x)$, notations for the second derivative include $\frac{d^2y}{dx^2}$, $f''(x)$, and y''. Higher order derivatives can be denoted $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$.

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Enduring Understandings (Students will understand that …)	**Learning Objectives** (Students will be able to …)	**Essential Knowledge** (Students will know that …)
EU 2.2: A function's derivative, which is itself a function, can be used to understand the behavior of the function.	**LO 2.2A:** Use derivatives to analyze properties of a function.	**EK 2.2A1:** First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection.
		EK 2.2A2: Key features of functions and their derivatives can be identified and related to their graphical, numerical, and analytical representations.
		EK 2.2A3: Key features of the graphs of f, f', and f'' are related to one another.
		EK 2.2A4: (BC) For a curve given by a polar equation $r = f(\theta)$, derivatives of r, x, and y with respect to θ and first and second derivatives of y with respect to x can provide information about the curve.
	LO 2.2B: Recognize the connection between differentiability and continuity.	**EK 2.2B1:** A continuous function may fail to be differentiable at a point in its domain.
		EK 2.2B2: If a function is differentiable at a point, then it is continuous at that point.
EU 2.3: The derivative has multiple interpretations and applications including those that involve instantaneous rates of change.	**LO 2.3A:** Interpret the meaning of a derivative within a problem.	**EK 2.3A1:** The unit for $f'(x)$ is the unit for f divided by the unit for x.
		EK 2.3A2: The derivative of a function can be interpreted as the instantaneous rate of change with respect to its independent variable.
	LO 2.3B: Solve problems involving the slope of a tangent line.	**EK 2.3B1:** The derivative at a point is the slope of the line tangent to a graph at that point on the graph.
		EK 2.3B2: The tangent line is the graph of a locally linear approximation of the function near the point of tangency.
	LO 2.3C: Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion.	**EK 2.3C1:** The derivative can be used to solve rectilinear motion problems involving position, speed, velocity, and acceleration.
		EK 2.3C2: The derivative can be used to solve related rates problems, that is, finding a rate at which one quantity is changing by relating it to other quantities whose rates of change are known.
		EK 2.3C3: The derivative can be used to solve optimization problems, that is, finding a maximum or minimum value of a function over a given interval.
		EK 2.3C4: (BC) Derivatives can be used to determine velocity, speed, and acceleration for a particle moving along curves given by parametric or vector-valued functions.

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Enduring Understandings (Students will understand that …)	**Learning Objectives** (Students will be able to …)	**Essential Knowledge** (Students will know that …)
EU 2.3: The derivative has multiple interpretations and applications including those that involve instantaneous rates of change. *(continued)*	**LO 2.3D:** Solve problems involving rates of change in applied contexts.	**EK 2.3D1:** The derivative can be used to express information about rates of change in applied contexts.
	LO 2.3E: Verify solutions to differential equations.	**EK 2.3E1:** Solutions to differential equations are functions or families of functions.
		EK 2.3E2: Derivatives can be used to verify that a function is a solution to a given differential equation.
	LO 2.3F: Estimate solutions to differential equations.	**EK 2.3F1:** Slope fields provide visual clues to the behavior of solutions to first order differential equations.
		EK 2.3F2: (BC) For differential equations, Euler's method provides a procedure for approximating a solution or a point on a solution curve.
EU 2.4: The Mean Value Theorem connects the behavior of a differentiable function over an interval to the behavior of the derivative of that function at a particular point in the interval.	**LO 2.4A:** Apply the Mean Value Theorem to describe the behavior of a function over an interval.	**EK 2.4A1:** If a function f is continuous over the interval $[a, b]$ and differentiable over the interval (a,b), the Mean Value Theorem guarantees a point within that open interval where the instantaneous rate of change equals the average rate of change over the interval.

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Big Idea 3: Integrals and the Fundamental Theorem of Calculus

Integrals are used in a wide variety of practical and theoretical applications. AP Calculus students should understand the definition of a definite integral involving a Riemann sum, be able to approximate a definite integral using different methods, and be able to compute definite integrals using geometry. They should be familiar with basic techniques of integration and properties of integrals. The interpretation of a definite integral is an important skill, and students should be familiar with area, volume, and motion applications, as well as with the use of the definite integral as an accumulation function. It is critical that students grasp the relationship between integration and differentiation as expressed in the Fundamental Theorem of Calculus — a central idea in AP Calculus. Students should be able to work with and analyze functions defined by an integral.

Enduring Understandings (Students will understand that …)	**Learning Objectives** (Students will be able to …)	**Essential Knowledge** (Students will know that …)
EU 3.1: Antidifferentiation is the inverse process of differentiation.	**LO 3.1A**: Recognize antiderivatives of basic functions.	**EK 3.1A1:** An antiderivative of a function f is a function g whose derivative is f. **EK 3.1A2:** Differentiation rules provide the foundation for finding antiderivatives.
EU 3.2: The definite integral of a function over an interval is the limit of a Riemann sum over that interval and can be calculated using a variety of strategies.	**LO 3.2A(a):** Interpret the definite integral as the limit of a Riemann sum. **LO 3.2A(b):** Express the limit of a Riemann sum in integral notation.	**EK 3.2A1:** A Riemann sum, which requires a partition of an interval I, is the sum of products, each of which is the value of the function at a point in a subinterval multiplied by the length of that subinterval of the partition. **EK 3.2A2:** The definite integral of a continuous function f over the interval $[a, b]$, denoted by $\int_a^b f(x)dx$, is the limit of Riemann sums as the widths of the subintervals approach 0. That is, $\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i$ where x_i^* is a value in the ith subinterval, Δx_i is the width of the ith subinterval, n is the number of subintervals, and $\max \Delta x_i$ is the width of the largest subinterval. Another form of the definition is $\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i$, where $\Delta x_i = \frac{b-a}{n}$ and x_i^* is a value in the ith subinterval. **EK 3.2A3:** The information in a definite integral can be translated into the limit of a related Riemann sum, and the limit of a Riemann sum can be written as a definite integral.

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Enduring Understandings (Students will understand that …)	**Learning Objectives** (Students will be able to …)	**Essential Knowledge** (Students will know that …)
EU 3.2: The definite integral of a function over an interval is the limit of a Riemann sum over that interval and can be calculated using a variety of strategies. *(continued)*	**LO 3.2B:** Approximate a definite integral.	**EK 3.2B1:** Definite integrals can be approximated for functions that are represented graphically, numerically, algebraically, and verbally.
		EK 3.2B2: Definite integrals can be approximated using a left Riemann sum, a right Riemann sum, a midpoint Riemann sum, or a trapezoidal sum; approximations can be computed using either uniform or nonuniform partitions.
	LO 3.2C: Calculate a definite integral using areas and properties of definite integrals.	**EK 3.2C1:** In some cases, a definite integral can be evaluated by using geometry and the connection between the definite integral and area.
		EK 3.2C2: Properties of definite integrals include the integral of a constant times a function, the integral of the sum of two functions, reversal of limits of integration, and the integral of a function over adjacent intervals.
		EK 3.2C3: The definition of the definite integral may be extended to functions with removable or jump discontinuities.
	LO 3.2D: (BC) Evaluate an improper integral or show that an improper integral diverges.	**EK 3.2D1: (BC)** An improper integral is an integral that has one or both limits infinite or has an integrand that is unbounded in the interval of integration.
		EK 3.2D2: (BC) Improper integrals can be determined using limits of definite integrals.
EU 3.3: The Fundamental Theorem of Calculus, which has two distinct formulations, connects differentiation and integration.	**LO 3.3A:** Analyze functions defined by an integral.	**EK 3.3A1:** The definite integral can be used to define new functions; for example, $f(x)=\int_0^x e^{-t^2}\,dt$.
		EK 3.3A2: If f is a continuous function on the interval $[a, b]$, then $\frac{d}{dx}\left(\int_a^x f(t)dt\right)=f(x)$, where x is between a and b.
		EK 3.3A3: Graphical, numerical, analytical, and verbal representations of a function f provide information about the function g defined as $g(x)=\int_a^x f(t)dt$.

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Enduring Understandings (Students will understand that …)	**Learning Objectives** (Students will be able to …)	**Essential Knowledge** (Students will know that …)
EU 3.3: The Fundamental Theorem of Calculus, which has two distinct formulations, connects differentiation and integration. *(continued)*	**LO 3.3B(a):** Calculate antiderivatives.	**EK 3.3B1:** The function defined by $F(x)=\int_a^x f(t)dt$ is an antiderivative of f.
	LO 3.3B(b): Evaluate definite integrals.	**EK 3.3B2:** If f is continuous on the interval $[a, b]$ and F is an antiderivative of f, then $\int_a^b f(x)dx = F(b) - F(a)$.
		EK 3.3B3: The notation $\int f(x)dx = F(x)+C$ means that $F'(x) = f(x)$, and $\int f(x)dx$ is called an indefinite integral of the function f.
		EK 3.3B4: Many functions do not have closed form antiderivatives.
		EK 3.3B5: Techniques for finding antiderivatives include algebraic manipulation such as long division and completing the square, substitution of variables, **(BC)** integration by parts, and nonrepeating linear partial fractions.
EU 3.4: The definite integral of a function over an interval is a mathematical tool with many interpretations and applications involving accumulation.	**LO 3.4A:** Interpret the meaning of a definite integral within a problem.	**EK 3.4A1:** A function defined as an integral represents an accumulation of a rate of change.
		EK 3.4A2: The definite integral of the rate of change of a quantity over an interval gives the net change of that quantity over that interval.
		EK 3.4A3: The limit of an approximating Riemann sum can be interpreted as a definite integral.
	LO 3.4B: Apply definite integrals to problems involving the average value of a function.	**EK 3.4B1:** The average value of a function f over an interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$.
	LO 3.4C: Apply definite integrals to problems involving motion.	**EK 3.4C1:** For a particle in rectilinear motion over an interval of time, the definite integral of velocity represents the particle's displacement over the interval of time, and the definite integral of speed represents the particle's total distance traveled over the interval of time.
		EK 3.4C2: (BC) The definite integral can be used to determine displacement, distance, and position of a particle moving along a curve given by parametric or vector-valued functions.

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Enduring Understandings (Students will understand that …)	**Learning Objectives** (Students will be able to …)	**Essential Knowledge** (Students will know that …)
EU 3.4: The definite integral of a function over an interval is a mathematical tool with many interpretations and applications involving accumulation. *(continued)*	**LO 3.4D:** Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.	**EK 3.4D1:** Areas of certain regions in the plane can be calculated with definite integrals. **(BC)** Areas bounded by polar curves can be calculated with definite integrals.
		EK 3.4D2: Volumes of solids with known cross sections, including discs and washers, can be calculated with definite integrals.
		EK 3.4D3: (BC) The length of a planar curve defined by a function or by a parametrically defined curve can be calculated using a definite integral.
	LO 3.4E: Use the definite integral to solve problems in various contexts.	**EK 3.4E1:** The definite integral can be used to express information about accumulation and net change in many applied contexts.
EU 3.5: Antidifferentiation is an underlying concept involved in solving separable differential equations. Solving separable differential equations involves determining a function or relation given its rate of change.	**LO 3.5A:** Analyze differential equations to obtain general and specific solutions.	**EK 3.5A1:** Antidifferentiation can be used to find specific solutions to differential equations with given initial conditions, including applications to motion along a line, exponential growth and decay, **(BC)** and logistic growth.
		EK 3.5A2: Some differential equations can be solved by separation of variables.
		EK 3.5A3: Solutions to differential equations may be subject to domain restrictions.
		EK 3.5A4: The function F defined by $F(x)=c+\int_a^x f(t)dt$ is a general solution to the differential equation $\frac{dy}{dx}=f(x)$, and $F(x)=y_0+\int_a^x f(t)dt$ is a particular solution to the differential equation $\frac{dy}{dx}=f(x)$ satisfying $F(a)=y_0$.
	LO 3.5B: Interpret, create, and solve differential equations from problems in context.	**EK 3.5B1:** The model for exponential growth and decay that arises from the statement "The rate of change of a quantity is proportional to the size of the quantity" is $\frac{dy}{dt}=ky$.
		EK 3.5B2: (BC) The model for logistic growth that arises from the statement "The rate of change of a quantity is jointly proportional to the size of the quantity and the difference between the quantity and the carrying capacity" is $\frac{dy}{dt}=ky(a-y)$.

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Big Idea 4: Series (BC)

The AP Calculus BC curriculum includes the study of series of numbers, power series, and various methods to determine convergence or divergence of a series. Students should be familiar with Maclaurin series for common functions and general Taylor series representations. Other topics include the radius and interval of convergence and operations on power series. The technique of using power series to approximate an arbitrary function near a specific value allows for an important connection to the tangent-line problem and is a natural extension that helps achieve a better approximation. The concept of approximation is a common theme throughout AP Calculus, and power series provide a unifying, comprehensive conclusion.

Enduring Understandings (Students will understand that …)	**Learning Objectives** (Students will be able to …)	**Essential Knowledge** (Students will know that …)
EU 4.1: The sum of an infinite number of real numbers may converge.	**LO 4.1A:** Determine whether a series converges or diverges.	**EK 4.1A1:** The *n*th partial sum is defined as the sum of the first *n* terms of a sequence.
		EK 4.1A2: An infinite series of numbers converges to a real number S (or has sum S), if and only if the limit of its sequence of partial sums exists and equals S.
		EK 4.1A3: Common series of numbers include geometric series, the harmonic series, and *p*-series.
		EK 4.1A4: A series may be absolutely convergent, conditionally convergent, or divergent.
		EK 4.1A5: If a series converges absolutely, then it converges.
		EK 4.1A6: In addition to examining the limit of the sequence of partial sums of the series, methods for determining whether a series of numbers converges or diverges are the *n*th term test, the comparison test, the limit comparison test, the integral test, the ratio test, and the alternating series test.
		EXCLUSION STATEMENT (EK 4.1A6): *Other methods for determining convergence or divergence of a series of numbers are not assessed on the AP Calculus AB or BC Exam. However, teachers may include these topics in the course if time permits.*

Enduring Understandings (Students will understand that …)	**Learning Objectives** (Students will be able to …)	**Essential Knowledge** (Students will know that …)
EU 4.1: The sum of an infinite number of real numbers may converge. *(continued)*	**LO 4.1B:** Determine or estimate the sum of a series.	**EK 4.1B1:** If a is a real number and r is a real number such that $\|r\| < 1$, then the geometric series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.
		EK 4.1B2: If an alternating series converges by the alternating series test, then the alternating series error bound can be used to estimate how close a partial sum is to the value of the infinite series.
		EK 4.1B3: If a series converges absolutely, then any series obtained from it by regrouping or rearranging the terms has the same value.
EU 4.2: A function can be represented by an associated power series over the interval of convergence for the power series.	**LO 4.2A:** Construct and use Taylor polynomials.	**EK 4.2A1:** The coefficient of the *n*th-degree term in a Taylor polynomial centered at $x = a$ for the function f is $\frac{f^{(n)}(a)}{n!}$.
		EK 4.2A2: Taylor polynomials for a function f centered at $x = a$ can be used to approximate function values of f near $x = a$.
		EK 4.2A3: In many cases, as the degree of a Taylor polynomial increases, the *n*th-degree polynomial will converge to the original function over some interval.
		EK 4.2A4: The Lagrange error bound can be used to bound the error of a Taylor polynomial approximation to a function.
		EK 4.2A5: In some situations where the signs of a Taylor polynomial are alternating, the alternating series error bound can be used to bound the error of a Taylor polynomial approximation to the function.
	LO 4.2B: Write a power series representing a given function.	**EK 4.2B1:** A power series is a series of the form $\sum_{n=0}^{\infty} a_n (x-r)^n$ where n is a non-negative integer, $\{a_n\}$ is a sequence of real numbers, and r is a real number.
		EK 4.2B2: The Maclaurin series for $\sin(x)$, $\cos(x)$, and e^x provide the foundation for constructing the Maclaurin series for other functions.
		EK 4.2B3: The Maclaurin series for $\frac{1}{1-x}$ is a geometric series.
		EK 4.2B4: A Taylor polynomial for $f(x)$ is a partial sum of the Taylor series for $f(x)$.

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Enduring Understandings (Students will understand that …)	**Learning Objectives** (Students will be able to …)	**Essential Knowledge** (Students will know that …)
EU 4.2: A function can be represented by an associated power series over the interval of convergence for the power series. *(continued)*	**LO 4.2B:** Write a power series representing a given function. *(continued)*	**EK 4.2B5:** A power series for a given function can be derived by various methods (e.g., algebraic processes, substitutions, using properties of geometric series, and operations on known series such as term-by-term integration or term-by-term differentiation).
	LO 4.2C: Determine the radius and interval of convergence of a power series.	**EK 4.2C1:** If a power series converges, it either converges at a single point or has an interval of convergence.
		EK 4.2C2: The ratio test can be used to determine the radius of convergence of a power series.
		EK 4.2C3: If a power series has a positive radius of convergence, then the power series is the Taylor series of the function to which it converges over the open interval.
		EK 4.2C4: The radius of convergence of a power series obtained by term-by-term differentiation or term-by-term integration is the same as the radius of convergence of the original power series.

AP Calculus AB and AP Calculus BC Instructional Approaches

The AP Calculus AB and AP Calculus BC courses are designed to help students develop a conceptual understanding of college-level calculus content, as well as proficiency in the skills and practices needed for mathematical reasoning and problem solving. After completing the course, students should be able to apply critical thinking, reasoning, and problem-solving skills in a variety of contexts; use calculus terminology and notations appropriately; and clearly communicate their findings using mathematical evidence and justifications.

When designing a plan to teach the course, teachers should keep in mind that in order for students to master the content and skills relevant to calculus, students need prerequisite content knowledge and skills. Addressing these conceptual gaps — particularly those relevant to algebra — will require ongoing formative assessment, strategic scaffolding, and targeted differentiation. Taking the time to plan ahead and anticipate these challenges will ultimately provide a stronger foundation for students' understanding of the concepts presented in the curriculum framework.

This section on instructional approaches provides teachers with recommendations for and examples of how to implement the curriculum framework in practical ways in the classroom.

I. Organizing the Course

The *AP Calculus AB and AP Calculus BC Curriculum Framework* presents a concept outline that is designed to build enduring understanding of the course content. While teachers typically address these concepts sequentially, the framework is designed to allow for flexibility in the instructional approaches teachers choose to incorporate. Three sample approaches organized by topic are shown in the table that follows.

Note that while each organizational approach has a particular emphasis, none are mutually exclusive from the others. For example, courses employing a technology-based approach would not focus entirely on technology, nor would courses designed around the other approaches neglect the use of technology. An AP Calculus classroom often incorporates elements from different approaches across various units of instruction. The table that follows notes the strengths for each type of approach and highlights ways in which each could incorporate strategies from and make connections to the others.

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Approach	Key Characteristics	Making Connections
Inquiry Organizing the course with an inquiry-based approach allows students to explore content through investigative activities such as experiments and hypothetical scenarios.	▸ Encourages the creation of knowledge versus the memorization of facts ▸ Provides opportunities for students to derive definitions and take ownership of concepts by exploring patterns and relationships ▸ Emphasizes questioning and discussion, with a focus on the "why" rather than just the "what"	**Technology** Students can conduct investigative activities using calculators, applets, or modeling software to visualize patterns and explore changes as they occur. **Applications** Students can conduct experiments relevant to their class, school, or community and use their findings to make generalizations to broader contexts.
Application A course organized with an applications-based approach emphasizes the use of real-world applications and problem solving in diverse contexts.	▸ Encourages exploration of concepts through real-world problem-solving scenarios ▸ Makes connections to career and industry applications ▸ Emphasizes modeling and communication of results to a broader audience	**Technology** Students can use calculators, applets, or modeling software populated with real-world data to explore relationships and solve problems within a particular context. **Inquiry** Students can solve problems presented as case studies or real-world investigations and then communicate their solutions as though presenting to a particular audience.
Technology Organizing the course using a technology-based approach means that instructional exercises and independent practice emphasize the use of technology to deepen understanding of course content. Technology can include graphing calculators, online simulators, interactive applets, and modeling software, among other tools.	▸ Allows students to explore and verify hypotheses formed by examining data and manipulating graphs. ▸ Allows students to "see" the concepts ▸ Incorporates both graphing calculators and modeling software ▸ Allows students to compare multiple representations of functions	**Inquiry** Students can use technology to explore patterns and relationships, use their findings to derive new information, and then verify that information again using technology. **Applications** Students can explore representations provided by technology as a way to visualize information relevant to real-world scenarios.

Students will benefit most when all three approaches are incorporated regularly throughout the course, allowing them to see how calculus concepts can be explored through inquiry, applied to real-world contexts, and visualized through the use of technology.

II. Linking the Practices and the Learning Objectives

The six Mathematical Practices for AP Calculus (MPACs) presented in the curriculum framework explicitly describe the practices students will need to apply in order to build conceptual understanding and demonstrate mastery of the learning objectives.

Teaching the learning objectives in connection to different practices

Each of the six mathematical practices contains a list of subskills that students must acquire in order to reach competency in that practice. Each learning objective in the curriculum framework can be tied to one or more of these subskills. Thus there are many opportunities for integrating these skills with the content of the course, as many mathematical practices will naturally align with more than one learning objective.

For example, the mathematical practice of interpreting one representational form from another (MPAC 4d) is reflected in Learning Objective 1.1B, where students may need to use either tables or graphs to estimate limits of functions:

Learning Objective 1.1B	**MPAC 4: Connecting multiple representations**
Estimate limits of functions	*Students can extract and interpret mathematical content from any presentation of a function (e.g., utilize information from a table of values).* – MPAC 4d

This same learning objective could also be taught with an emphasis on confirming that the conditions for a hypothesis have been satisfied (MPAC 1b), because in order to determine the limit of a function, students must demonstrate awareness of the conditions under which a limit exists:

Learning Objective 1.1B	**MPAC 1: Reasoning with definitions and theorems**
Estimate limits of functions	*Students can confirm that hypotheses have been satisfied in order to apply the conclusion of a theorem.* – MPAC 1b

Scaffolding practices across multiple learning objectives

The sequential nature of the learning objectives within a big idea provides multiple opportunities to apply mathematical practices in various contexts and scaffold the development of students' critical-thinking, reasoning, and problem-solving skills throughout the course.

For example, Learning Objective 1.2B is often addressed early in the course, and students could be asked to produce examples and counterexamples to clarify their understanding of those theorems (MPAC 1f). This same practice could then be revisited with increasing levels of complexity at multiple points later in the course; for instance, when addressing learning objectives 2.2B and 3.1A:

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MPAC 1: Reasoning with definitions and theorems

Students can produce examples and counterexamples to clarify understanding of definitions, to investigate whether converses of theorems are true or false, or to test conjectures. (MPAC 1f)

- **Scaffolding opportunity #1**

  LO 1.2B *Determine the applicability of important calculus theorems using continuity.*

- **Scaffolding opportunity #2**

  LO 2.2B *Recognize the connection between differentiability and continuity.*

- **Scaffolding opportunity #3**

  LO 3.1A *Recognize antiderivatives of basic functions.*

When planning the integration of these practices, teachers should take special note of which MPACs could also help to scaffold algebraic computational and reasoning skills. For example, students who struggle with connecting their results to the question being asked might benefit from instructional activities that emphasize MPAC 3f at multiple points and in a variety of contexts:

MPAC 3f
Students can connect the results of algebraic/computational processes to the question asked.

LO 1.1C *Determine the limits of functions.*	→	**LO 2.1C** *Calculate derivatives.*	→	**LO 2.3B** *Solve problems involving the slope of a tangent line.*

III. Teaching the Broader Skills

The MPACs help students build conceptual understanding of calculus topics and develop the skills that will be valuable in subsequent math courses. These practices are also critical for helping students develop a broader set of critical thinking skills that can be applied beyond the scope of the course. Through the use of guided questioning, discussion techniques, and other instructional strategies, teachers can help students practice justification, reasoning, modeling, interpretation, drawing conclusions, building arguments, and applying what they know in new contexts, providing an important foundation for students' college and career readiness.

The table that follows provides examples of MPACs and strategies that help to support the development of each of these broader skills. See section IV for a glossary that defines and explains the purpose of each strategy.

Broader skill	Students can display this by (sample MPACs)	Questioning and instructional cues	Other strategies to develop proficiency
Justification	**MPAC 1: Reasoning with definitions and theorems** ▸ Using definitions and theorems to build arguments, to justify conclusions or answers, and to prove results (1a) ▸ Confirming that hypotheses have been satisfied in order to apply the conclusion of a theorem (1b) ▸ Producing examples and counterexamples to clarify understanding of definitions, to investigate whether converses of theorems are true or false, or to test conjectures (1f) **MPAC 6: Communicating** ▸ Clearly presenting methods, reasoning, justifications, and conclusions (6a) ▸ Using accurate and precise language and notation (6b)	*How do you know … ?* *How could we test … ?* *Show me an example of a solution that would NOT work in this context.*	Error analysis Critique reasoning Sharing and responding Think-pair-share
Reasoning	**MPAC 1: Reasoning with definitions and theorems** ▸ Using definitions and theorems to build arguments, to justify conclusions or answers, and to prove results (1a) ▸ Confirming that hypotheses have been satisfied in order to apply the conclusion of a theorem (1b) **MPAC 2: Connecting concepts** ▸ Relating the concept of a limit to all aspects of calculus (2a) ▸ Using the connection between concepts or processes to solve problems (2b) ▸ Identifying a common underlying structure in problems involving different contextual situations (2d) **MPAC 4: Connecting multiple representations** ▸ Identifying how mathematical characteristics of functions are related in different representations (4c) **MPAC 5: Building notational fluency** ▸ Connecting notation to definitions (5b) **MPAC 6: Communicating** ▸ Clearly presenting methods, reasoning, justifications, and conclusions (6a) ▸ Using accurate and precise language and notation (6b) ▸ Explaining the connections among concepts (6d) ▸ Analyzing, evaluating, and comparing the reasoning of others (6f)	*Under what conditions … ?* *How is this related to … ?* *What would happen if … ?* *How is this similar to (or different from) … ?* *What patterns do you see?*	Quickwrite Note-taking Look for a pattern Construct an argument Graphic organizer Think aloud Critique reasoning Debriefing Sharing and responding Think-pair-share

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Broader skill	Students can display this by (sample MPACs)	Questioning and instructional cues	Other strategies to develop proficiency
Modeling	**MPAC 1: Reasoning with definitions and theorems** ▶ Developing conjectures based on exploration with technology (1e) **MPAC 2: Connecting concepts** ▶ Connecting concepts to their visual representations with and without technology (2c) **MPAC 4: Connecting multiple representations** ▶ Associating tables, graphs, and symbolic representations of functions (4a) ▶ Developing concepts using graphical, symbolical, verbal, or numerical representations with and without technology (4b) ▶ Identifying how mathematical characteristics of functions are related in different representations (4c) ▶ Constructing one representational form from another (4e) ▶ Considering multiple representations of a function (graphical, numerical, analytical, and verbal) to select or construct a useful representation for solving a problem (4f) **MPAC 5: Building notational fluency** ▶ Connecting notation to different representations (graphical, numerical, analytical, and verbal) (5c) **MPAC 6: Communicating** ▶ Using accurate and precise language and notation (6b)	*What would a graph of this equation look like?* *How could this graph be represented as an equation?* *How can this situation be represented in a diagram?* *Why is ___ a more appropriate representation than ___ ?*	Use manipulatives Graph and switch Note-taking Create representations Debriefing Ask the expert Sharing and responding Think-pair-share

Broader skill	Students can display this by (sample MPACs)	Questioning and instructional cues	Other strategies to develop proficiency
Interpretation	**MPAC 1: Reasoning with definitions and theorems** ▸ Interpreting quantifiers in definitions and theorems (1d) **MPAC 4: Connecting multiple representations** ▸ Extracting and interpreting mathematical content from any presentation of a function (4d) ▸ Constructing one representational form from another (4e) **MPAC 5: Building notational fluency** ▸ Assigning meaning to notation, accurately interpreting the notation in a given problem and across different contexts (5d) **MPAC 6: Communicating** ▸ Using accurate and precise language and notation (6b) ▸ Explaining the meaning of expressions, notation, and results in terms of a context (including units) (6c) ▸ Critically interpreting and accurately reporting information provided by technology (6e)	*What does* ___ *mean?* *How is this similar to (or different from) … ?* *What units are appropriate?*	Notation read-aloud Note-taking Ask the expert Sharing and responding Think-pair-share
Drawing conclusions	**MPAC 1: Reasoning with definitions and theorems** ▸ Using definitions and theorems to build arguments, to justify conclusions or answers, and to prove results (1a) **MPAC 4: Connecting multiple representations** ▸ Extracting and interpreting mathematical content from any presentation of a function (4d) **MPAC 6: Communicating** ▸ Clearly presenting methods, reasoning, justifications, and conclusions (6a) ▸ Using accurate and precise language and notation (6b) ▸ Explaining the meaning of expressions, notation, and results in terms of a context (including units) (6c) ▸ Critically interpreting and accurately reporting information provided by technology (6e)	*What patterns do you see?* *What would we expect to happen based on this information?* *What does the solution mean in the context of this problem?* *How can we confirm that this solution is correct?*	Look for a pattern Predict and confirm Identify a subtask Guess and check Work backward Think aloud Quickwrite Critique reasoning Sharing and responding Think-pair-share

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Broader skill	Students can display this by (sample MPACs)	Questioning and instructional cues	Other strategies to develop proficiency
Building arguments	**MPAC 1: Reasoning with definitions and theorems** ▶ Using definitions and theorems to build arguments, to justify conclusions or answers, and to prove results (1a) ▶ Producing examples and counterexamples to clarify understanding of definitions, to investigate whether converses of theorems are true or false, or to test conjectures (1f) **MPAC 3: Implementing algebraic/computational processes** ▶ Sequencing algebraic/computational procedures logically (3b) ▶ Attending to precision graphically, numerically, analytically, and verbally, and specifying units of measure (3e) ▶ Connecting the results of algebraic/computational processes to the question asked (3f) **MPAC 4: Connecting multiple representations** ▶ Extracting and interpreting mathematical content from any presentation of a function (4d) **MPAC 5: Building notational fluency** ▶ Connecting notation to different representations (graphical, numerical, analytical, and verbal) (5c) ▶ Assigning meaning to notation, accurately interpreting the notation in a given problem and across different contexts (5d) **MPAC 6: Communicating** ▶ Clearly presenting methods, reasoning, justifications, and conclusions (6a) ▶ Using accurate and precise language and notation (6b) ▶ Explaining the meaning of expressions, notation, and results in terms of a context (including units) (6c) ▶ Explaining the connections among concepts (6d) ▶ Eritically interpreting and accurately reporting information provided by technology (6e)	*What is your hypothesis?* *What line of reasoning did you use to … ?* *How does this step build to the step that follows?* *What does* ___ *mean?* *What evidence do you have to support … ?* *What can you conclude from the evidence?*	Construct an argument Create representations Critique reasoning Error analysis Quickwrite Sharing and responding Think aloud

Broader skill	Students can display this by (sample MPACs)	Questioning and instructional cues	Other strategies to develop proficiency
Application	**MPAC 1: Reasoning with definitions and theorems** ▸ Applying definitions and theorems in the process of solving a problem (1c) **MPAC 3: Implementing algebraic/computational processes** ▸ Selecting appropriate mathematical strategies (3a) ▸ Sequencing algebraic/computational procedures logically (3b) ▸ Completing algebraic/computational processes correctly (3c) ▸ Applying technology strategically to solve problems (3d) ▸ Attending to precision graphically, numerically, analytically, and verbally and specifying units of measure (3e) ▸ Connecting the results of algebraic/computational processes to the question asked (3f) **MPAC 4: Connecting multiple representations** ▸ Considering multiple representations of a function (graphical, numerical, analytical, and verbal) to select or construct a useful representation for solving a problem (4f) **MPAC 5: Building notational fluency** ▸ Knowing and using a variety of notations (5a) ▸ Assigning meaning to notation, accurately interpreting the notation in a given problem and across different contexts (5d) **MPAC 6: Communicating** ▸ Using accurate and precise language and notation (6b)	*What is the problem asking us to find?* *What are the conditions given?* *Can we make a reasonable prediction?* *What information do you need … ?* *Have we solved a problem similar to this?* *What would be a simplified version of this problem?* *What steps are needed?* *When would this be used?* *Did you use all of the information?* *Is there any information that was not needed?* *Does this answer the question being asked?* *Is this solution reasonable? How do you know?*	Model questions Discussion groups Predict and confirm Create a plan Simplify the problem Identify a subtask Guess and check Work backward Marking the text Paraphrasing Think aloud Ask the expert Sharing and responding Think-pair-share

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IV. Representative Instructional Strategies

The *AP Calculus AB and AP Calculus BC Curriculum Framework* outlines the concepts and skills students must master by the end of the courses. In order to address those concepts and skills effectively, teachers must incorporate into their daily lessons and activities a variety of instructional approaches and best practices — strategies that research has shown to have a positive impact on student learning.

The table below provides a definition and explanation for each of the strategies referenced in section III, along with an example of its application in the context of a calculus classroom.

Strategy	Definition	Purpose	Example
Ask the expert	Students are assigned as "experts" on problems they have mastered; groups rotate through the expert stations to learn about problems they have not yet mastered.	Provides opportunities for students to share their knowledge and learn from one another	When learning rules of differentiation, the teacher assigns students as "experts" on product rule, quotient rule, chain rule, and derivatives of transcendental functions. Students rotate through stations in groups, working with the station expert to complete a series of problems using the corresponding rule.
Construct an argument	Students use mathematical reasoning to present assumptions about mathematical situations, support conjectures with mathematically relevant and accurate data, and provide a logical progression of ideas leading to a conclusion that makes sense.	Helps develop the process of evaluating mathematical information, developing reasoning skills, and enhancing communication skills in supporting conjectures and conclusions	This strategy can be used with word problems that do not lend themselves to immediate application of a formula or mathematical process. The teacher can provide distance and velocity graphs that represent a motorist's behavior through several towns on a map and ask students to construct a mathematical argument either in defense of or against a police officer's charge of speeding, given a known speed limit.
Create a plan	Students analyze the tasks in a problem and create a process for completing the tasks by finding the information needed, interpreting data, choosing how to solve a problem, communicating the results, and verifying accuracy.	Assists in breaking tasks into smaller parts and identifying the steps needed to complete the entire task	Given an optimization problem that asks for a choice between two boxes with different dimensions but the same cross-sectional perimeter, students identify the steps needed to determine which box will hold the most candy. This involves selecting an appropriate formula, differentiating the resulting function, applying the second derivative test, and interpreting the results.
Create representations	Students create pictures, tables, graphs, lists, equations, models, and/or verbal expressions to interpret text or data.	Helps organize information using multiple ways to present data and answer a question or show a problem's solution.	In order to evaluate limits, the teacher can introduce a variety of methods, including constructing a graph, creating a table, directly substituting a given value into the function, or applying an algebraic process.

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Strategy	Definition	Purpose	Example
Critique reasoning	Through collaborative discussion, students respond to the arguments of others and question the use of mathematical terminology, assumptions, and conjectures to improve understanding and justify and communicate conclusions.	Helps students learn from each other as they make connections between mathematical concepts and learn to verbalize their understanding and support their arguments with reasoning and data that make sense to peers.	Given a table that lists a jogger's velocity at five different times during her workout, students explain the meaning of the definite integral of the absolute value of the velocity function between the first and the last time recorded. As students discuss their responses in groups, they learn how to communicate specific concepts and quantities using mathematical notation and terminology.
Debriefing	Students discuss the understanding of a concept to lead to a consensus on its meaning.	Helps clarify misconceptions and deepen understanding of content.	In order to discern the difference between average rate of change and instantaneous rate of change, students roll a ball down a simplified ramp and measure the distance the ball travels over time, every second for 5 seconds. Plotting the points and sketching a curve of best fit, students discuss how they might determine the average velocity of the ball over the 5 seconds and then the instantaneous velocity of the ball at 3 seconds. A discussion in which students address the distinction between the ball's velocity between two points and its velocity at a single particular time would assist in clarifying the concept and mathematical process of arriving at the correct answers.
Discussion groups	Students work within groups to discuss content, create problem solutions, and explain and justify a solution.	Aids understanding through the sharing of ideas, interpretation of concepts, and analysis of problem scenarios	Once students learn all methods of integration and choose which is the most appropriate given a particular function, they can discuss in small groups, with pencils down, why a specific method should be used over another.
Error analysis	Students analyze an existing solution to determine whether (or where) errors have occurred.	Allows students to troubleshoot errors and focus on solutions that may arise when they do the same procedures themselves.	When students begin to evaluate definite integrals, they can analyze their answers and troubleshoot any errors that might lead to a negative area when there is a positive accumulation.
Graph and switch	Generating a graph (or sketch of a graph) to model a certain function, then switch calculators (or papers) to review each other's solutions.	Allows students to practice creating different representations of functions and both give and receive feedback on each other's work.	As students learn about integration and finding the area under a curve, they can use calculators to shade in the appropriate area between lower and upper limits while calculating the total accumulation. Since input keystrokes are critical in obtaining the correct numerical value, students calculate their own answers, share their steps with a partner, and receive feedback on their calculator notation and final answer.

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Strategy	Definition	Purpose	Example
Graphic organizer	Students arrange information into charts and diagrams.	Builds comprehension and facilitates discussion by representing information in visual form.	In order to determine the location of relative extrema for a function, students construct a sign chart or number line while applying the first derivative test, marking where the first derivative is positive or negative and determining where the original function is increasing or decreasing.
Guess and check	Students guess the solution to a problem and then check that the guess fits the information in the problem and is an accurate solution.	Allows exploration of different ways to solve a problem; guess and check may be used when other strategies for solving are not obvious.	Teachers can encourage students to employ this strategy for drawing a graphical representation of a given function, given written slope statements and/or limit notation. For example, given two sets of statements that describe the same function, students sketch a graph of the function described from the first statement and check it against the second statement.
Identify a subtask	Students break a problem into smaller pieces whose outcomes lead to a solution.	Helps to organize the pieces of a complex problem and reach a complete solution.	After providing students with the rates in which rainwater flows into and out of a drainpipe, students may be asked to find how many cubic feet of water flow into it during a specific time period, and whether the amount of water in the pipe is increasing or decreasing at a particular instance. Students would begin by distinguishing functions from each other and determining whether differentiation or integration is necessary; they would then perform the appropriate calculations and verify whether they have answered the question.
Look for a pattern	Students observe information or create visual representations to find a trend.	Helps to identify patterns that may be used to make predictions.	Patterns can be detected when approximating area under a curve using Riemann sums. Students calculate areas using left and right endpoint rectangles, midpoint rectangles, and trapezoids, increasing and decreasing the width in order to determine the best method for approximation.
Marking the text	Students highlight, underline, and/or annotate text to focus on key information to help understand the text or solve the problem.	Helps the student identify important information in the text and make notes in the text about the interpretation of tasks required and concepts to apply to reach a solution.	This strategy can be used with problems that involve related rates. Students read through a given problem, underline the given static and changing quantities, list these quantities, and use the quantities to label a sketch that models the situation given in the problem. Students then use this information to substitute for variables in a differential equation.

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Strategy	Definition	Purpose	Example
Model questions	Students answer items from released AP Calculus Exams.	Provides rigorous practice and assesses students' ability to apply multiple mathematical practices on content presented as either a multiple-choice or a free-response question.	After learning how to construct slope fields, students practice by completing free-response questions in which they are asked to sketch slope fields for given differential equations at points indicated.
Notation read aloud	Students read symbols and notational representations aloud.	Helps students to accurately interpret symbolic representations.	This strategy can be used to introduce new symbols and mathematical notation to ensure that students learn proper terminology from the start. For example, after introducing summation notation the teacher can ask students to write or say aloud the verbal translation of a given sum.
Note-taking	Students create a record of information while reading a text or listening to a speaker.	Helps in organizing ideas and processing information.	Students can write down verbal descriptions of the steps needed to solve a differential equation so that a record of the process can be referred to at a later point in time.
Paraphrasing	Students restate in their own words the essential information in a text or problem description.	Assists with comprehension, recall of information, and problem solving.	After reading a mathematical definition from a textbook, students can express the definition in their own words. For example, with parametric equations students explain the difference between y being a function of x directly and x and y both being functions of a parameter t.
Predict and confirm	Students make conjectures about what results will develop in an activity and confirm or modify the conjectures based on outcomes.	Stimulates thinking by making, checking, and correcting predictions based on evidence from the outcome.	Given two sets of cards with functions and the graphs of their derivatives, students attempt to match the functions with their appropriate derivative. Students then calculate the derivatives of the functions using specific rules and graph the derivatives using calculators to confirm their original match selection.
Quickwrite	Students write for a short, specific amount of time about a designated topic.	Helps generate ideas in a short time.	To help synthesize concepts after having learned how to calculate the derivative of a function at a point, students list as many real-world situations as possible in which knowing the instantaneous rate of change of a function is advantageous.
Sharing and responding	Students communicate with another person or a small group of peers who respond to a proposed problem solution.	Gives students the opportunity to discuss their work with peers, make suggestions for improvement to the work of others, and/or receive appropriate and relevant feedback on their own work.	Given tax-rate schedules for single taxpayers in a specific year, students construct functions to represent the amount of tax paid for taxpayers in specific tax brackets. Then students come together in a group to review the constructed functions, make any necessary corrections, and build and graph a single piecewise function to represent the tax-rate schedule for single taxpayers for the specific year.

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Strategy	Definition	Purpose	Example
Simplify the problem	Students use friendlier numbers or functions to help solve a problem.	Provides insight into the problem or the strategies needed to solve the problem.	When applying the chain rule for differentiation or u-substitution for integration, the teacher reviews how to proceed when there is no "inner function," before addressing composite functions.
Think aloud	Students talk through a difficult problem by describing what the text means.	Helps in comprehending the text, understanding the components of a problem, and thinking about possible paths to a solution.	In order to determine if a series converges or diverges, students ask themselves a series of questions out loud to identify series characteristics and corresponding tests (e.g., ratio, root, integral, limit comparison) that are appropriate for determining convergence.
Think-Pair-Share	Students think through a problem alone, pair with a partner to share ideas, then share results with the class.	Enables the development of initial ideas that are then tested with a partner in preparation for revising ideas and sharing them with a larger group.	Given the equation of a discontinuous function, students think of ways to make the function continuous and adjust the given equation to establish such continuity. Then students pair with a partner to share their ideas before sharing out with the whole class.
Use manipulatives	Students use objects to examine relationships between the information given.	Provides a visual representation of data that supports comprehension of information in a problem.	To visualize the steps necessary to find the volume of a solid with a known cross section, students build a physical model on a base with a standard function using foam board or weighted paper to construct several cross sections.
Work backward	Students trace a possible answer back through the solution process to the starting point.	Provides another way to check possible answers for accuracy.	Students can check whether they have found a correct antiderivative by differentiating their answer and comparing it to the original function.

V. Communicating in Mathematics

Each year the Chief Reader Reports for the AP Calculus Exams indicate that students consistently struggle with interpretation, justification, and assigning meaning to solutions within the context of a given problem. For this reason, teachers should pay particular attention to the subskills listed under MPAC 6: Communicating, as these make explicit the discipline-specific communication practices in which calculus students must be able to engage.

Students often need targeted support to develop these skills, so teachers should remind their students that **communicating a solution** is just as important as **finding a solution**, because the true value of a solution lies in the fact that it can be conveyed to a broader audience.

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Teachers should also reinforce that when students are asked to provide reasoning or a justification for their solution, a quality response will include:

- a logical sequence of steps;
- an argument that explains why those steps are appropriate; and
- an accurate interpretation of the solution (with units) in the context of the situation.

In order to help their students develop these communication skills, teachers can:

- have students practice explaining their solutions orally to a small group or to the class;
- present an incomplete argument or explanation and have students supplement it for greater clarification; and
- provide sentence starters, template guides, and tips to help scaffold the writing process.

Teachers also need to remind students that the approach to communicating a solution will, in some cases, depend on the context of the forum or the audience being addressed. For example, a justification on an AP free-response question could possibly include more symbolic notations and a greater level of detail than a narrative description provided for a team project.

VI. Using Formative Assessment to Address Challenge Areas

Formative assessment is a process used to monitor student learning and provide ongoing feedback so that students can improve.[2] Unlike summative assessments, formative assessments do not result in a score or grade because the goal is instead to provide specific, detailed information about what students know and understand in order to inform the learning process.

When teachers use robust formative-assessment strategies, they have a better understanding of their students' learning needs and how those needs could be addressed. For AP Calculus specifically, teacher surveys and student assessment data indicate that gaps in algebraic understanding often contribute to the challenges students experience with foundational concepts such as:

- theorems
- the chain rule
- related rates
- optimization
- the analysis of functions
- area and volume

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[2] **https://www.cmu.edu/teaching/assessment/basics/formative-summative.html**

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In order to mitigate these challenges, teachers must design their course in a way that incorporates both a rigorous approach to formative assessment and a plan for addressing critical areas of need. There are several steps teachers can take in order to mitigate these challenges and support their students' success, including:

Understanding What Students Know

The process of addressing students' misconceptions and gaps in understanding begins with assessing what they already know. This can be done in a variety of ways, such as:

- **Preassessments:** These are assessments administered before teaching a particular topic or unit. It is recommended that teachers begin by examining the upcoming learning objectives, then consider what prerequisite knowledge and skills their students should have in order to ultimately be successful at those objectives. Questions do not necessarily need to be on-level for the course as it may be informative for a teacher to see, for example, whether a particular algebra skill has been mastered before moving on to course-level material. Note that preassessments are particularly helpful when they include questions of varying difficulty so that teachers can develop lessons and activities appropriate for students at different levels of mastery.
- **Student self-analysis:** Teachers provide students with a set of questions that address the content and skills for a particular unit and ask them to rate their ability to solve each one. For example:

	How confident are you in your ability to solve this?	**If you answered "I know how to solve this," use the space below to solve:**
For what value of k, if any, is $\int_0^\infty kxe^{-2x}\,dx = 1$? (A) $\frac{1}{4}$ (B) 1 (C) 4 (D) There is no such value of k.	☐ I don't know how to solve this. ☐ I may know how to solve this, but I could use some assistance. ☐ I know how to solve this.	

This is a low-stakes exercise that allows students to realistically consider their own level of mastery, while providing the teacher with valuable information about students' skills and confidence.

Addressing gaps in understanding

After determining where students are in terms of their content and skills development, teachers can design instructional resources and implement strategies that provide support and address existing gaps. For example:

- **Supplemental resources:** For independent practice and supplemental guidance on particular topics, teachers provide additional resources such as worksheets, online tutorials, textbook readings, and samples of student work.

- **Breaking down activities into subtasks:** For students struggling with a particularly long or challenging type of problem, teachers provide a brief guide that takes students through one of the exercises by identifying subtasks that break the problem up into smaller steps. For instance, if the ultimate goal of the problem is to determine the height of the water in a tank when the height is changing at a given rate defined by a function over an interval, then the subtasks could be to first find the derivative and then substitute values into that function.
- **Structured note-taking:** To help students process information being provided through text readings or direct instruction, teachers provide a note-taking sheet that scaffolds the information with headings, fill-in-the-blank sentences, graphic organizers, space to work out examples, and reminder tips.
- **Graphic organizers:** Teachers use organizers such as charts, Venn diagrams, and other representations to help students visualize information and processes.
- **Self-check assignments:** Teachers provide independent practice with self-checking mechanisms embedded into the task. One way to do this is to include an appendix that provides the correct answers and steps for each problem so students can assess their own progress. Another way is to provide a scrambled list of answers without indicating which problems they are for; students can then see whether their answer is one of those listed, and if their solution is not there they know to revisit the problem using a different approach.

Assessing learning while teaching

Formative assessment occurs in real time and provides information to teachers about whether students understand the information being presented. Incorporating strategies to gauge student understanding **during instruction** allows teachers to make adjustments and correct misunderstandings before they become ongoing challenges that impact student learning of other concepts. These strategies can include:

- **Checks for understanding:** Using hand signals, journal prompts, exit tickets, homework checks, or another approach to assess student learning of a particular topic.
- **Debriefings:** Guiding a discussion with targeted questions in order to deepen students' understanding of a particular topic.

It is also recommended that teachers spiral back to previously covered topics, as this provides additional opportunities to assess retention and reinforce student learning.

Providing feedback

It is important to provide students with real-time feedback both during the learning process and after a formative assessment has occurred. Students who receive specific, meaningful, and timely feedback are more likely to learn from their mistakes and avoid making those errors again in the future.

Effective feedback has the following characteristics:

- It is provided as soon as possible after the error occurs.

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- It addresses the nature of the error using language that is clear and specific.
- It provides actionable steps and/or examples of how to address the error.

For example, consider the feedback statements below:

Ineffective Feedback	**Effective Feedback**
$f(x)$ does not mean to multiply.	$f(x)$ means that x is the independent variable associated with the function f. To find the value of f, substitute a value in for x.
The derivative of $y = 3x^2 \sin x$ is not $\frac{dy}{dx}3x^2 * \frac{dy}{dx}\sin x$.	When finding the derivative of a product of functions, the derivative is calculated as *the first function times the derivative of the second, plus the second function times the derivative of the first*. In this case, the first function is $f(x) = 3x^2$ and the second function is $g(x) = \sin x$, so the derivative is $(3x^2)\left(\frac{d}{dx}\sin x\right) + (\sin x)\left(\frac{d}{dx}3x^2\right)$.
The Mean Value Theorem is not applicable because $f(x)$ is not continuous on the closed interval.	Although $f(x)$ has a limit at $x = 4$, there is a hole at that point so the function is not continuous. Therefore the Mean Value Theorem cannot be applied.

VII. Building a Pipeline for Success

Teachers should take note of areas that appear to present broader challenges or to trigger recurring student misunderstandings, as addressing these will require a more long-term strategy such as:

Communicating with the school's vertical team

Teachers should seek the advice and support of colleagues and administrators, particularly those who are involved with designing the curriculum for prerequisite courses. Scheduling regular check-in meetings will allow for discussion of concerns and the development of collaborative solutions. Areas of focus for these sessions may include:

- **Content across the curriculum:** What is being taught in each course and how do those topics relate to one another or build towards subsequent courses?
- **Assessment:** Do the current assessments reflect the learning objectives?
- **Challenging concepts:** What topics do teachers struggle to teach or do students struggle to learn? What are some common student misconceptions surrounding those topics? How can these challenges be mitigated?
- **Vocabulary coordination across the curriculum:** Are teachers using consistent vocabulary when addressing the same topic? Are students able to describe mathematical terms using everyday language?

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- **Notation coordination across the curriculum:** What are the notations that cause students difficulties? Are symbolic representations (e.g., parentheses) being used consistently from one course to another?

Planning for in-classroom support

Engaging in professional reflections and noting areas for improvement are critical to maintaining an effective instructional practice. After each lesson, teachers should write down observations about what worked and brainstorm ways to make adjustments the next time that lesson is taught. Having informal, one-on-one conversations with students will also provide additional insights into which parts of the lesson were engaging, what strategies helped them make connections, and areas where they could use additional support.

VIII. Using Graphing Calculators and Other Technologies in AP Calculus

The use of a graphing calculator is considered an integral part of the AP Calculus courses, and it is required on some portions of the exams. Professional mathematics organizations such as the National Council of Teachers of Mathematics (NCTM), the Mathematical Association of America (MAA), and the National Academy of Sciences (NAS) Board on Mathematical Sciences and Their Applications have strongly endorsed the use of calculators in mathematics instruction and testing.

Graphing calculators are valuable tools for achieving multiple components of the Mathematical Practices for AP Calculus, including using technology to develop conjectures, connecting concepts to their visual representations, solving problems, and critically interpreting and accurately reporting information. The AP Calculus Program also supports the use of other technologies that are available to students and encourages teachers to incorporate technology into instruction in a variety of ways as a means of facilitating discovery and reflection.

Appropriate examples of graphing calculator use in AP Calculus include, but certainly are not limited to, zooming to reveal local linearity, constructing a table of values to conjecture a limit, developing a visual representation of Riemann sums approaching a definite integral, graphing Taylor polynomials to understand intervals of convergence for Taylor series, or drawing a slope field and investigating how the choice of initial condition affects the solution to a differential equation.

IX. Other Resources for Strengthening Teacher Practice

The College Board provides support for teachers through a variety of tools, resources, and professional development opportunities, including:

a. **AP Teacher Community:** The online community where AP teachers discuss teaching strategies, share resources, and connect with each other: **https://apcommunity.collegeboard.org.**

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b. **Teaching and Assessing AP Calculus:** A collection of free, online professional development modules that provide sample questions to help teachers understand expectations for the AP Calculus exam, and resources to help them implement key instructional strategies in the classroom.

c. **APSI Workshops:** These one-week training sessions begin in the summer of the launch year and will provide in-depth support regarding the new course updates and targeted instructional strategies.

d. **"Try This! Calculus Teaching Tips":** An online article explaining a variety of in-class and out-of-class calculus activities that support student engagement through active learning: **http://apcentral.collegeboard.com/apc/members/courses/teachers_corner/9748.html.**

e. ***Principles to Actions:*** A publication by the National Council of Teachers of Mathematics that includes eight research-based teaching practices to support a high-quality mathematics education for all students: **https://www.nctm.org/uploadedFiles/Standards_and_Positions/PtAExecutiveSummary.pdf.**

The AP Calculus Exams

Exam Information

Students take either the AP Calculus AB Exam or the AP Calculus BC Exam. The exams, which are identical in format, consist of a multiple-choice section and a free-response section, as shown below.

Section	Part	Graphing Calculator	Number of Questions	Time	Percentage of Total Exam Score
Section I: Multiple Choice	Part A	Not permitted	30	60 minutes	**50%**
	Part B	Required	15	45 minutes	
	TOTAL		**45**	**1 hour, 45 minutes**	
Section II: Free Response	Part A	Required	2	30 minutes	**50%**
	Part B	Not permitted	4	60 minutes	
	TOTAL		**6**	**1 hour, 30 minutes**	

Student performance on these two parts will be compiled and weighted to determine an AP Exam score. Each section of the exam counts toward 50 percent of the student's score. Points are not deducted for incorrect answers or unanswered questions.

Exam questions assess the learning objectives detailed in the course outline; as such, they require a strong conceptual understanding of calculus in conjunction with the application of one or more of the mathematical practices. Although topics in subject areas such as algebra, geometry, and precalculus are not explicitly assessed, students must have mastered the relevant preparatory material in order to apply calculus techniques successfully and accurately.

The multiple-choice sections of the AP Calculus Exams are designed for broad coverage of the content for AP Calculus. Multiple-choice questions are discrete, as opposed to appearing in question sets, and the questions do not appear in the order in which topics are addressed in the curriculum framework. Each part of the multiple-choice section is timed. Students may not return to questions in Part A of the multiple-choice section once they have begun Part B.

Free-response questions provide students with an opportunity to demonstrate their knowledge of correct mathematical reasoning and thinking. In most cases, an answer without supporting work will receive no credit; students are required to articulate the reasoning and methods that support their answer. Some questions will ask students to justify an answer or discuss whether a theorem can be applied. Each part of the free-response section is timed, and students may use a graphing

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calculator only for Part A. During the timed portion for Part B of the free-response section, students are allowed to return to working on Part A questions, though without the use of a graphing calculator.

Calculus AB Subscore for the Calculus BC Exam

Common topics are assessed at the same conceptual level on both of the AP Calculus Exams. Students who take the AP Calculus BC Exam receive an AP Calculus AB subscore based on their performance on the portion of the exam devoted to Calculus AB topics (approximately 60 percent of the exam). The Calculus AB subscore is designed to give students as well as colleges and universities feedback on how the student performed on the AP Calculus AB topics on the AP Calculus BC Exam.

Calculator Use on the Exams

Both the multiple-choice and free-response sections of the AP Calculus Exams include problems that require the use of a graphing calculator. A graphing calculator appropriate for use on the exams is expected to have the built-in capability to do the following:

1. Plot the graph of a function within an arbitrary viewing window
2. Find the zeros of functions (solve equations numerically)
3. Numerically calculate the derivative of a function
4. Numerically calculate the value of a definite integral

One or more of these capabilities should provide the sufficient computational tools for successful development of a solution to any AP Calculus AB or BC Exam question that requires the use of a calculator. Care is taken to ensure that the exam questions do not favor students who use graphing calculators with more extensive built-in features.

Students are expected to bring a graphing calculator with the capabilities listed above to the exams. AP teachers should check their own students' calculators to ensure that the required conditions are met. Students and teachers should keep their calculators updated with the latest available operating systems. Information is available on calculator company websites. A list of acceptable calculators can be found at AP Central.

Note that requirements regarding calculator use help ensure that all students have sufficient computational tools for the AP Calculus Exams. Exam restrictions should not be interpreted as restrictions on classroom activities.

Completing Section II: Free-Response Questions

- Show all of your work, even though a question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

- Your work must be expressed in standard mathematical notation rather than calculator syntax. For example, $\int_1^5 x^2 dx$ may not be written as fnInt(x^2, x, 1, 5).
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If you use decimal approximations in calculations, your work will be scored on accuracy. Unless otherwise specified, your final answers should be accurate to three places after the decimal point.
- Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

Sample Exam Questions

The sample questions that follow illustrate the relationship between the *AP Calculus AB and AP Calculus BC Curriculum Framework* and the redesigned AP Calculus Exams and serve as examples of the types of questions that will appear on the exams. Sample questions addressing the new content of the courses have been deliberately included; as such, the topic distribution of these questions is not indicative of the distribution on the actual exam.

Each question is accompanied by a table containing the main learning objective(s), essential knowledge statement(s), and Mathematical Practices for AP Calculus that the question addresses. In addition, each free-response question is accompanied by an explanation of how the relevant Mathematical Practices for AP Calculus can be applied in answering the question. The information accompanying each question is intended to aid in identifying the focus of the question, with the underlying assumption that learning objectives, essential knowledge statements, and MPACs other than those listed may also partially apply. Note that in the cases where multiple learning objectives, essential knowledge statements, or MPACs are provided for a multiple-choice question, the primary one is listed first.

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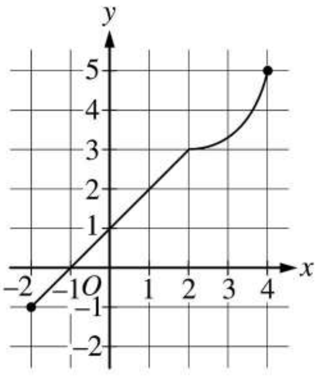
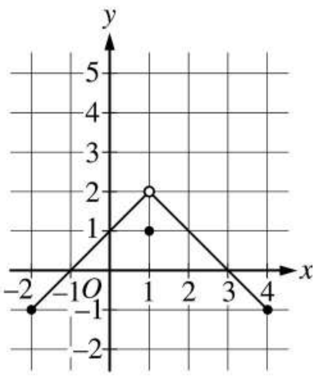
AP Calculus AB Sample Exam Questions

Multiple Choice: Section I, Part A

A calculator may not be used on questions on this part of the exam.



Graph of f



Graph of g

1. The graphs of the functions f and g are shown above. The value of $\lim_{x \to 1} f(g(x))$ is

(A) 1

(B) 2

(C) 3

(D) nonexistent

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 1.1C: Determine limits of functions.	**EK 1.1C1:** Limits of sums, differences, products, quotients, and composite functions can be found using the basic theorems of limits and algebraic rules.	**MPAC 4:** Connecting multiple representations **MPAC 2:** Connecting concepts

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2. $\lim\limits_{x \to 0} \dfrac{7x - \sin x}{x^2 + \sin(3x)} =$

(A) 6

(B) 2

(C) 1

(D) 0

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 1.1C: Determine limits of functions.	**EK 1.1C3:** Limits of the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ may be evaluated using L'Hospital's Rule.	**MPAC 3:** Implementing algebraic/computational processes **MPAC 5:** Building notational fluency

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3. If $f(x) = \sin(\ln(2x))$, then $f'(x) =$

(A) $\dfrac{\sin(\ln(2x))}{2x}$

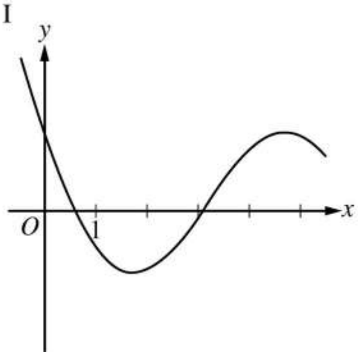
(B) $\dfrac{\cos(\ln(2x))}{x}$

(C) $\dfrac{\cos(\ln(2x))}{2x}$

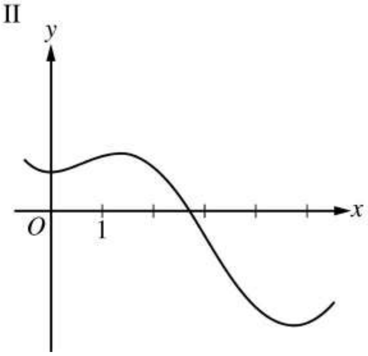
(D) $\cos\left(\dfrac{1}{2x}\right)$

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 2.1C: Calculate derivatives.	**EK 2.1C4:** The chain rule provides a way to differentiate composite functions.	**MPAC 3:** Implementing algebraic/computational processes **MPAC 5:** Building notational fluency

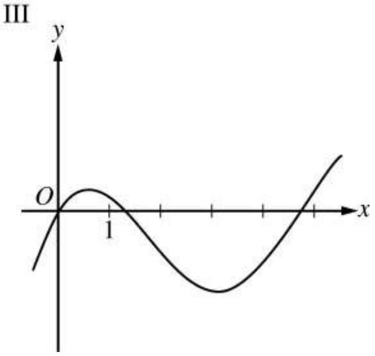
I



II



III



4. Three graphs labeled I, II, and III are shown above. One is the graph of f, one is the graph of f', and one is the graph of f''. Which of the following correctly identifies each of the three graphs?

	f	f'	f''
(A)	I	II	III
(B)	II	I	III
(C)	II	III	I
(D)	III	I	II

Learning Objective	**Essential Knowledge**	**Mathematical Practice for AP Calculus**
LO 2.2A: Use derivatives to analyze properties of a function.	**EK 2.2A3:** Key features of the graphs of f, f', and f'' are related to one another.	**MPAC 2:** Connecting concepts **MPAC 4:** Connecting multiple representations

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5. The local linear approximation to the function g at $x = \frac{1}{2}$ is $y = 4x + 1$. What is the value of $g\left(\frac{1}{2}\right) + g'\left(\frac{1}{2}\right)$?

(A) 4

(B) 5

(C) 6

(D) 7

Learning Objective	**Essential Knowledge**	**Mathematical Practice for AP Calculus**
LO 2.3B: Solve problems involving the slope of a tangent line.	**EK 2.3B2:** The tangent line is the graph of a locally linear approximation of the function near the point of tangency.	**MPAC 2:** Connecting concepts **MPAC 1:** Reasoning with definitions and theorems

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6. For time $t \geq 0,$ the velocity of a particle moving along the x-axis is given by $v(t) = (t-5)(t-2)^2.$ At what values of t is the acceleration of the particle equal to 0?

(A) 2 only

(B) 4 only

(C) 2 and 4

(D) 2 and 5

Learning Objective	**Essential Knowledge**	**Mathematical Practice for AP Calculus**
LO 2.3C: Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion.	**EK 2.3C1:** The derivative can be used to solve rectilinear motion problems involving position, speed, velocity, and acceleration.	**MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes
LO 2.1C: Calculate derivatives.	**EK 2.1C3:** Sums, differences, products, and quotients of functions can be differentiated using derivative rules.	

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7. The cost, in dollars, to shred the confidential documents of a company is modeled by C, a differentiable function of the weight of documents in pounds. Of the following, which is the best interpretation of $C'(500) = 80$?

(A) The cost to shred 500 pounds of documents is \$80.

(B) The average cost to shred documents is $\frac{80}{500}$ dollar per pound.

(C) Increasing the weight of documents by 500 pounds will increase the cost to shred the documents by approximately \$80.

(D) The cost to shred documents is increasing at a rate of \$80 per pound when the weight of the documents is 500 pounds.

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 2.3A: Interpret the meaning of a derivative within a problem.	**EK 2.3A1:** The unit for $f'(x)$ is the unit for f divided by the unit for x.	**MPAC 2:** Connecting concepts **MPAC 5:** Building notational fluency

8. Which of the following integral expressions is equal to $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sqrt{1+\frac{3k}{n}}\cdot\frac{1}{n}\right)$?

(A) $\int_0^1 \sqrt{1+3x}\,dx$

(B) $\int_0^3 \sqrt{1+x}\,dx$

(C) $\int_1^4 \sqrt{x}\,dx$

(D) $\frac{1}{3}\int_0^3 \sqrt{x}\,dx$

Learning Objective	**Essential Knowledge**	**Mathematical Practice for AP Calculus**
LO 3.2A(b): Express the limit of a Riemann sum in integral notation.	**EK 3.2A2:** The definite integral of a continuous function f over the interval $[a, b]$, denoted by $\int_a^b f(x)\,dx$, is the limit of Riemann sums as the widths of the subintervals approach 0. That is, $\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i$ where x_i^* is a value in the *i*th subinterval, Δx_i is the width of the *i*th subinterval, *n* is the number of subintervals, and $\max \Delta x_i$ is the width of the largest subinterval. Another form of the definition is $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i$, where $\Delta x_i = \frac{b-a}{n}$ and x_i^* is a value in the *i*th subinterval.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 5:** Building notational fluency

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9. $f(x) = \begin{cases} x & \text{for } x < 2 \\ 3 & \text{for } x \geq 2 \end{cases}$

If f is the function defined above, then $\int_{-1}^{4} f(x)\,dx$ is

(A) $\frac{9}{2}$

(B) $\frac{15}{2}$

(C) $\frac{17}{2}$

(D) undefined

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 3.2C: Calculate a definite integral using areas and properties of definite integrals.	**EK 3.2C3:** The definition of the definite integral may be extended to functions with removable or jump discontinuities.	**MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes

10. $\int e^x \cos\left(e^x + 1\right) dx =$

(A) $\sin\left(e^x + 1\right) + C$

(B) $e^x \sin\left(e^x + 1\right) + C$

(C) $e^x \sin\left(e^x + x\right) + C$

(D) $\frac{1}{2}\cos^2\left(e^x + 1\right) + C$

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 3.3B(a): Calculate antiderivatives.	**EK 3.3B5:** Techniques for finding antiderivatives include algebraic manipulation such as long division and completing the square, substitution of variables, **(BC)** integration by parts, and nonrepeating linear partial fractions.	**MPAC 3:** Implementing algebraic/computational processes **MPAC 5:** Building notational fluency

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11. At time t, a population of bacteria grows at the rate of $5e^{0.2t} + 4t$ grams per day, where t is measured in days. By how many grams has the population grown from time $t = 0$ days to $t = 10$ days?

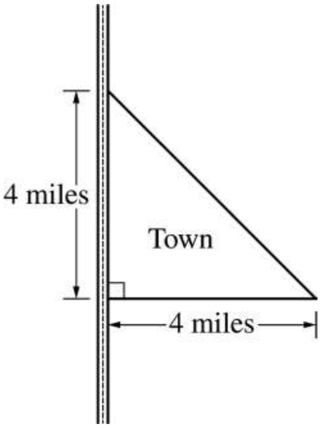
(A) $5e^2 + 40$

(B) $5e^2 + 195$

(C) $25e^2 + 175$

(D) $25e^2 + 375$

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 3.4A: Interpret the meaning of a definite integral within a problem.	**EK 3.4A2:** The definite integral of the rate of change of a quantity over an interval gives the net change of that quantity over that interval.	**MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes



12. The right triangle shown in the figure above represents the boundary of a town that is bordered by a highway. The population density of the town at a distance of x miles from the highway is modeled by $D(x)=\sqrt{x+1}$, where $D(x)$ is measured in thousands of people per square mile. According to the model, which of the following expressions gives the total population, in thousands, of the town?

(A) $\int_0^4 \sqrt{x+1}\,dx$

(B) $\int_0^4 8\sqrt{x+1}\,dx$

(C) $\int_0^4 x\sqrt{x+1}\,dx$

(D) $\int_0^4 (4-x)\sqrt{x+1}\,dx$

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 3.4A: Interpret the meaning of a definite integral within a problem.	**EK 3.4A3:** The limit of an approximating Riemann sum can be interpreted as a definite integral.	**MPAC 2:** Connecting concepts **MPAC 5:** Building notational fluency

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13. Which of the following is the solution to the differential equation $\frac{dy}{dx} = y\sec^2 x$ with the initial condition $y\left(\frac{\pi}{4}\right) = -1$?

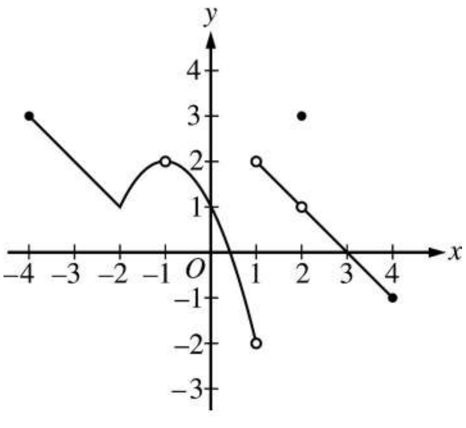
(A) $y = -e^{\tan x}$

(B) $y = -e^{(-1+\tan x)}$

(C) $y = -e^{\left(\sec^3 x - 2\sqrt{2}\right)/3}$

(D) $y = -\sqrt{2\tan x - 1}$

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 3.5A: Analyze differential equations to obtain general and specific solutions.	**EK 3.5A2:** Some differential equations can be solved by separation of variables.	**MPAC 3:** Implementing algebraic/computational processes **MPAC 2:** Connecting concepts



Graph of f

14. The graph of the function f is shown in the figure above. For how many values of x in the open interval $(-4, 4)$ is f discontinuous?

(A) one

(B) two

(C) three

(D) four

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 1.2A: Analyze functions for intervals of continuity or points of discontinuity.	**EK 1.2A3:** Types of discontinuities include removable discontinuities, jump discontinuities, and discontinuities due to vertical asymptotes.	**MPAC 2:** Connecting concepts **MPAC 4:** Connecting multiple representations

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15.

x	0	1	2
$f(x)$	5	2	–7
$f'(x)$	–2	–5	–14

The table above gives selected values of a differentiable and decreasing function *f* and its derivative. If *g* is the inverse function of *f*, what is the value of $g'(2)$?

(A) $-\frac{1}{5}$

(B) $-\frac{1}{14}$

(C) $\frac{1}{5}$

(D) 5

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 2.1C: Calculate derivatives.	**EK 2.1C6:** The chain rule can be used to find the derivative of an inverse function, provided the derivative of that function exists.	**MPAC 3:** Implementing algebraic/computational processes **MPAC 4:** Connecting multiple representations

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Multiple Choice: Section I, Part B

A graphing calculator is required for some questions on this part of the exam.

16. The derivative of the function f is given by $f'(x) = -\frac{x}{3} + \cos\left(x^2\right)$. At what values of x does f have a relative minimum on the interval $0 < x < 3$?

(A) 1.094 and 2.608

(B) 1.798

(C) 2.372

(D) 2.493

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 2.2A: Use derivatives to analyze properties of a function.	**EK 2.2A1:** First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection.	**MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes

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17. The second derivative of a function g is given by $g''(x) = 2^{-x^2} + \cos x + x$. For $-5 < x < 5$, on what open intervals is the graph of g concave up?

(A) $-5 < x < -1.016$ only

(B) $-1.016 < x < 5$ only

(C) $0.463 < x < 2.100$ only

(D) $-5 < x < 0.463$ and $2.100 < x < 5$

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 2.2A: Use derivatives to analyze properties of a function.	**EK 2.2A1:** First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection.	**MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes

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18. The temperature, in degrees Fahrenheit $\left(^\circ\mathrm{F}\right)$, of water in a pond is modeled by the function H given by $H(t) = 55 - 9\cos\left(\frac{2\pi}{365}(t+10)\right)$, where t is the number of days since January 1 $(t=0)$. What is the instantaneous rate of change of the temperature of the water at time $t = 90$ days?

(A) 0.114°F/day

(B) 0.153°F/day

(C) 50.252°F/day

(D) 56.350°F/day

Learning Objective	**Essential Knowledge**	**Mathematical Practice for AP Calculus**
LO 2.3D: Solve problems involving rates of change in applied contexts.	**EK 2.3D1:** The derivative can be used to express information about rates of change in applied contexts.	**MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes

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19.

x	0	2	4	8
$f(x)$	3	4	9	13
$f'(x)$	0	1	1	2

The table above gives values of a differentiable function f and its derivative at selected values of x. If h is the function given by $h(x) = f(2x)$, which of the following statements must be true?

(I) h is increasing on $2 < x < 4$.

(II) There exists c, where $0 < c < 4$, such that $h(c) = 12$.

(III) There exists c, where $0 < c < 2$, such that $h'(c) = 3$.

(A) II only

(B) I and III only

(C) II and III only

(D) I, II, and III

Learning Objective	**Essential Knowledge**	**Mathematical Practice for AP Calculus**
LO 2.4A: Apply the Mean Value Theorem to describe the behavior of a function over an interval.	**EK 2.4A1:** If a function f is continuous over the interval $[a, b]$ and differentiable over the interval (a, b), the Mean Value Theorem guarantees a point within that open interval where the instantaneous rate of change equals the average rate of change over the interval.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 4:** Connecting multiple representations
LO 1.2B: Determine the applicability of important calculus theorems using continuity.	**EK 1.2B1:** Continuity is an essential condition for theorems such as the Intermediate Value Theorem, the Extreme Value Theorem, and the Mean Value Theorem.	

20. Let h be the function defined by $h(x)=\dfrac{1}{\sqrt{x^5+1}}$. If g is an antiderivative of h and $g(2)=3$, what is the value of $g(4)$?

(A) -0.020

(B) 0.152

(C) 3.031

(D) 3.152

Learning Objective	**Essential Knowledge**	**Mathematical Practice for AP Calculus**
LO 3.3B(b): Evaluate definite integrals.	**EK 3.3B2:** If f is continuous on the interval $[a, b]$ and F is an antiderivative of f, then $\int_a^b f(x)dx=F(b)-F(a)$.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 2:** Connecting concepts

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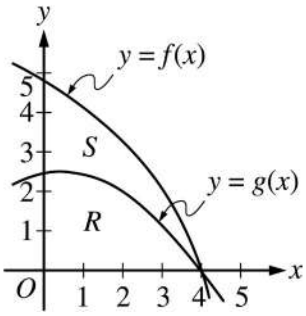
Free Response: Section II, Part A

A graphing calculator is required for problems on this part of the exam.



1. Let R be the region in the first quadrant bounded by the graph of g, and let S be the region in the first quadrant between the graphs of f and g, as shown in the figure above. The region in the first quadrant bounded by the graph of f and the coordinate axes has area 12.142. The function g is given by $g(x) = \left(\sqrt{x+6}\right)\cos\left(\frac{\pi x}{8}\right)$, and the function f is not explicitly given. The graphs of f and g intersect at the point $(4, 0)$.

   (A) Find the area of S.

   (B) A solid is generated when S is revolved about the horizontal line $y = 5$. Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid.

   (C) Region R is the base of an art sculpture. At all points in R at a distance x from the y-axis, the height of the sculpture is given by $h(x) = 4 - x$. Find the volume of the art sculpture.

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 3.2C: Calculate a definite integral using areas and properties of definite integrals.	**EK 3.2C2:** Properties of definite integrals include the integral of a constant times a function, the integral of the sum of two functions, reversal of limits of integration, and the integral of a function over adjacent intervals.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes **MPAC 4:** Connecting multiple representations **MPAC 5:** Building notational fluency **MPAC 6:** Communicating
LO 3.4D: Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.	**EK 3.4D1:** Areas of certain regions in the plane can be calculated with definite integrals. **(BC)** Areas bounded by polar curves can be calculated with definite integrals.	
LO 3.4D: Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.	**EK 3.4D2:** Volumes of solids with known cross sections, including discs and washers, can be calculated with definite integrals.	

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Free Response: Section II, Part B

2.

t (minutes)	0	3	5	6	9
$r(t)$ (rotations per minute)	72	95	112	77	50

Rochelle rode a stationary bicycle. The number of rotations per minute of the wheel of the stationary bicycle at time t minutes during Rochelle's ride is modeled by a differentiable function r for $0 \le t \le 9$ minutes. Values of $r(t)$ for selected values of t are shown in the table above.

(A) Estimate $r'(4)$. Show the computations that lead to your answer. Indicate units of measure.

(B) Is there a time t, for $3 \le t \le 5$, at which $r(t)$ is 106 rotations per minute? Justify your answer.

(C) Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\int_0^9 r(t)\,dt$. Using correct units, explain the meaning of $\int_0^9 r(t)\,dt$ in the context of the problem.

(D) Sarah also rode a stationary bicycle. The number of rotations per minute of the wheel of the stationary bicycle at time t minutes during Sarah's ride is modeled by the function s, defined by $s(t) = 40 + 20\pi \sin\left(\frac{\pi t}{18}\right)$ for $0 \le t \le 9$ minutes. Find the average number of rotations per minute of the wheel of the stationary bicycle for $0 \le t \le 9$ minutes.

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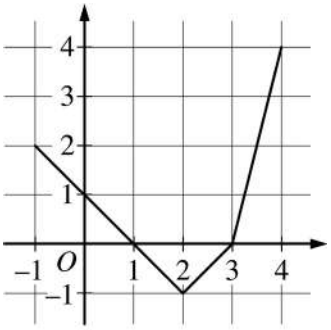
Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 1.2B: Determine the applicability of important calculus theorems using continuity.	**EK 1.2B1:** Continuity is an essential condition for theorems such as the Intermediate Value Theorem, the Extreme Value Theorem, and the Mean Value Theorem.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes **MPAC 4:** Connecting multiple representations **MPAC 5:** Building notational fluency **MPAC 6:** Communicating
LO 2.1B: Estimate derivatives.	**EK 2.1B1:** The derivative at a point can be estimated from information given in tables or graphs.	
LO 3.2B: Approximate a definite integral.	**EK 3.2B2:** Definite integrals can be approximated using a left Riemann sum, a right Riemann sum, a midpoint Riemann sum, or a trapezoidal sum; approximations can be computed using either uniform or nonuniform partitions.	
LO 3.3B(b): Evaluate definite integrals.	**EK 3.3B2:** If f is continuous on the interval $[a, b]$ and F is an antiderivative of f, then $\int_a^b f(x)\,dx = F(b) - F(a)$.	
LO 3.4A: Interpret the meaning of a definite integral within a problem.	**EK 3.4A2:** The definite integral of the rate of change of a quantity over an interval gives the net change of that quantity over that interval.	
LO 3.4B: Apply definite integrals to problems involving the average value of a function.	**EK 3.4B1:** The average value of a function f over an interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.	
LO 3.4E: Use the definite integral to solve problems in various contexts.	**EK 3.4E1:** The definite integral can be used to express information about accumulation and net change in many applied contexts.	



Graph of f

3. Let f be a continuous function defined on the closed interval $-1 \le x \le 4$. The graph of f, consisting of three line segments, is shown above. Let g be the function defined by $g(x) = 5 + \int_2^x f(t)\,dt$ for $-1 \le x \le 4$.

   (A) Find $g(4)$.

   (B) On what intervals is g increasing? Justify your answer.

   (C) On the closed interval $-1 \le x \le 4$, find the absolute minimum value of g and find the absolute maximum value of g. Justify your answers.

   (D) Let $h(x) = x \cdot g(x)$. Find $h'(2)$.

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 2.1C: Calculate derivatives.	**EK 2.1C3:** Sums, differences, products, and quotients of functions can be differentiated using derivative rules.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes **MPAC 4:** Connecting multiple representations **MPAC 5:** Building notational fluency **MPAC 6:** Communicating
LO 2.2A: Use derivatives to analyze properties of a function.	**EK 2.2A1:** First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection.	
LO 3.2C: Calculate a definite integral using areas and properties of definite integrals.	**EK 3.2C1:** In some cases, a definite integral can be evaluated by using geometry and the connection between the definite integral and area.	
LO 3.3A: Analyze functions defined by an integral.	**EK 3.3A3:** Graphical, numerical, analytical, and verbal representations of a function f provide information about the function g defined as $g(x)=\int_a^x f(t)\,dt$.	

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Answers and Rubrics (AB)

Answers to Multiple-Choice Questions

1	C
2	B
3	B
4	C
5	D
6	C
7	D
8	A
9	B
10	A
11	C
12	D
13	B
14	C
15	A
16	C
17	B
18	B
19	C
20	D

Rubrics for Free-Response Questions

Question 1

Solutions	Point Allocation
(A) Area of region $S = (\text{Area under } f) - (\text{Area under } g)$ $= 12.142 - \int_0^4 g(x)\,dx = 12.142 - 6.938$ $= 5.204$ (or 5.203)	$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{uses area under } f \\ 1 : \text{answer} \end{cases}$
(B) Volume $= \pi \int_0^4 \left((5 - g(x))^2 - (5 - f(x))^2 \right) dx$	$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$
(C) Volume $= \int_0^4 ((4 - x)g(x))\,dx = 17.243$	$3 : \begin{cases} 2 : \text{integral} \\ 1 : \text{answer} \end{cases}$

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Question 2

Solutions	Point Allocation
(A) $r'(4) \approx \frac{r(5) - r(3)}{5 - 3} = \frac{112 - 95}{2} = \frac{17}{2}$ rotations per minute per minute	1 : answer with units
(B) r is differentiable $\Rightarrow$ r is continuous on $3 \le t \le 5$. $r(3) = 95 < 106 < 112 = r(5)$ Therefore, by the Intermediate Value Theorem, there is a time t, $3 \le t \le 5$, such that $r(t) = 106$.	$2 : \begin{cases} 1 : r(3) < 106 < r(5) \\ 1 : \text{conclusion, using IVT} \end{cases}$
(C) $\int_0^9 r(t)\,dt \approx (3-0)\cdot r(0) + (5-3)\cdot r(3) + (6-5)\cdot r(5) + (9-6)\cdot r(6)$ $= 3(72) + 2(95) + 1(112) + 3(77) = 749$ $\int_0^9 r(t)\,dt$ is the total number of rotations of the wheel of the stationary bicycle over the time interval $0 \le t \le 9$ minutes.	$3 : \begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{approximation} \\ 1 : \text{explanation} \end{cases}$
(D) $\frac{1}{9}\int_0^9 s(t)\,dt = \frac{1}{9}\int_0^9 \left(40 + 20\pi \sin\left(\frac{\pi t}{18}\right)\right) dt$ $= \frac{1}{9}\left[40t - 360\cos\left(\frac{\pi t}{18}\right)\right]_0^9$ $= \frac{1}{9}\left(360 - 360\cos\left(\frac{\pi}{2}\right)\right) - \frac{1}{9}(0 - 360\cos(0))$ $= \frac{720}{9} = 80$ rotations per minute	$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

Question 3

<table>
<tr><th>Solutions</th><th>Point Allocation</th></tr>
<tr><td>(A) $g(4) = 5 + \int_2^4 f(t)\,dt = 5 + \left(-\frac{1}{2}\right) + 2 = \frac{13}{2}$</td><td>1 : answer</td></tr>
<tr><td>(B) $g'(x) = f(x)$
The function g is increasing on the intervals $-1 \le x \le 1$ and $3 \le x \le 4$ because $g' = f$ is nonnegative on these intervals.</td><td>$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$</td></tr>
<tr><td>(C) $g'(x) = f(x) = 0 \Rightarrow x = 1, x = 3$
<table>
<tr><th>x</th><th>$g(x)$</th></tr>
<tr><td>-1</td><td>$\frac{7}{2}$</td></tr>
<tr><td>1</td><td>$\frac{11}{2}$</td></tr>
<tr><td>3</td><td>$\frac{9}{2}$</td></tr>
<tr><td>4</td><td>$\frac{13}{2}$</td></tr>
</table>
The absolute minimum value of g is $\frac{7}{2}$, and the absolute maximum value of g is $\frac{13}{2}$.</td><td>$4 : \begin{cases} 1 : \text{considers } g'(x) = 0 \\ 1 : \text{identifies } x = 1 \text{ and } x = 3 \text{ as candidates} \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$</td></tr>
<tr><td>(D) $h'(x) = 1 \cdot g(x) + x \cdot g'(x)$
$h'(2) = 1 \cdot g(2) + 2 \cdot g'(2) = 1(5) + 2(-1) = 3$</td><td>$2 : \begin{cases} 1 : h'(x) \\ 1 : h'(2) \end{cases}$</td></tr>
</table>

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AP Calculus BC Sample Exam Questions

Multiple Choice: Section I, Part A

A calculator may not be used on questions on this part of the exam.

1. A curve is defined by the parametric equations $x(t) = 3e^{2t}$ and $y(t) = e^{3t} - 1$. What is $\frac{d^2y}{dx^2}$ in terms of t?

(A) $\frac{1}{12e^t}$

(B) $\frac{1}{9e^t}$

(C) $\frac{e^t}{2}$

(D) $\frac{3e^t}{4}$

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 2.1C: Calculate derivatives.	**EK 2.1C7: (BC)** Methods for calculating derivatives of real-valued functions can be extended to vector-valued functions, parametric functions, and functions in polar coordinates.	**MPAC 3:** Implementing algebraic/ computational processes **MPAC 2:** Connecting concepts

2.

$x_0 = 0$	$f(x_0) = 2$
$x_1 = 2$	$f(x_1) \approx 6$
$x_2 = 4$	$f(x_2) \approx 10$

Consider the differential equation $\frac{dy}{dx} = \frac{Ax^2 + 4}{y}$, where A is a constant.

Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 2$. Euler's method, starting at $x = 0$ with a step size of 2, is used to approximate $f(4)$. Steps from this approximation are shown in the table above. What is the value of A ?

(A) $\frac{1}{2}$

(B) 2

(C) 5

(D) $\frac{13}{2}$

Learning Objective	**Essential Knowledge**	**Mathematical Practice for AP Calculus**
LO 2.3F: Estimate solutions to differential equations.	**EK 2.3F2: (BC)** For differential equations, Euler's method provides a procedure for approximating a solution or a point on a solution curve.	**MPAC 4:** Connecting multiple representations **MPAC 3:** Implementing algebraic/ computational processes

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3. $\displaystyle\int \frac{12}{(x-1)(x-5)}\,dx =$

(A) $-3\ln|x-1| + 3\ln|x-5| + C$

(B) $-2\ln|x-1| + 2\ln|x-5| + C$

(C) $3\ln|x-1| - 3\ln|x-5| + C$

(D) $12\ln|x-1| + 12\ln|x-5| + C$

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 3.3B(a): Calculate antiderivatives.	**EK 3.3B5:** Techniques for finding antiderivatives include algebraic manipulation such as long division and completing the square, substitution of variables, **(BC)** integration by parts, and nonrepeating linear partial fractions.	**MPAC 3:** Implementing algebraic/ computational processes **MPAC 5:** Building notational fluency

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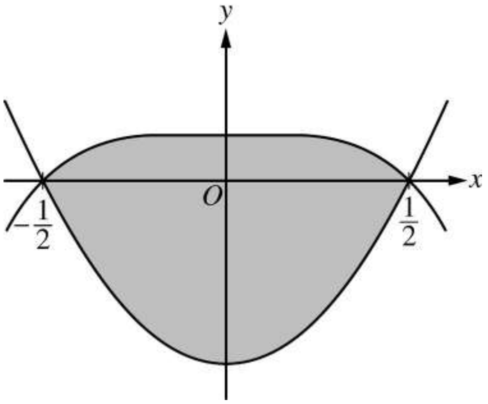

4. The shaded region in the figure above is bounded by the graphs of $y = x^2 - \frac{1}{4}$ and $y = \frac{1}{16} - x^4$ for $-\frac{1}{2} \le x \le \frac{1}{2}$. Which of the following expressions gives the perimeter of the region?

(A) $2\int_0^{1/2} \sqrt{4x^2 + 16x^6}\, dx$

(B) $2\int_0^{1/2} \sqrt{1 + 4x^2 + 16x^6}\, dx$

(C) $2\int_0^{1/2} \sqrt{1 + 4x^2}\, dx + 2\int_0^{1/2} \sqrt{1 + 16x^6}\, dx$

(D) $2\int_0^{1/2} \sqrt{1 + \left(x^2 - \frac{1}{4}\right)^2}\, dx + 2\int_0^{1/2} \sqrt{1 + \left(\frac{1}{16} - x^4\right)^2}\, dx$

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 3.4D: Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.	**EK 3.4D3: (BC)** The length of a planar curve defined by a function or by a parametrically defined curve can be calculated using a definite integral.	**MPAC 2:** Connecting concepts **MPAC 4:** Connecting multiple representations

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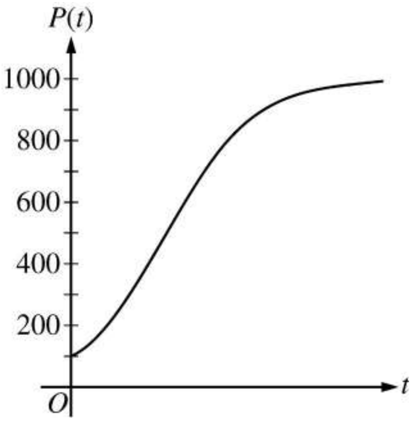
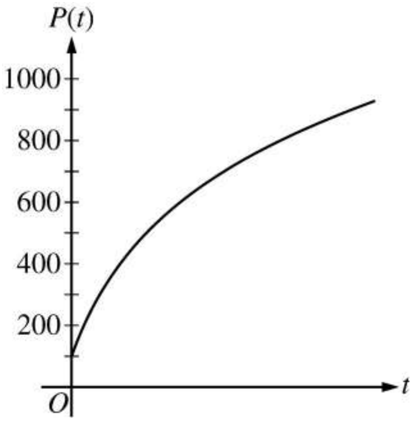
5. The number of fish in a lake is modeled by the function P that satisfies the differential equation $\frac{dP}{dt} = 0.003P(1000 - P)$, where t is the time in years. Which of the following could be the graph of $y = P(t)$?

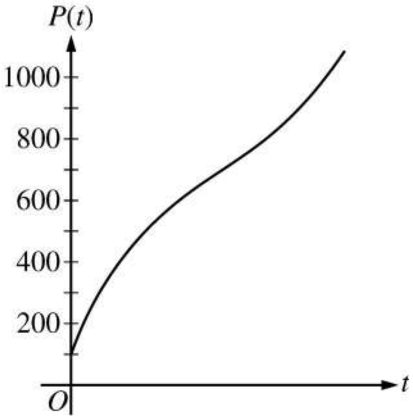
(A)



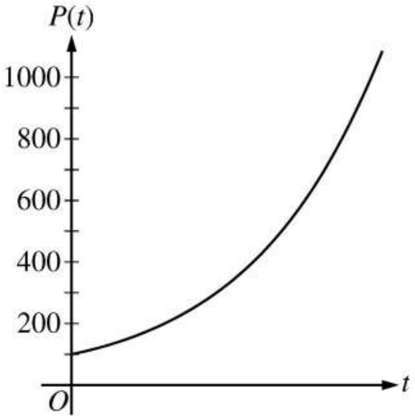
(B)



(C)



(D)



Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 3.5B: Interpret, create and solve differential equations from problems in context.	**EK 3.5B2: (BC)** The model for logistic growth that arises from the statement "The rate of change of a quantity is jointly proportional to the size of the quantity and the difference between the quantity and the carrying capacity" is $\frac{dy}{dt} = ky(a - y)$.	**MPAC 2:** Connecting concepts **MPAC 4:** Connecting multiple representations

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6. Which of the following series is absolutely convergent?

(A) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n}$

(B) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$

(C) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n+1}$

(D) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{2}\right)^{n}$

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 4.1A: Determine whether a series converges or diverges.	**EK 4.1A4:** A series may be absolutely convergent, conditionally convergent, or divergent.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 2:** Connecting concepts

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7. Which of the following series cannot be shown to converge using the limit comparison test with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$?

(A) $\sum_{n=1}^{\infty} \frac{4}{3n^2 - n}$

(B) $\sum_{n=1}^{\infty} \frac{15}{\sqrt{n^4 + 5}}$

(C) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$

(D) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 4.1A: Determine whether a series converges or diverges.	**EK 4.1A6:** In addition to examining the limit of the sequence of partial sums of the series, methods for determining whether a series of numbers converges or diverges are the *n*th term test, the comparison test, the limit comparison test, the integral test, the ratio test, and the alternating series test.	**MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes

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8. The third-degree Taylor polynomial for the function f about $x = 0$ is $T(x) = 3 - 4x + 2x^2 - 3x^3$. Which of the following tables gives the values of f and its first three derivatives at $x = 0$?

(a)

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
0	3	−8	6	−12

(b)

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
0	3	−4	2	−3

(c)

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
0	3	−4	4	−18

(d)

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
0	3	−4	4	−9

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 4.2A: Construct and use Taylor polynomials.	**EK 4.2A1:** The coefficient of the nth-degree term in a Taylor polynomial centered at $x = a$ for the function f is $\frac{f^{(n)}(a)}{n!}$.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 4:** Connecting multiple representations

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9. What is the interval of convergence for the power series $\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{n \cdot 3^n}(x-4)^n$?

(A) $-3 < x < 3$

(B) $-3 < x \leq 3$

(C) $1 < x < 7$

(D) $1 < x \leq 7$

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 4.2C: Determine the radius and interval of convergence of a power series.	**EK 4.2C2:** The ratio test can be used to determine the radius of convergence of a power series.	**MPAC 3:** Implementing algebraic/computational processes **MPAC 1:** Reasoning with definitions and theorems
LO 4.1A: Determine whether a series converges or diverges.	**EK 4.1A6:** In addition to examining the limit of the sequence of partial sums of the series, methods for determining whether a series of numbers converges or diverges are the *n*th term test, the comparison test, the limit comparison test, the integral test, the ratio test, and the alternating series test.	

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Multiple Choice: Section I, Part B

A graphing calculator is required for some questions on this part of the exam.

10. For time $t \geq 0$ seconds, the position of an object traveling along a curve in the *xy*-plane is given by the parametric equations $x(t)$ and $y(t)$, where $\frac{dx}{dt} = t^2 + 3$ and $\frac{dy}{dt} = t^3 + t$. At what time *t* is the speed of the object 10 units per second?

(A) 1.675

(B) 1.813

(C) 4.217

(D) 10.191

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 2.3C: Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion.	**EK 2.3C4: (BC)** Derivatives can be used to determine velocity, speed, and acceleration for a particle moving along curves given by parametric or vector-valued functions.	**MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/ computational processes

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11. A particle moving in the *xy*-plane has velocity vector given by $v(t) = \left\langle e^{\sin t}, 5t^2 \right\rangle$ for time $t \geq 0$. What is the magnitude of the displacement of the particle between time $t = 1$ and $t = 2$?

(A) 3.778

(B) 11.954

(C) 11.992

(D) 15.001

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 3.4C: Apply definite integrals to problems involving motion.	**EK 3.4C2: (BC)** The definite integral can be used to determine displacement, distance, and position of a particle moving along a curve given by parametric or vector-valued functions.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 3:** Implementing algebraic/computational processes

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12. Consider the series $\sum_{n=0}^{\infty}(-1)^n a_n$, where $a_n > 0$ for all n. Which of the following conditions guarantees that the series converges?

(A) $\lim_{n \to \infty} a_n = 0$

(B) $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} < 1$

(C) $a_{n+1} < a_n$ for all n

(D) $\int_0^{\infty} f(x)\,dx$ converges, where $f(n) = a_n$ for all n

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 4.1A: Determine whether a series converges or diverges.	**EK 4.1A6:** In addition to examining the limit of the sequence of partial sums of the series, methods for determining whether a series of numbers converges or diverges are the *n*th term test, the comparison test, the limit comparison test, the integral test, the ratio test, and the alternating series test.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 5:** Building notational fluency
LO 4.1A: Determine whether a series converges or diverges.	**EK 4.1A5:** If a series converges absolutely, then it converges.	

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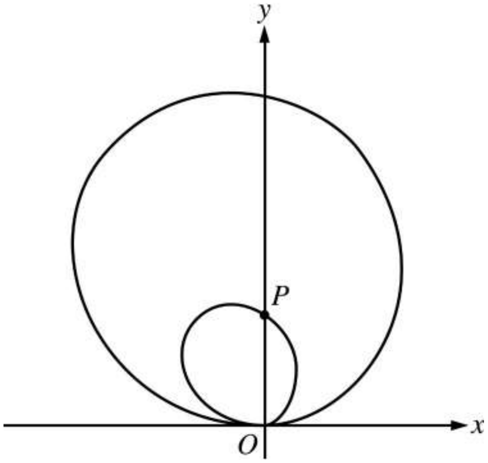
Free Response: Section II, Part A

A graphing calculator is required for problems on this part of the exam.



1. Let r be the function given by $r(\theta) = 3\theta \sin\theta$ for $0 \le \theta \le 2\pi$. The graph of r in polar coordinates consists of two loops, as shown in the figure above. Point P is on the graph of r and the y-axis.

   (A) Find the rate of change of the x-coordinate with respect to θ at the point P.

   (B) Find the area of the region between the inner and outer loops of the graph.

   (C) The function r satisfies $\frac{dr}{d\theta} = 3\sin\theta + 3\theta\cos\theta$. For $0 \le \theta \le 2\pi$, find the value of θ that gives the point on the graph that is farthest from the origin. Justify your answer.

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 2.2A: Use derivatives to analyze properties of a function.	**EK 2.2A4: (BC)** For a curve given by a polar equation $r = f(\theta)$, derivatives of r, x, and y with respect to θ and first and second derivatives of y with respect to x can provide information about the curve.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes **MPAC 4:** Connecting multiple representations **MPAC 5:** Building notational fluency **MPAC 6:** Communicating
LO 2.3C: Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion.	**EK 2.3C3:** The derivative can be used to solve optimization problems, that is, finding a maximum or minimum value of a function over a given interval.	
LO 3.4D: Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.	**EK 3.4D1:** Areas of certain regions in the plane can be calculated with definite integrals. **(BC)** Areas bounded by polar curves can be calculated with definite integrals.	

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Free Response: Section II, Part B

No calculator is allowed for problems on this part of the exam.

2. Consider the function f given by $f(x) = xe^{-2x}$ for all $x \geq 0$.

(A) Find $\lim\limits_{x \to \infty} f(x)$.

(B) Find the maximum value of f for $x \geq 0$. Justify your answer.

(C) Evaluate $\int_0^\infty f(x)\,dx$, or show that the integral diverges.

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 1.1D: Deduce and interpret behavior of functions using limits.	**EK 1.1D2:** Relative magnitudes of functions and their rates of change can be compared using limits.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes **MPAC 4:** Building notational fluency **MPAC 6:** Communicating
LO 2.2A: Use derivatives to analyze properties of a function.	**EK 2.2A1:** First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection.	
LO 3.2D: (BC) Evaluate an improper integral or show that an improper integral diverges.	**EK 3.2D2: (BC)** Improper integrals can be determined using limits of definite integrals.	
LO 3.3B(b): Evaluate definite integrals.	**EK 3.3B5:** Techniques for finding antiderivatives include algebraic manipulation such as long division and completing the square, substitution of variables, **(BC)** integration by parts, and nonrepeating linear partial fractions.	

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3. The function f is defined by the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n(n+1)} = 1 + \frac{x-2}{3 \cdot 2} + \frac{(x-2)^2}{3^2 \cdot 3} + \frac{(x-2)^3}{3^3 \cdot 4} + \cdots + \frac{(x-2)^n}{3^n(n+1)} + \cdots$$

for all real numbers x for which the series converges.

(A) Determine the interval of convergence of the power series for f. Show the work that leads to your answer.

(B) Find the value of $f''(2)$.

(C) Use the first three nonzero terms of the power series for f to approximate $f(1)$. Use the alternating series error bound to show that this approximation differs from $f(1)$ by less than $\frac{1}{100}$.

Learning Objective	Essential Knowledge	Mathematical Practice for AP Calculus
LO 4.1A: Determine whether a series converges or diverges.	**EK 4.1A3:** Common series of numbers include geometric series, the harmonic series, and *p*-series.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes **MPAC 5:** Building notational fluency **MPAC 6:** Communicating
LO 4.1A: Determine whether a series converges or diverges.	**EK 4.1A4:** A series may be absolutely convergent, conditionally convergent, or divergent.	
LO 4.1B: Determine or estimate the sum of a series.	**EK 4.1B2:** If an alternating series converges by the alternating series test, then the alternating series error bound can be used to estimate how close a partial sum is to the value of the infinite series.	
LO 4.2A: Construct and use Taylor polynomials.	**EK 4.2A1:** The coefficient of the nth-degree term in a Taylor polynomial centered at $x = a$ for the function f is $\frac{f^{(n)}(a)}{n!}$.	
LO 4.2C: Determine the radius and interval of convergence of a power series.	**EK 4.2C1:** If a power series converges, it either converges at a single point or has an interval of convergence.	
LO 4.2C: Determine the radius and interval of convergence of a power series.	**EK 4.2C2:** The ratio test can be used to determine the radius of convergence of a power series.	
LO 4.2C: Determine the radius and interval of convergence of a power series.	**EK 4.2C3:** If a power series has a positive radius of convergence, then the power series is the Taylor series of the function to which it converges over the open interval.	

Answers and Rubrics (BC)

Answers to Multiple-Choice Questions

1	A
2	B
3	A
4	C
5	A
6	D
7	D
8	C
9	C
10	B
11	B
12	B

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Rubrics for Free-Response Questions

Question 1

<table>
<tr><th>Solutions</th><th>Point Allocation</th></tr>
<tr><td>(A) $x = r\cos\theta = 3\theta\sin\theta\cos\theta$
At point P, $\theta = \frac{\pi}{2}$.
$\left.\frac{dx}{d\theta}\right|_{\theta=\pi/2} = -4.712$</td><td>$3 : \begin{cases} 1 : x(\theta) \\ 1 : \text{uses } \theta = \frac{\pi}{2} \\ 1 : \text{answer} \end{cases}$</td></tr>
<tr><td>(B) Area $= \frac{1}{2}\int_{\pi}^{2\pi}(r(\theta))^2\,d\theta - \frac{1}{2}\int_{0}^{\pi}(r(\theta))^2\,d\theta = 139.528$</td><td>$3 : \begin{cases} 2 : \text{expression for area} \\ 1 : \text{answer} \end{cases}$</td></tr>
<tr><td>(C) $3\sin\theta + 3\theta\cos\theta = 0 \Rightarrow \theta = 2.028758,\ \theta = 4.913180$
<table>
<tr><th>θ</th><th>$r(\theta)$</th></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>2.028758</td><td>5.459117</td></tr>
<tr><td>4.913180</td><td>−14.443410</td></tr>
<tr><td>2π</td><td>0</td></tr>
</table>
The value $\theta = 4.913$ gives the point on the graph that is farthest from the origin.</td><td>$3 : \begin{cases} 1 : \text{sets } \frac{dr}{d\theta} = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$</td></tr>
</table>

Question 2

Solutions	Point Allocation
(A) $\lim\limits_{x\to\infty} xe^{-2x} = \lim\limits_{x\to\infty} \frac{x}{e^{2x}} = 0$	1 : answer
(B) $f'(x) = e^{-2x} + x\left(-2e^{-2x}\right)$ $= e^{-2x}(1-2x)$ f' exists for all $x > 0$. $f'(x) = 0 \Rightarrow x = \frac{1}{2}$ Because $f'(x) > 0$ for $0 < x < \frac{1}{2}$ and $f'(x) < 0$ for $x > \frac{1}{2}$, the maximum value of $f(x)$ for $x \geq 0$ is $f\left(\frac{1}{2}\right) = \frac{1}{2}e^{-2\cdot\frac{1}{2}} = \frac{1}{2e}$.	$4 : \begin{cases} 2 : f'(x) \\ 1 : \text{identifies } x = \frac{1}{2} \text{ as a candidate} \\ 1 : \text{answer with justification} \end{cases}$
(C) $u = x \qquad dv = e^{-2x}\,dx$ $du = dx \qquad v = -\frac{1}{2}e^{-2x}$ $\int_0^b xe^{-2x}dx = -\frac{x}{2}e^{2x}\Big\vert_0^b + \int_0^b \frac{1}{2}e^{-2x}dx$ $= \left[\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}\right]_0^b$ $= \left[-e^{-2x}\left(\frac{x}{2} + \frac{1}{4}\right)\right]_0^b$ $\int_0^\infty xe^{-2x}\,dx = \lim\limits_{b\to\infty}\int_0^b xe^{-2x}\,dx = \lim\limits_{b\to\infty}\left[-e^{-2x}\left(\frac{x}{2} + \frac{1}{4}\right)\right]_0^b$ $= \lim\limits_{b\to\infty}\left[-\frac{1}{e^{2b}}\left(\frac{b}{2} + \frac{1}{4}\right) - (-1)\left(\frac{1}{4}\right)\right] = \frac{1}{4}$	$4 : \begin{cases} 2 : \text{antiderivative} \\ 1 : \text{limit as } b \to \infty \\ 1 : \text{answer} \end{cases}$

Question 3

Solutions	Point Allocation
(A) $\left\lvert \frac{(x-2)^{n+1}}{3^{n+1}(n+2)} \cdot \frac{3^n(n+1)}{(x-2)^n} \right\rvert = \left\lvert \frac{n+1}{n+2} \cdot \frac{x-2}{3} \right\rvert$ $\lim_{n\to\infty} \left\lvert \frac{n+1}{n+2} \cdot \frac{x-2}{3} \right\rvert = \frac{1}{3}\lvert x-2 \rvert$ $\frac{1}{3}\lvert x-2\rvert < 1 \Rightarrow \lvert x-2\rvert < 3 \Rightarrow -1 < x < 5$ The series converges when $-1 < x < 5$. When $x = -1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. This is the alternating harmonic series, which converges conditionally. When $x = 5$, the series is $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$. This is the harmonic series, which diverges. Therefore, the interval of convergence is $-1 \le x < 5$.	$5 : \left\{ \begin{array}{l} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{identifies interior of interval of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{array} \right.$
(B) The power series given is the Taylor series for f about $x = 2$. Thus, $\frac{f''(2)}{2!} = \frac{1}{3^2 \cdot 3} \Rightarrow f''(2) = \frac{2}{27}$	1 : answer

(C) $f(1) \approx 1 + \frac{1-2}{3 \cdot 2} + \frac{(1-2)^2}{3^2 \cdot 3} = 1 - \frac{1}{6} + \frac{1}{27} = \frac{47}{54}$ The power series for *f* evaluated at $x = 1$ is an alternating series whose terms decrease in absolute value to 0. The alternating series error bound is the absolute value of the fourth term of the series. $\left\lvert f(1) - \frac{47}{54} \right\rvert < \left\lvert \frac{(1-2)^3}{3^3 \cdot 4} \right\rvert = \frac{1}{108} < \frac{1}{100}$	$3 : \begin{cases} 1 : \text{approximation} \\ 1 : \text{uses the fourth term as an error bound} \\ 1 : \text{analysis} \end{cases}$

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