Notes 8B Day 1 – Doubling Time

Exponential growth leads to repeated doublings (see Graph in Notes 8A) and exponential decay leads to repeated halvings. In this unit we'll be converting between growth (or decay) rates and doubling (or halving) times.

Doubling Time: the time it takes for a quantity that is growing exponentially to double.

Consider an initial population of 10,000 that grows with a doubling time of 10 years.

- In 10 years, or one doubling time, the population increases by a factor of 2, to a new population of 10,000 x 2 =20,000.
- In 20 years, or 2 doubling times, the population increases by a factor of 2^2, to a new population of 10,000 x 4 = 40,000.
- In 30 years, or 3 doubling times, the population increases by a factor of 2^3, to a new population of 10,000 x 8 = 80,000.

Calculations with Doubling Time

After t time, the final amount of a quantity (that's growing exponentially), with a doubling time of T can be found by:

$$y = a \bullet 2^{\frac{t}{T}}$$

y=final amount a=initial amount t=time that has passed T=doubling time $2^{\frac{t}{T}}$ = growth factor

1) Compound interest produces exponential growth because an interest bearing account grows by the same percentage each year. Suppose your bank account has a doubling time of 13 years. By what factor does your balance increase in 50 years?

2) The world population doubled from 3 billion in 1960 to 6 billion in 2000. Suppose that world population continued to grow (from 2000 on) with a doubling time of 40 years. What would the population be in 2030? In 2200?

3) The number of ants in John's ant farm is growing exponentially. At 8:00 AM the ant farm had 1500 ants. Ten hours later, the ant population had doubled.

a) If the ant population continues to grow at this same rate, how many ants will there be after 1 week (from the 8:00 count)?

b) By what factor will the ant population increase in 24 hours?

Consider an ecological study of a prairie dog community. The community contains 100 prairie dogs when the study begins, and researchers soon determine that the population is increasing at a rate of 10% per month. That is, each month the population grows to 110% of, or 1.1 times, its previous value. Table 8.3 tracks the population growth (rounded to the nearest whole number).

TABLE 8.3 Growth of a Prairie Dog Community

Month	Population	Month	Population
0	100	8	$(1.1)^8 \times 100 = 214$
1	$(1.1)^1 \times 100 = 110$	9	$(1.1)^9 \times 100 = 236$
2	$(1.1)^2 \times 100 = 121$	10	$(1.1)^{10} \times 100 = 259$
3	$(1.1)^3 \times 100 = 133$	11	$(1.1)^{11} \times 100 = 285$
4	$(1.1)^4 \times 100 = 146$	12	$(1.1)^{12} \times 100 = 314$
5	$(1.1)^5 \times 100 = 161$	13	$(1.1)^{13} \times 100 = 345$
6	$(1.1)^6 \times 100 = 177$	14	$(1.1)^{14} \times 100 = 380$
7	$(1.1)^7 \times 100 = 195$	15	$(1.1)^{15} \times 100 = 418$

Use the table above to find the doubling time for this prairie dog community.

Approximate Doubling Time Formula (Rule of 70)

For a quantity growing exponentially at a rate of P% per time period, the doubling time is *approximately*

$$T \approx \frac{70}{P\%}$$

* Use the actual percent for P

This approximation works best for small growth rates and breaks down for growth rates over about 15%.

Use the rule of 70 to find the doubling time for this prairie dog community in the previous problem.

4) A town's population was about 1.8 million in 2000 and was growing at a rate of about 1.4% per year. What is the approximate doubling time at this growth rate? By what factor will the population increase in 80 years?

4.5) The number of weeds in my backyard is growing exponentially. In six days the number of weeds has doubled. What is the average percentage growth rate per day during this period?

Notes 8B Day 2 – Exponential Decay and Half-Life

Exponential decay occurs whenever a quantity decreases by the same percentage in every fixed time period (for example 20% every year).The amount of the quantity repeatedly decreases to half its amount, with each halving occurring in a time called the **half-life.**

Radioactive plutonium-239 (Pu-239) has a half-life of about 24,000 years. Suppose 100- pound of Pu-239 is deposited at a nuclear waste site.

- In 24,000 years, or one half-life, the amount of Pu-239 declines to ½ the original amount, or to (1/2) x 100 = 50 pounds
- In 48,000 years, or two half-lives, the amount of Pu-239 declines to $(½)^2$ the original amount, or to (1/4) x 100 = 25 pounds
- In 72,000 years, or three half-lives, the amount of Pu-239 declines to $(½)^3$ the original amount, or to (1/8) x 100 = 12.5 pounds

Calculations with Half-Life

After t time, the final amount of a quantity (that's exponentially decaying), with a half-life time of T can be found by:

$$y = a \bullet \left(\frac{1}{2}\right)^{\frac{t}{T}}$$

y=final amount a=initial amount t=time that has passed T=half-life $\left(\frac{1}{2}\right)^{\frac{t}{T}}$ = fraction of the initial amount that remains

Example 5 Carbon-14 Decay

Radioactive carbon-14 has a half-life of about 5700 years. It collects in organisms only while they are alive. Once they are dead, it only decays. What fraction of the carbon-14 in an animal bone still remains 1000 years after the animal had died?

Example 6 Plutonium After 100,000 Years

Suppose that 100 pounds of Pu-239 is deposited are a nuclear waste site. How much of it will still be present in 100,000 years?

Approximate Half-Life Formula (rule of 70)

For a quantity decaying exponentially at a rate of P% per time period, the half-life is *approximately*

$$T \approx \frac{70}{P\%}$$

* Use the actual percent for P

This approximation works best for small decay rates and breaks down for decay rates over about 15%.

Example 7 Devaluation of Currency

Suppose that inflation causes the value of the Russian ruble to fall at a rate of 12% per year (relative to the dollar). At this rate, approximately how long does it take for the ruble to lose half its value?

Exact Formulas for Doubling Time and Half-Life

The approximate doubling time and half-life formulas are useful because they are easy to remember. However, for more precise work or for cases of larger rates where the approximate formulas break down, we need to exact formulas, given below. In Unit 9C, we will see how they are derived. These formulas use the fractional growth rate, defined as r = P/100, with r positive for growth and negative for decay. For example, if the percentage growth rate is 5% per year, the fractional growth rate is r = 0.05 per year. For a 5% *decay* rate per year, the fractional growth rate is r = -0.05 per year.

Exact Doubling Time and Half-Life Formulas

For an exponentially growing quantity with a fractional growth rate *r*, the doubling time is

$$T_{\text{double}} = \frac{\log_{10} 2}{\log_{10}(1+r)}$$

For an exponentially decaying quantity, we use a *negative* value for *r* (for example, if the decay rate is P = 5% per year, we set r = -0.05 per year). The half-life is

$$T_{\text{half}} = -\frac{\log_{10} 2}{\log_{10}(1+r)}$$

Note that *the units of time used for T and r must be the same*. For example, if the fractional growth rate is 0.05 per month, then the doubling time will also be measured in months. Also note that the formulas ensure that both T_{double} and T_{half} have positive values.

Example 8 Large Growth Rate

A population of rats is growing at a rate of 80% per month. Find the exact doubling time for this growth rate and compare it to the doubling time found with the approximate doubling time formula.

Example 9 Ruble Revisited

Suppose the Russian ruble is falling in value against the dollar at 12% per year. Using the exact half-life formula, determine how long it takes the ruble to lose half its value. Compare your answer to the approximate answer found in Example 7.