Section 5.7

Integration: "Rectilinear Motion Revisited Using Integration"

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Introduction

- In Section 4.6 we used the derivative to find velocity and acceleration for a particle in rectilinear motion.

  $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$

- In this section, we will use the integral to reverse the process.

$s(t) = \int v(t)dt$ and $v(t) = \int a(t)dt$

Example

Suppose a particle moves with velocity v(t) = cos πt along a coordinate line. Assuming that the particle has coordinate s = 4 at time t = 0, find its position function.

<u>Solution:</u>

The position function is s(t) = $\int v(t)dt = \int \cos \pi t \; dt = \frac{1}{\pi}\sin \pi t + C$

Now, substitute in (0,4) to solve for the specific C: $s(0) = \frac{1}{\pi}\sin 0 + C = 4$

$\frac{1}{\pi} * 0 + C = 4$

$C = 4$

$s(t) = \frac{1}{\pi}\sin \pi t + 4$

Computing Displacement by Integration

- Since displacement is final position minus initial position, it can be written as follows in integral form:

$$\begin{bmatrix} \text{displacement} \\ \text{over the time} \\ \text{interval } [t_0, t_1] \end{bmatrix} = \int_{t_0}^{t_1} v(t)\,dt = \int_{t_0}^{t_1} s'(t)\,dt = s(t_1) - s(t_0)$$

- This is a special case of a form of the Fundamental Theorem of Calculus from section 5.6:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Computing Distance Traveled by Integration

- Distance traveled is different than displacement because it is the total of all of the distances traveled in both positive and negative directions.
- Therefore, we must integrate the absolute value of the velocity function:

$$\begin{bmatrix} \text{distance traveled} \\ \text{during time} \\ \text{interval } [t_0, t_1] \end{bmatrix} = \int_{t_0}^{t_1} |v(t)|\, dt$$

- NOTE: Integrating velocity over a time interval produces displacement, and integrating speed over a time produces distance traveled.

Example

Example 2 Suppose that a particle moves on a coordinate line so that its velocity at time t is $v(t) = t^2 - 2t$ m/s (Figure 5.7.3).

(a) Find the displacement of the particle during the time interval $0 \le t \le 3$.

(b) Find the distance traveled by the particle during the time interval $0 \le t \le 3$.

Solution (a). From (3) the displacement is

$$\int_0^3 v(t)\,dt = \int_0^3 (t^2 - 2t)\,dt = \left[\frac{t^3}{3} - t^2\right]_0^3 = 0$$

Thus, the particle is at the same position at time $t = 3$ as at $t = 0$.

Solution (b). The velocity can be written as $v(t) = t^2 - 2t = t(t-2)$, from which we see that $v(t) \le 0$ for $0 \le t \le 2$ and $v(t) \ge 0$ for $2 \le t \le 3$. Thus, it follows from (4) that the distance traveled is

$$\int_0^3 |v(t)|\,dt = \int_0^2 -v(t)\,dt + \int_2^3 v(t)\,dt$$

$$= \int_0^2 -(t^2 - 2t)\,dt + \int_2^3 (t^2 - 2t)\,dt$$

$$= -\left[\frac{t^3}{3} - t^2\right]_0^2 + \left[\frac{t^3}{3} - t^2\right]_2^3 = \frac{4}{3} + \frac{4}{3} = \frac{8}{3} \text{ m}$$

Use a and b as given in question.

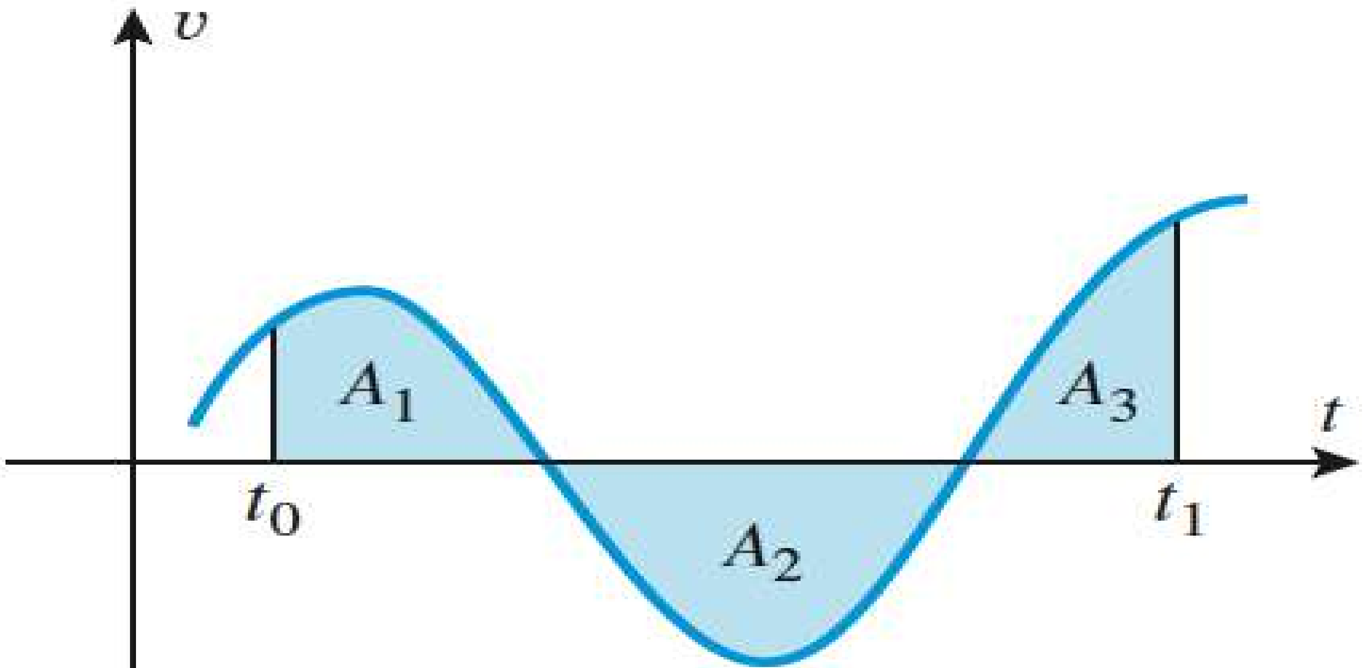
First determine where the particle turns around, then use Theorem 5.5.5 to break into parts.

Analyzing the Velocity vs. Time Curve

- As you hopefully saw in the last example, the integral is the "net signed area" under the velocity curve v(t) between time zero t_0 and time one t_1 (see graph on next slide) which gives you displacement.

- The "total area" under v(t) between those times gives you distance traveled.

Distance Traveled vs. Displacement



Notice the sign change on A_2 for distance traveled.

$$A_1 - A_2 + A_3 = \text{displacement}$$
$$A_1 + A_2 + A_3 = \text{distance traveled}$$

Constant Acceleration

- When acceleration is constant, we can work backwards to find formulas for position and velocity as long as we know the position and velocity at some point in time.
- Example: Suppose that an intergalactic spacecraft uses a sail and the "solar wind" to produce a constant acceleration of 0.032 m/s^2 (a = .032). Assuming that the spacecraft has a velocity of 10,000 m/s (v(0) = 10,000) when the said is first raised, how far will the spacecraft travel in 1 hour?
- NOTE: s(0) = 0
- Remember: $s(t) = \int v(t)dt$ and $v(t) = \int a(t)dt$
- Therefore, v(t) = .032t + C which is the integral of accel.

10,000 = .032 (0) + C so C = 10,000 gives

v(t) = .032t + 10,000 and

s(t) = .032t^2/2 + 10,000t + C_2 the integral of v(t)

0 = .016(0)^2 + 10,000(0) + C_2 so C_2 = 0 gives

s(t) = .016t^2 +10,000t and since 1 hour = 3600 sec

s(3600) = .016(3600)^2 + 10,000(3600) is apprx 36,200,000 meters

General Case of Constant Acceleration

- We can use the same method from the previous example to find general formulas for velocity and position when acceleration is constant by integrating acceleration:

$$v(t) = \int a(t)\,dt = \int a\,dt = at + C_1 \tag{8}$$

To determine the constant of integration C_1 we apply initial condition (7) to this equation to obtain

$$v_0 = v(0) = a \cdot 0 + C_1 = C_1$$

Substituting this in (8) and putting the constant term first yields

$$v(t) = v_0 + at$$

Since v_0 is constant, it follows that

$$s(t) = \int v(t)\,dt = \int (v_0 + at)\,dt = v_0 t + \tfrac{1}{2}at^2 + C_2 \tag{9}$$

To determine the constant C_2 we apply initial condition (6) to this equation to obtain

$$s_0 = s(0) = v_0 \cdot 0 + \tfrac{1}{2}a \cdot 0 + C_2 = C_2$$

Substituting this in (9) and putting the constant term first yields

$$s(t) = s_0 + v_0 t + \tfrac{1}{2}at^2$$

Free-Fall Model

- Motion that occurs when an object near the Earth is imparted some initial velocity (up or down) and thereafter moves along a vertical line is called free-fall motion.
- We assume the only force acting on the object is the Earth's gravity which is constant (when sufficiently close to Earth). NOTE: We are disregarding air resistance and gravitational pull from the moon, etc. for now.
- A particle with free-fall motion has constant acceleration in the downward direction (9.8 meters/second^2 or 32 feet/second^2).
- Therefore, the formulas developed on the previous slide apply and a = -acceleration due to gravity (g).

5.7.2 CONSTANT ACCELERATION If a particle moves with constant acceleration a along an s-axis, and if the position and velocity at time $t = 0$ are s_0 and v_0, respectively, then the position and velocity functions of the particle are

$$s(t) = s_0 + v_0 t + \tfrac{1}{2}at^2 \quad (10)$$

$$v(t) = v_0 + at \quad (11)$$

Examples

- There are examples on page 381 that my be helpful.
- They are similar to the spacecraft example, but include gravity.

Paris from atop the Eiffel Tower