Trigonometric Functions

4.1

Radian and Degree Measure

What You Should Learn

- Describe angles
- Use radian measure
- Use degree measure and convert between degree and radian measure
- Use angles to model and solve real-life problems

Angles

Angles

Read this slide and the next, but do not copy them down:

As derived from the Greek language, the word **trigonometry** means “measurement of triangles.” Initially, trigonometry dealt with relationships among the sides and angles of triangles and was used in the development of astronomy, navigation, and surveying.

With the development of calculus and the physical sciences in the 17th century, a different perspective arose—one that viewed the classic trigonometric relationships as *functions* with the set of real numbers as their domains.

Angles

Consequently, the applications of trigonometry expanded to include a vast number of physical phenomena involving rotations and vibrations, including the following.

sound waves

light rays

planetary orbits

vibrating strings

pendulums

orbits of atomic particles

Angles

The approach in this text incorporates *both* perspectives, starting with angles and their measure.

An **angle** is determined by rotating a ray (half-line) about its endpoint. The starting position of the ray is the **initial side** of the angle, and the position after rotation is the **terminal side**, as shown in Figure 4.1.

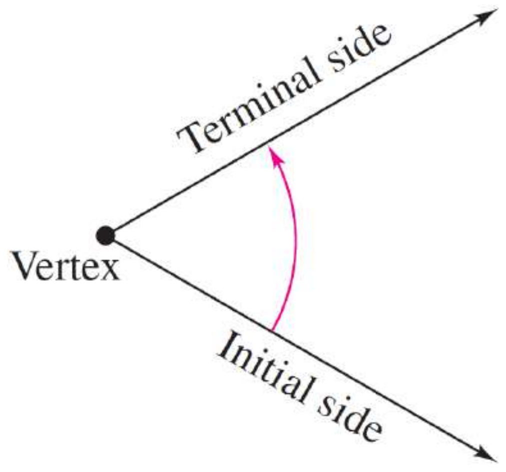


Figure 4.1

Angles

The endpoint of the ray is the **vertex** of the angle. This perception of an angle fits a coordinate system in which the origin is the vertex and the initial side coincides with the positive *x*-axis. Such an angle is in **standard position,** as shown in Figure 4.2.

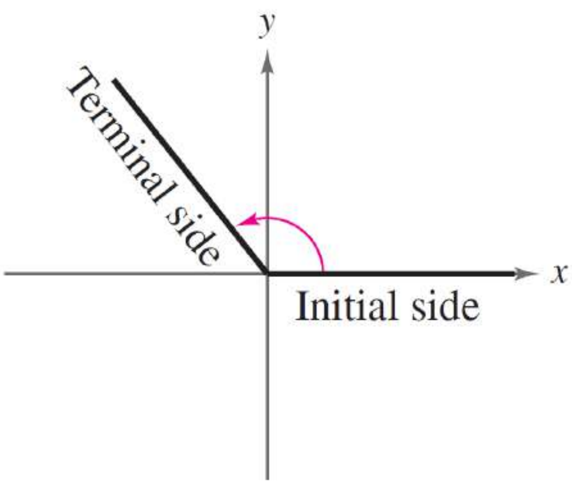


Figure 4.2

Angles

Positive angles are generated by counterclockwise rotation, and **negative angles** by clockwise rotation, as shown in Figure 4.3.

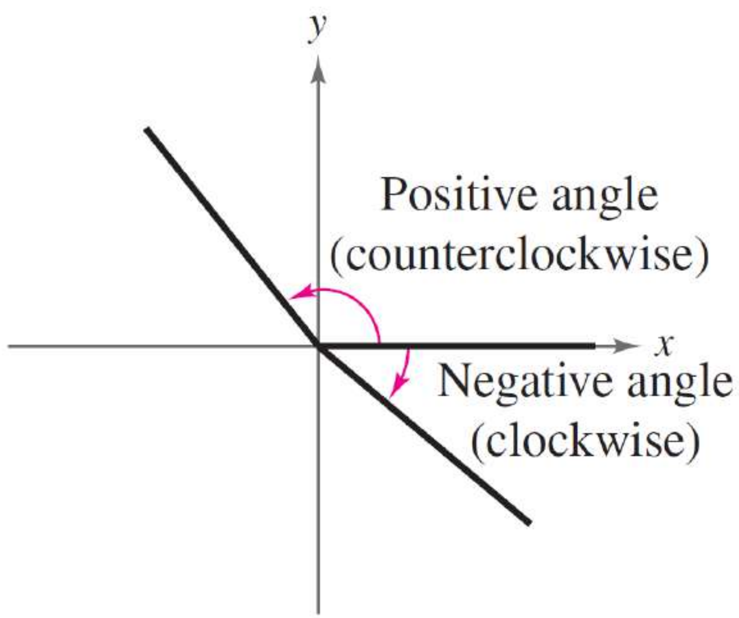


Figure 4.3

Angles

Angles are labeled with Greek letters such as α (alpha), β (beta), and (theta), as well as uppercase letters such as *A*,*B*, and *C*. In Figure 4.4, note that angles α and β have the same initial and terminal sides. Such angles are **coterminal.**

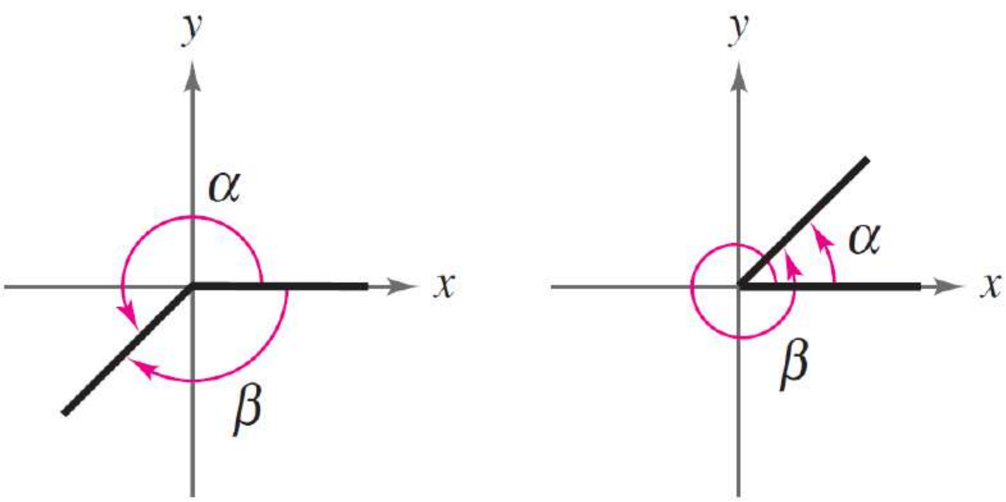


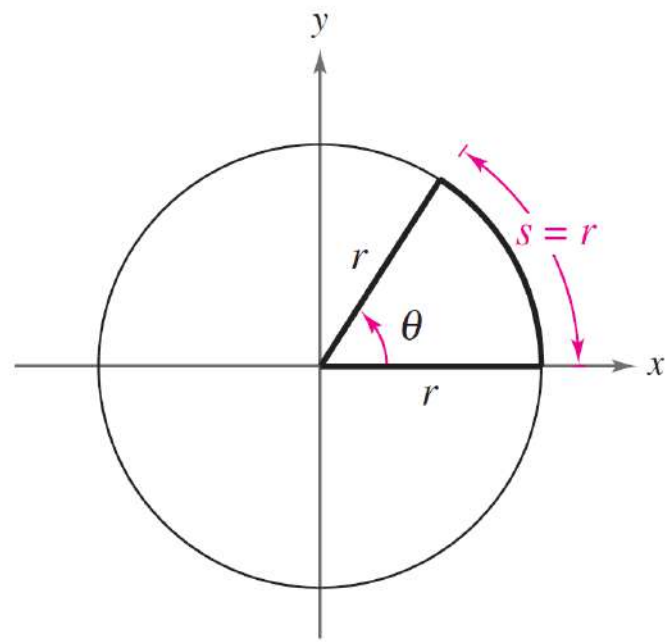
Figure 4.4

Radian Measure

Radian Measure

The **measure of an angle** is determined by the amount of rotation from the initial side to the terminal side. One way to measure angles is in *radians*.

This type of measure is especially useful in calculus. To define a radian, you can use a **central angle** of a circle, one whose vertex is the center of the circle, as shown in Figure 4.5.



Arc length = radius when θ = 1 radian.

Figure 4.5

Radian Measure

Definition of Radian

One **radian** (rad) is the measure of a central angle θ that intercepts an arc s equal in length to the radius r of the circle. (See Figure 4.5.) Algebraically this means that

$$\theta = \frac{s}{r}$$

where θ is measured in radians.

Because the circumference of a circle is $2\pi r$ units, it follows that a central angle of one full revolution (counterclockwise) corresponds to an arc length of

$s = 2\pi r$.

Radian Measure

Moreover, because

$2\pi \approx 6.28$

there are just over six radius lengths in a full circle, as shown in Figure 4.6. Because the units of measure for and are the same, the ratio

$$\frac{s}{r}$$

has no units—it is simply a real number.

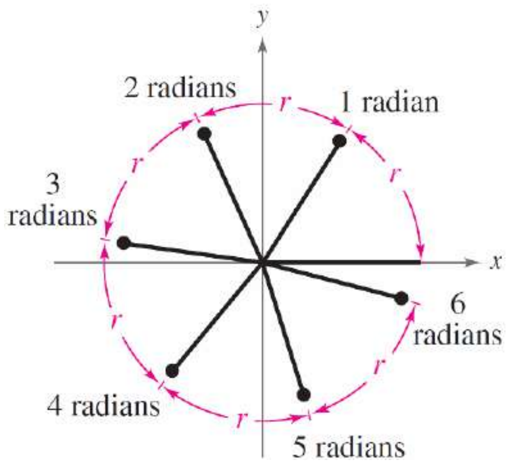


Figure 4.6

Radian Measure

The four quadrants in a coordinate system are numbered I, II, III, and IV. Figure 4.8 shows which angles between 0 and 2π lie in each of the four quadrants. Note that angles between 0 and $\pi/2$ and are **acute** and that angles between $\pi/2$ and π are **obtuse.**

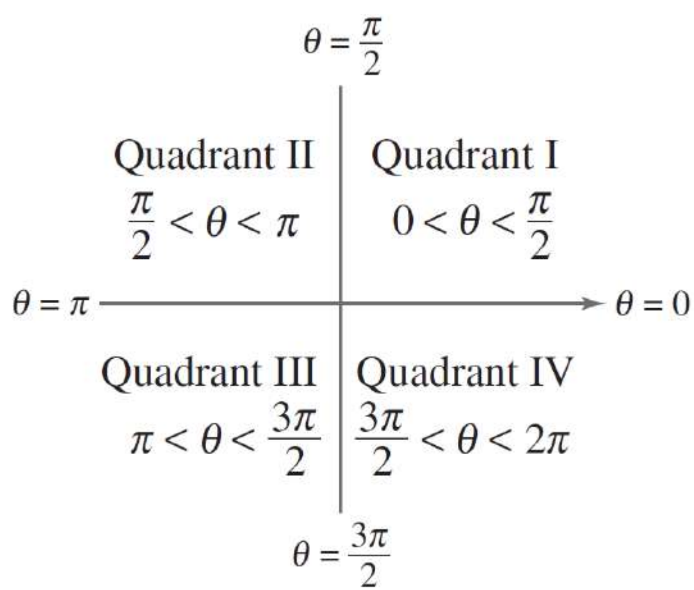


Figure 4.8

Radian Measure

Two angles are coterminal when they have the same initial and terminal sides. For instance, the angles 0 and 2π are coterminal, as are the angles $\pi/6$ and $13\pi/6$.

A given angle θ has infinitely many coterminal angles. For instance $\theta = \pi/6$, is coterminal with

, where is n an in $\frac{\pi}{6} + 2n\pi$

Example 1 – *Sketching and Finding Conterminal Angels*

a. For the positive angle, $\theta = \frac{13\pi}{6}$ 2π to obtain a coterminal angle.

$$\frac{13\pi}{6} - 2\pi = \frac{\pi}{6}$$

See Figure 4.9

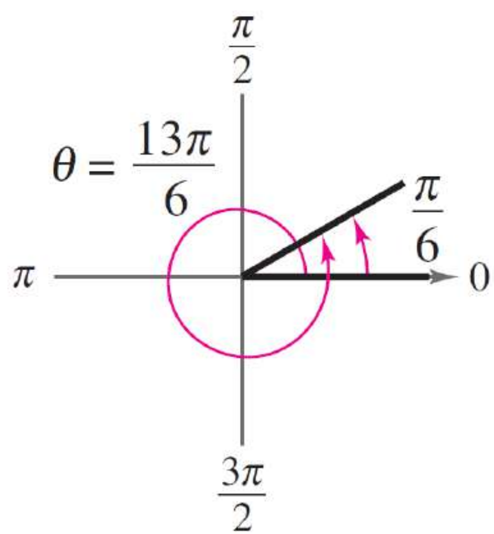


Figure 4.9

Example 1 – *Sketching and Finding Conterminal Angels*

b. For the positive angle $\theta = \frac{3\pi}{4}$ ubtract 2π to obtain a coterminal angle.

$$\frac{3\pi}{4} - 2\pi = -\frac{5\pi}{4}$$

See Figure 4.10.

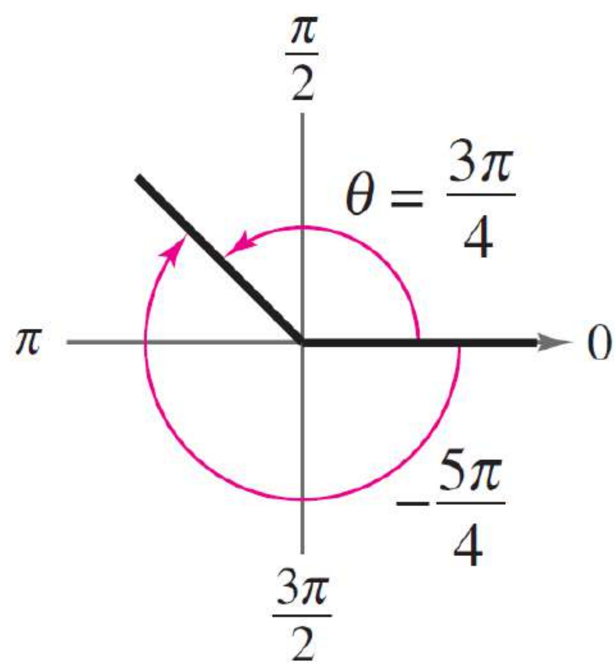


Figure 4.10

Example 1 – *Sketching and Finding Conterminal Angels*

c. For the negative angle, $\theta = -\frac{2\pi}{3}$ to obtain a coterminal angle.

$$-\frac{2\pi}{3} + 2\pi = \frac{4\pi}{3}$$

See Figure 4.11

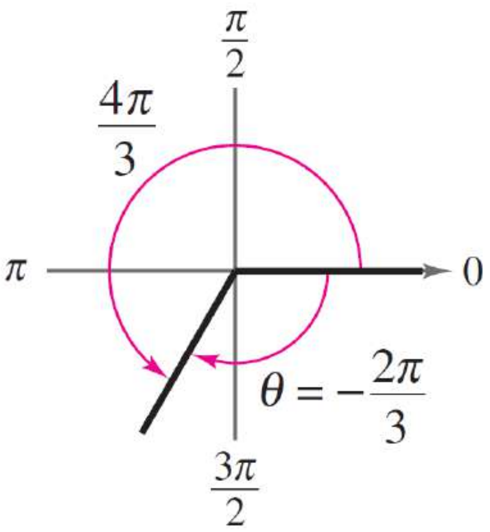


Figure 4.11

Degree Measure

Degree Measure

A second way to measure angles is in terms of **degrees,** denoted by the symbol °. A measure of one degree (1°) is equivalent to a rotation of $\frac{1}{360}$ of a complete revolution about the vertex. To measure angles, it is convenient to mark degrees on the circumference of a circle, as shown in Figure 4.12.

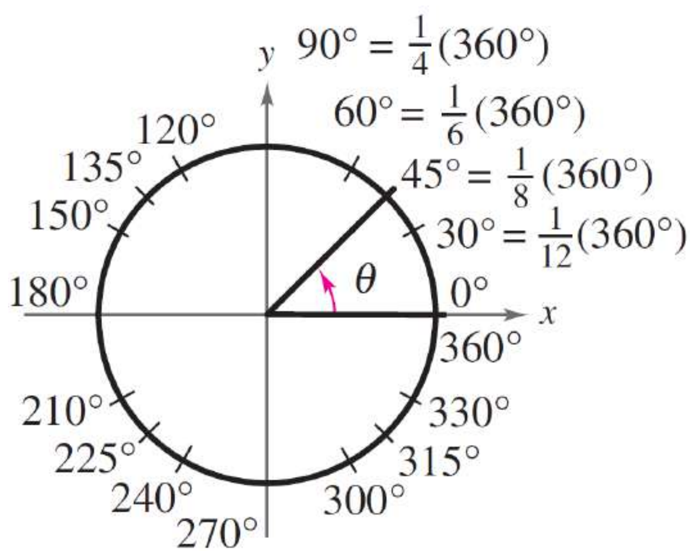


Figure 4.12

Degree Measure

So, a full revolution (counterclockwise) corresponds to 360° a half revolution to 180°, a quarter revolution to 90° and so on.

Because 2π radians corresponds to one complete revolution, degrees and radians are related by the equations

$360° = 2\pi$ rad and $180° = \pi$ rad.

Degree Measure

From the second equation, you obtain

$$1° = \frac{\pi}{180}\text{ rad} \quad \text{and } 1 \text{ rad} = \frac{180°}{\pi}$$

which lead to the following conversion rules which you can use with a bridge like you do in chemistry.

Conversions Between Degrees and Radians

1. To convert degrees to radians, multiply degrees by $\frac{\pi \text{ rad}}{180°}$.

2. To convert radians to degrees, multiply radians by $\frac{180°}{\pi \text{ rad}}$.

To apply these two conversion rules, use the basic relationship π rad $= 180°$. (See Figure 4.13.)

Degree Measure

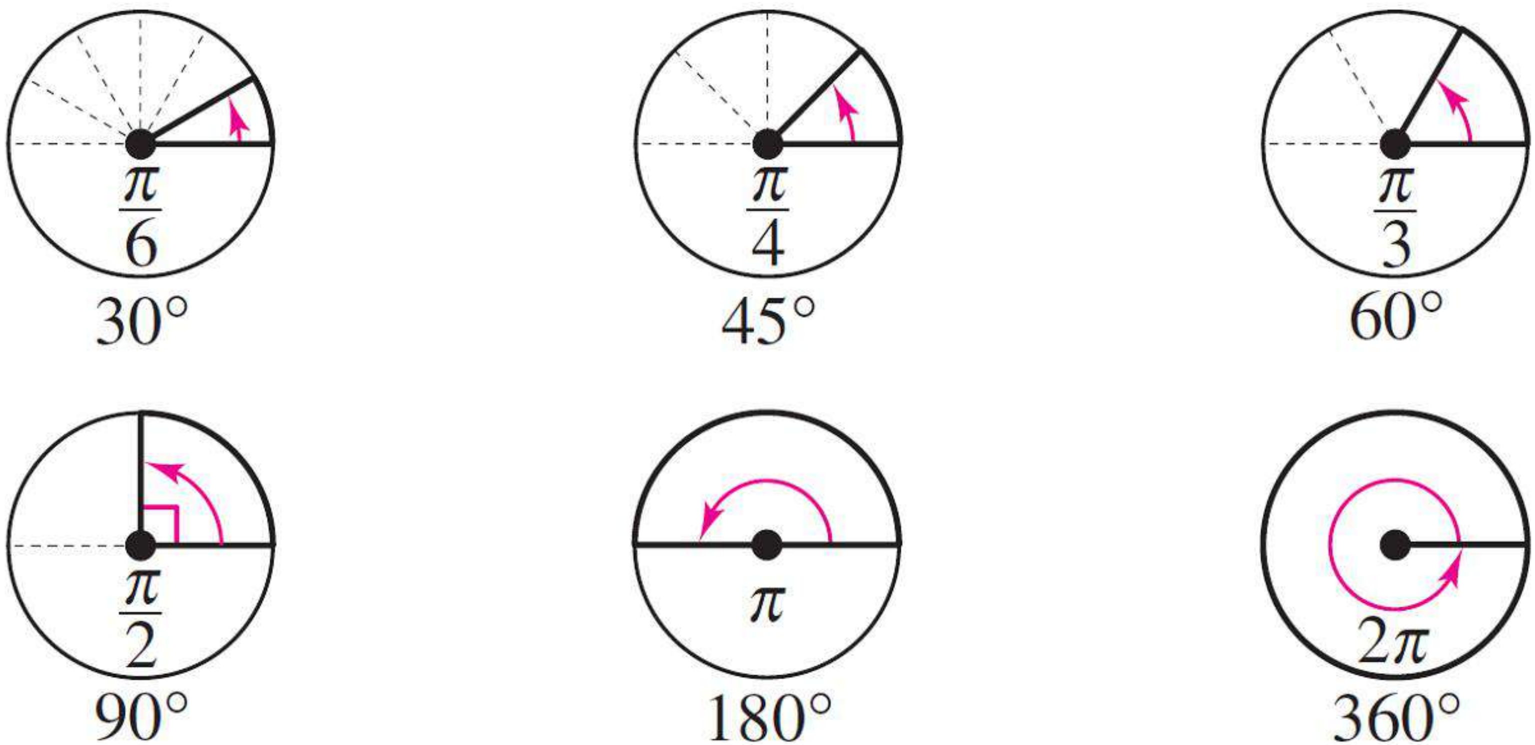


Figure 4.13

Example 2 – *Converting From Degrees to Radians*

a. $135° = (135\ \text{deg})\left(\dfrac{\pi\ \text{rad}}{180\ \text{deg}}\right) = \dfrac{3\pi}{4}$ radians — Multiply by $\dfrac{\pi}{180}$.

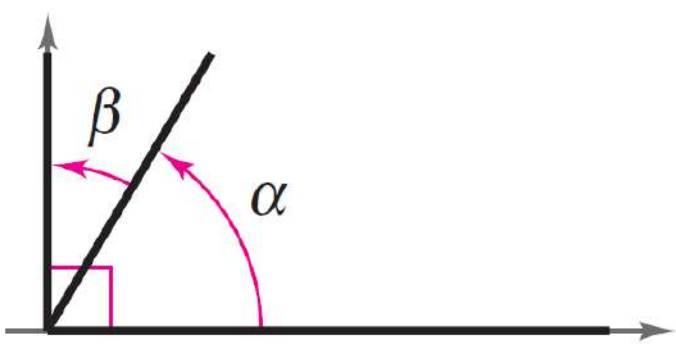
b. $540° = (540\ \text{deg})\left(\dfrac{\pi\ \text{rad}}{180\ \text{deg}}\right) = 3\pi$ radians — Multiply by $\dfrac{\pi}{180}$.

c. $-270° = (-270\ \text{deg})\left(\dfrac{\pi\ \text{rad}}{180\ \text{deg}}\right) = -\dfrac{3\pi}{2}$ radians — Multiply by $\dfrac{\pi}{180}$.

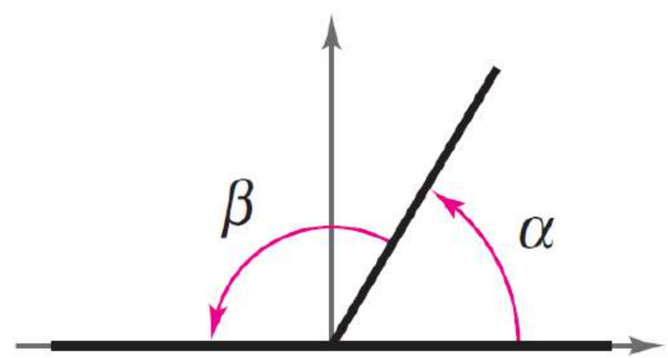
Degree Measure

Two positive angles α and β are **complementary** (complements of each other) when their sum is 90° (or $\pi/2$)

Two positive angles are **supplementary** (supplements of each other) when their sum is 180° (or π).
(See Figure 4.14.)



Complementary angles



Supplementary angles

Figure 4.14

Example 4 – *Complementary and Supplementary Angels*

If possible, find the complement and supplement of each angle.

a. 72°**b.**148°**c.** **d.** $\frac{2\pi}{5}$ $\frac{4\pi}{5}$

Solution:

a. The complement is

90° – 72° =18°.

The supplement is

180° – 72° = 108°.

Example 4 – *Solution*

cont'd

b. Because 148° is greater than 90° it has no complement. (Remember that complements are *positive* angles.)

The supplement is

180° – 148° = 32°.

c. The complement is

$$\frac{\pi}{2} - \frac{2\pi}{5} = \frac{5\pi}{10} - \frac{4\pi}{10} = \frac{\pi}{10}.$$

The supplement is

$$\pi - \frac{2\pi}{5} = \frac{5\pi}{5} - \frac{2\pi}{5} = \frac{3\pi}{5}.$$

Example 4 – *Solution*

cont'd

d. Because $4\pi/5$ is greater than $\pi/2$ it has no complement. The supplement is

$$\pi - \frac{4\pi}{5} = \frac{5\pi}{5} - \frac{4\pi}{5} = \frac{\pi}{5}.$$

Linear and Angular Speed

Linear and Angular Speed

The *radian measure* formula

$$\theta = \frac{s}{r}$$

can be used to measure arc length along a circle.

Arc Length

For a circle of radius r, a central angle θ intercepts an arc of length s given by

$$s = r\theta \qquad \text{Length of circular arc}$$

where θ is measured in radians. Note that if $r = 1$, then $s = \theta$, and the radian measure of θ equals the arc length.

Example 5 – *Finding Arc Length*

A circle has a radius of 4 inches. Find the length of the arc intercepted by a central angle of 240° as shown in Figure 4.15.

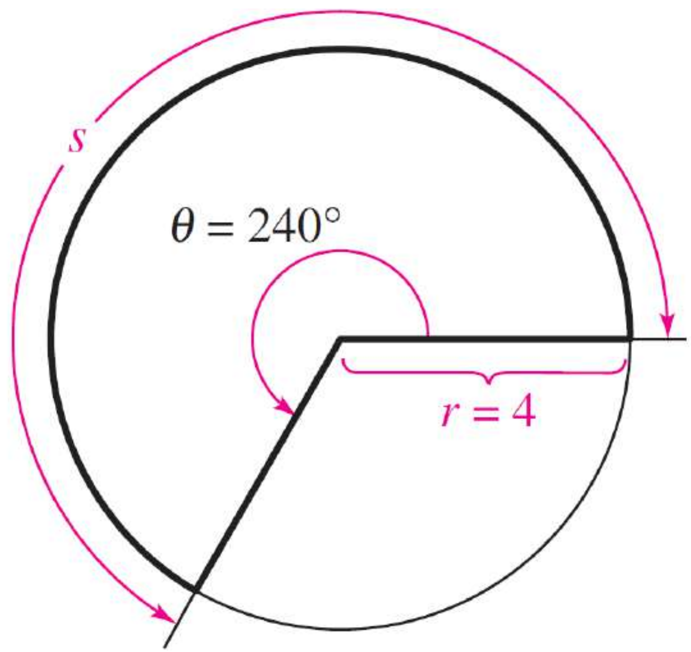


Figure 4.15

Example 5 – *Solution*

To use the formula

$s = r\theta$

first convert 240° to radian measure.

$$240^\circ = (240 \text{ deg})\left(\frac{\pi \text{ rad}}{180 \text{ deg}}\right) = \frac{4\pi}{3} \text{ radians}$$

Then, using a radius of $r = 4$ inches, you can find the arc length to be

$$s = r\theta = 4\left(\frac{4\pi}{3}\right) = \frac{16\pi}{3} \approx 16.76 \text{ inches.}$$

Note that the units for $r\theta$ are determined by the units for r because θ is given in radian measure and therefore has no units.

Linear and Angular Speed

Linear and Angular Speed

Consider a particle moving at a constant speed along a circular arc of radius r. If s is the length of the arc traveled in time t, then the **linear speed** of the particle is

$$\text{Linear speed} = \frac{\text{arc length}}{\text{time}} = \frac{s}{t}.$$

Moreover, if θ is the angle (in radian measure) corresponding to the arc length s, then the **angular speed** of the particle is

$$\text{Angular speed} = \frac{\text{central angle}}{\text{time}} = \frac{\theta}{t}.$$

Linear speed measures how fast the particle moves, and angular speed measures how fast the angle changes.

Example 6 – *Finding Linear Speed*

The second hand of a clock is 10.2 centimeters long, as shown in Figure 4.16. Find the linear speed of the tip of this second hand.

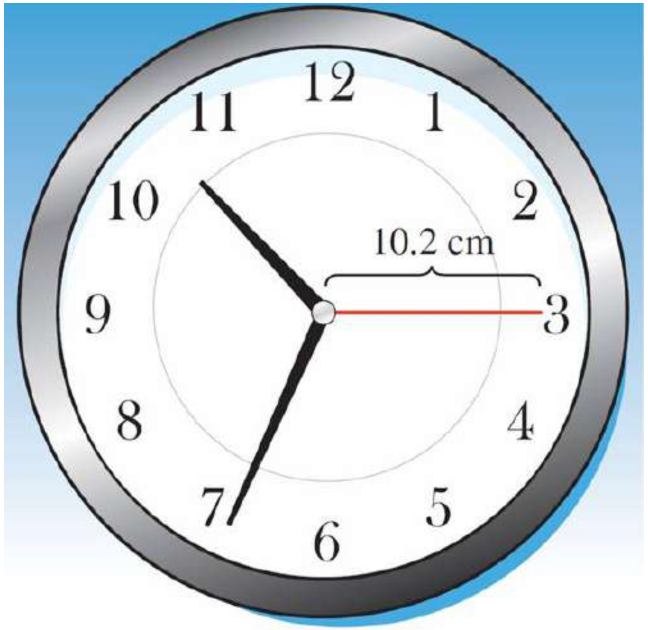


Figure 4.16

Example 6 – *Solution*

In one revolution, the arc length traveled is

$s = 2\pi r$

$= 2\pi(10.2)$ Substitute for r.

$= 20.4\pi$ centimeters.

The time required for the second hand to travel this distance is

t =1 minute = 60 seconds.

Example 6 – *Solution*

cont'd

So, the linear speed of the tip of the second hand is

$$\text{Linear speed} = \frac{s}{t}$$

$$= \frac{20.4\pi \text{ centimeters}}{60 \text{ seconds}}$$

$\approx$ 1.07 centimeters per second.