Precalc takeaways Name:

Algebra 2 Unit 5: Rational Functions and Expressions

Standards

A2 5.1 I can graph a transformation of 1/x without technology and describe its features.

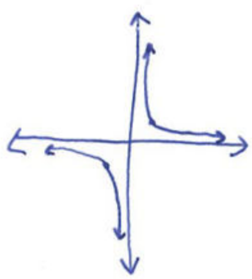
A2 5.2 I can find domains, vertical asymptotes, and removable discontinuities of rational functions.

A2 5.3 I can find horizontal and oblique asymptotes of rational functions.

A2 5.4 I can add, subtract, multiply, divide, and simplify rational functions, indicating domain restrictions.

A2 5.5 I can solve rational equations.

Features of the function $\boxed{f(x) = \dfrac{1}{x}}$



anchor points: (1,1) and (-1,-1)

asymptotes: x-axis (horizontal asymptote)
y-axis (vertical asymptote)

domain: $x \neq 0$ (x can be any number but zero)
$(-\infty, 0) \cup (0, \infty)$

range: $y \neq 0$ (y can be anything but zero)

this is an odd function

Features of the function $\boxed{f(x) = k + \dfrac{a}{x-h}}$ parent function: $f(x) = \frac{1}{x}$

- translated up k units, so the horizontal asymptote is $y=k$
- translated to the right h units, so the vertical asymptote is $x=h$
- dilated (stretched) up & down away from the horizontal asymptote by a factor of a.
- can reflect the whole graph by making a negative, like $\frac{-1}{x}$

Definition of **rational function**

a function of the form $f(x) = \dfrac{P(x)}{Q(x)}$,

where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$

Simplifying rational expressions

by factoring

- factor the numerator and denominator
- cancel matching factors
- BUT there is a hole at the x-values that make canceled factors 0.

example: $\dfrac{x^2+5x+6}{x^2+6x+8}$ factor & cancel

$$\frac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x+4)}$$

$$\frac{x+3}{x+4}, \quad x \neq -2$$

by factoring the denominator and then dividing

- factor the denominator
- divide the numerator by each factor in the denominator (using polynomial division)
- if remainder is zero, it is divisible and the factor can be canceled BUT there is a hole at the x-value that makes the factor 0.
- if the remainder is not zero, it is not divisible and the factor cannot be canceled.

example: $\dfrac{x^3+2x^2-9x-18}{x^2+6x+8}$ factor

$$\frac{x^3+2x^2-9x-18}{(x+4)(x+2)}$$

divide by $x+2$

$$\begin{array}{r} x^2 - 9 \\ x+2 \overline{\smash{)}\, x^3+2x^2-9x-18} \\ -(x^3+2x^2) \qquad\qquad \\ \hline 0 \; -9x-18 \\ -(-9x-18) \\ \hline 0 \end{array}$$

divisible!

$$\rightarrow \frac{(x^2-9)\cancel{(x+2)}}{(x+4)\cancel{(x+2)}}$$

divide by $x+4$

$$\begin{array}{r} x - 4 \\ x+4 \overline{\smash{)}\, x^2+0x-9} \\ -(x^2+4x) \quad \\ \hline -4x-9 \\ -(-4x-16) \\ \hline 7 \end{array}$$

← not divisible

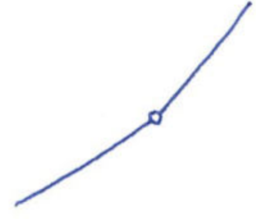
answer:

$$\frac{x^2-9}{x+4}, \quad x \neq -2$$

removable discontinuities a.k.a. holes

when a factor can be canceled, there is still a "divide by zero" problem for the x-value that makes the factor zero. this creates a <u>hole</u> a.k.a. <u>removable discontinuity</u> in the function.

it looks like this in the graph:



vertical asymptote(s) $x = \#$

after simplifying, there is a vertical asymptote at every x-value that makes the denominator zero.

factor the denominator to find those x-values.

horizontal asymptote $y = \#$

degree of numerator < degree of denominator

horizontal asymptote at $y=0$

degree of numerator = degree of denominator

- put the numerator and denominator in standard form.
- divide the leading coefficients

example: $\dfrac{3x^2-2x+4}{2x^2+6}$ horizontal asymptote at $y=\frac{3}{2}$

slant asymptotes a.k.a. oblique asymptotes $y = mx+b$

after simplifying, if degree of numerator > degree of denominator, there is an asymptote that is a more complex function, like a slanted line.

- divide the numerator by the denominator (with polynomial division)
- there should be a remainder, unless you forgot to simplify first
- the quotient is the slant asymptote.

example: $\dfrac{x^2+5x+7}{x+2}$

$$
\begin{array}{r l}
 & x+3 \\
x+2 \,\big) & x^2+5x+7 \\
 & -(x^2+2x) \\
\hline
 & 3x+7 \\
 & -(3x+6) \\
\hline
 & 1
\end{array}
$$

slant asymptote: $y=x+3$

note: sometimes the asymptote could be a parabola or a higher-degree polynomial. then it is called a curvilinear asymptote

Multiplying rational expressions

multiply across and then simplify

domain restriction: any x that makes the denominator zero at any point (even before simplifying) is not a possible x-value.

example: $\frac{x^2(x-3)}{x+1} \cdot \frac{5}{x} \xrightarrow{\text{multiply across}} \frac{5x^2(x-3)}{x(x+1)} \xrightarrow{\text{simplify}} \boxed{\frac{5x(x-3)}{x+1}, \; x \neq 0, \; x \neq -1}$

domain restrictions

Dividing rational expressions

"keep, change, flip" a.k.a. multiply by the reciprocal

check for domain restrictions at every step, including before & after flipping

example: $\frac{x-2}{x+2} \div \frac{x+5}{x(x+2)} \xrightarrow{\text{keep, change, flip}} \frac{x-2}{x+2} \cdot \frac{x(x+2)}{x+5} \xrightarrow{\text{multiply across}} \frac{x(x-2)\cancel{(x+2)}}{\cancel{(x+2)}(x+5)}$

$\longrightarrow \boxed{\frac{x(x-2)}{x+5}, \; x \neq -2, x \neq 0, x \neq -5}$

domain restrictions

Adding or subtracting rational expressions

you need a common denominator to add/subtract fractions.

you can get a common denominator by multiplying all the different denominators together, but if you factor the denominators first, you may be able to find a simpler one.

multiply each fraction by a clever form of 1 to make the denominators the same.

check for domain restrictions at every step.

example: $\frac{3}{x+7} + \frac{4}{x-4} \xrightarrow{\text{clever form of 1}} \frac{3}{x+7} \cdot \frac{x-4}{x-4} + \frac{4}{x-4} \cdot \frac{x+7}{x+7}$

$\xrightarrow{\text{multiply across}} \frac{3(x-4)}{(x+7)(x-4)} + \frac{4(x+7)}{(x+7)(x-4)} \xrightarrow{\text{add}} \frac{3(x-4)+4(x+7)}{(x+7)(x-4)}$

$\xrightarrow{\text{distribute}} \frac{3x-12+4x+28}{(x+7)(x-4)} \xrightarrow{\text{combine like terms}} \boxed{\frac{7x+16}{(x+7)(x-4)}, \; x \neq -7, x \neq 4}$

domain restrictions

Solving rational equations

- factor all the denominators
- find a common denominator
- multiply each term by a clever form of 1 so that every term has the same (common) denominator
- multiply both sides by the common denominator, which makes the denominators disappear
- solve as if you were in IM2 / IM1 / 8th grade
- check to make sure solutions work
  - → plug solutions into <u>original equation</u>
  - → look for zeros of denominators

example:

$$\frac{x}{x+3} - \frac{4}{x-2} = \frac{-5x^2}{x^2+x-6}$$

factor denominator

$$\frac{x}{x+3} - \frac{4}{x-2} = \frac{-5x^2}{(x+3)(x-2)}$$

clever forms of 1

$$\frac{x}{x+3}\cdot\frac{x-2}{x-2} - \frac{4}{x-2}\cdot\frac{x+3}{x+3} = \frac{-5x^2}{(x+3)(x-2)}$$

$$\frac{x(x-2)}{(x+3)(x-2)} - \frac{4(x+3)}{(x-2)(x+3)} = \frac{-5x^2}{(x+3)(x-2)}$$

multiply both sides by common denominator

$$x(x-2) - 4(x+3) = -5x^2$$

distribute

solving like IM2

$$x^2 - 2x - 4x - 12 = -5x^2$$
$$+5x^2 \qquad +5x^2$$

$$\frac{6x^2 - 6x - 12}{6} = \frac{0}{6}$$

factor

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2 \text{ or } x = -1$$

↑ extraneous because of $x-2$ denominator

$x = -1$

<u>Extraneous solutions</u>

solutions you get while solving, but are not really solutions because they make a denominator zero.