2

Polynomial and Rational Functions

2.6

Rational Functions and Asymptotes

What You Should Learn

- Find the domains of rational functions.
- Find vertical and horizontal asymptotes of graphs of rational functions.
- Use rational functions to model and solve real-life problems.

Introduction to Rational Functions

Introduction to Rational Functions

A **rational function** can be written in the form

$$f(x) = \frac{N(x)}{D(x)}$$

where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.

In general, the *domain* of a rational function of includes all real numbers except x-values that make the denominator zero. Much of the discussion of rational functions will focus on their graphical behavior near these x-values.

Example 1 – *Finding the Domain of a Rational Function*

Read this slide and the next, but do not write them down:

Find the domain of $f(x) = 1/x$ and discuss the behavior of f near any excluded x-values.

Solution:

Because the denominator is zero when $x = 0$, the domain of f is all real numbers except $x = 0$. To determine the behavior of f near this excluded value, evaluate $f(x)$ to the left and right of $x = 0$, as indicated in the following tables.

x	-1	-0.5	-0.1	-0.01	-0.001	$\to 0$
$f(x)$	-1	-2	-10	-100	-1000	$\to -\infty$

x	$0 \leftarrow$	0.001	0.01	0.1	0.5	1
$f(x)$	$\infty \leftarrow$	1000	100	10	2	1

Example 1 – *Solution*

cont'd

From the table, note that as x approaches 0 *from the left*, $f(x)$ decreases without bound. In contrast, as x approaches 0 *from the right*, $f(x)$ increases without bound.

Because $f(x)$ decreases without bound from the left and increases without bound from the right, you can conclude that f is not continuous. The graph of is shown in Figure 2.36.

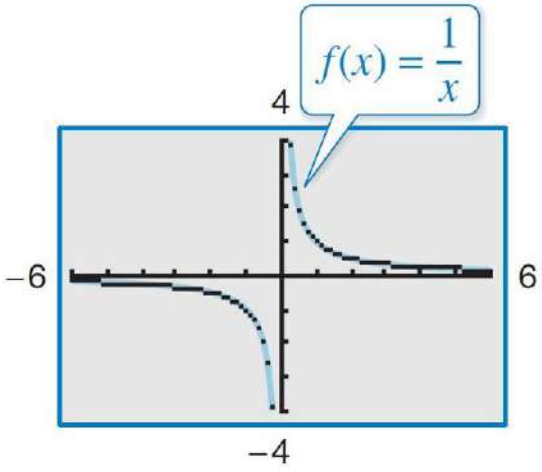

Figure 2.36

Vertical and Horizontal Asymptotes

Vertical and Horizontal Asymptotes

In Example 1, the behavior of near $x = 0$ is denoted as follows.

$$\underbrace{f(x) \to -\infty \text{ as } x \to 0^-}_{}$$

$f(x)$ decreases without bound as x approaches 0 from the left.

$$\underbrace{f(x) \to \infty \text{ as } x \to 0^+}_{}$$

$f(x)$ increases without bound as x approaches 0 from the right.

The line $x = 0$ is a **vertical asymptote** of the graph of f as shown in Figure 2.37.

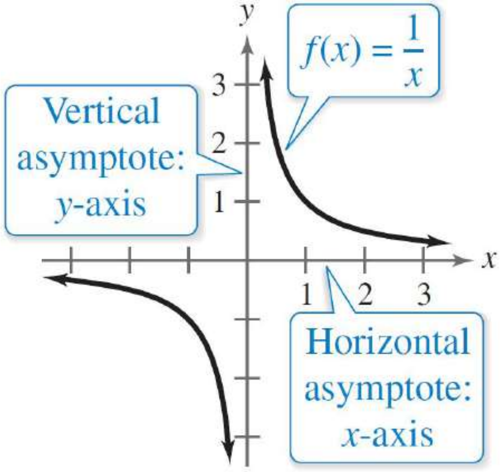

Figure 2.37

Vertical and Horizontal Asymptotes

From this figure you can see that the graph of *f* also has a **horizontal asymptote**—the line $y = 0$. This means the values of

$$f(x) = \frac{1}{x}$$

approach zero as increases or decreases without bound.

$$\underbrace{f(x) \to 0 \text{ as } x \to -\infty}$$

f(*x*) approaches 0 as x decreases without bound.

$$\underbrace{f(x) \to 0 \text{ as } x \to \infty}$$

f(*x*) approaches 0 as x decreases without bound.

Vertical and Horizontal Asymptotes

Definition of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the graph of f when

$$f(x) \to \infty \quad \text{or} \quad f(x) \to -\infty$$

as $x \to a$, either from the right or from the left.

2. The line $y = b$ is a **horizontal asymptote** of the graph of f when

$$f(x) \to b$$

as $x \to \infty$ or $x \to -\infty$.

Vertical and Horizontal Asymptotes

The graphs of *f*(*x*) = 1/*x* in Figure 2.37 and *f*(*x*) = (2*x* + 1)/(*x* + 1) in Figure 2.38 (a) are **hyperbolas.**

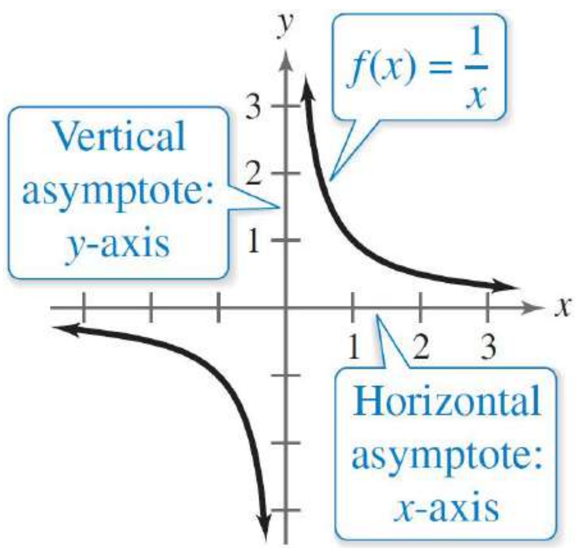

Figure 2.37

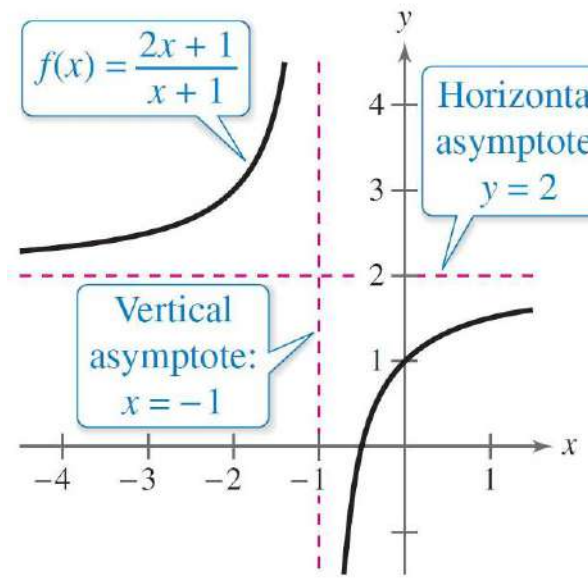

Figure 2.38(a)

Vertical and Horizontal Asymptotes

Vertical and Horizontal Asymptotes of a Rational Function

Let f be the rational function

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

where $N(x)$ and $D(x)$ have no common factors.

1. The graph of f has *vertical* asymptotes at the zeros of $D(x)$.
2. The graph of f has at most one *horizontal* asymptote determined by comparing the degrees of $N(x)$ and $D(x)$.

   a. If $n < m$, then the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.

   b. If $n = m$, then the graph of f has the line

   $$y = \frac{a_n}{b_m}$$

   as a horizontal asymptote, where a_n is the leading coefficient of the numerator and b_m is the leading coefficient of the denominator.

   c. If $n > m$, then the graph of f has no horizontal asymptote.

Example 2 – *Finding Vertical and Horizontal Asymptotes*

Find all asymptotes of the graph of each rational function.

a. $f(x) = \dfrac{2x}{3x^2 + 1}$ **b.** $f(x) = \dfrac{2x^2}{x^2 - 1}$

Solution:

a. For this rational function, the degree of the numerator is *less than* the degree of the denominator, so the graph has the line $y = 0$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x.

$3x^2 + 1 = 0$ Set denominator equal to zero.

Example 1 – *Solution*

cont'd

Because this equation has no real solutions, you can conclude that the graph has no vertical asymptote. The graph of the function is shown in Figure 2.39.

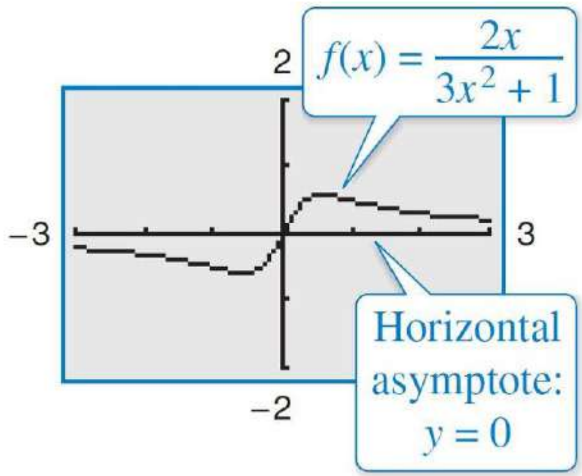

Figure 2.39

Example 1 – *Solution*

cont'd

b. For this rational function, the degree of the numerator is *equal* to the degree of the denominator. The leading coefficient of the numerator is 2 and the leading coefficient of the denominator is 1, so the graph has the line $y = 2$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x.

$x^2 - 1 = 0$ Set denominator equal to zero.

$(x + 1)(x - 1) = 0$ Factor.

$x + 1 = 0 \quad x = -1$ Set 1st factor equal to 0.

$x - 1 = 0 \quad x = 1$ Set 2nd factor equal to 0.

Example 1 – *Solution*

cont'd

This equation has two real solutions, $x = -1$ and $x = 1$, so the graph has the lines $x = -1$ and $x = 1$ as vertical asymptotes, as shown in Figure 2.40.

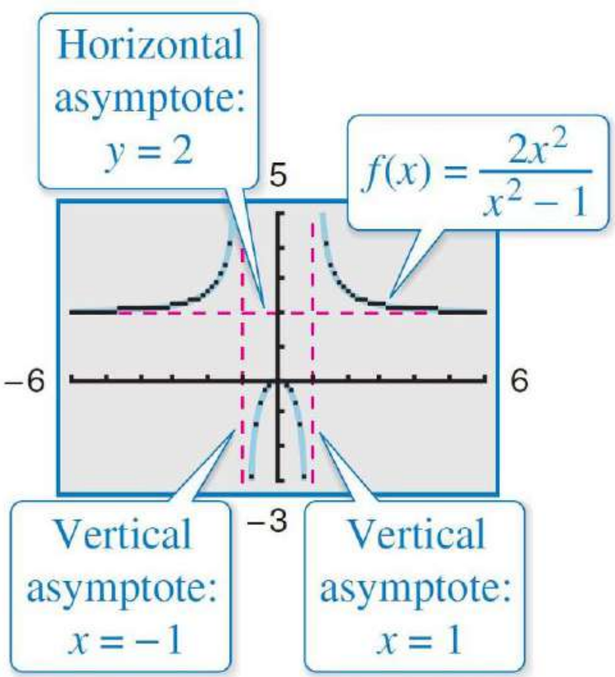

Figure 2.40

Application

Application

Read this slide, but do not write it down:

There are many examples of asymptotic behavior in real life. For instance, Example 5 shows how a vertical asymptote can be used to analyze the cost of removing pollutants from smokestack emissions.

For the next slide, please paraphrase the question. Do not copy it word for word.

Example 5 – *Cost-Benefit Model*

A utility company burns coal to generate electricity. The cost C (in dollars) of removing p% of the smokestack pollutants is given by

$$C = \frac{80{,}000p}{100 - p}$$

for $0 \le p < 100$. Use a graphing utility to graph this function. You are a member of a state legislature that is considering a law that would require utility companies to remove 90% of the pollutants from their smokestack emissions. The current law requires 85% removal. How much additional cost would the utility company incur as a result of the new law?

Example 5 – *Solution*

The graph of this function is shown in Figure 2.42. Note that the graph has a vertical asymptote at

p = 100.

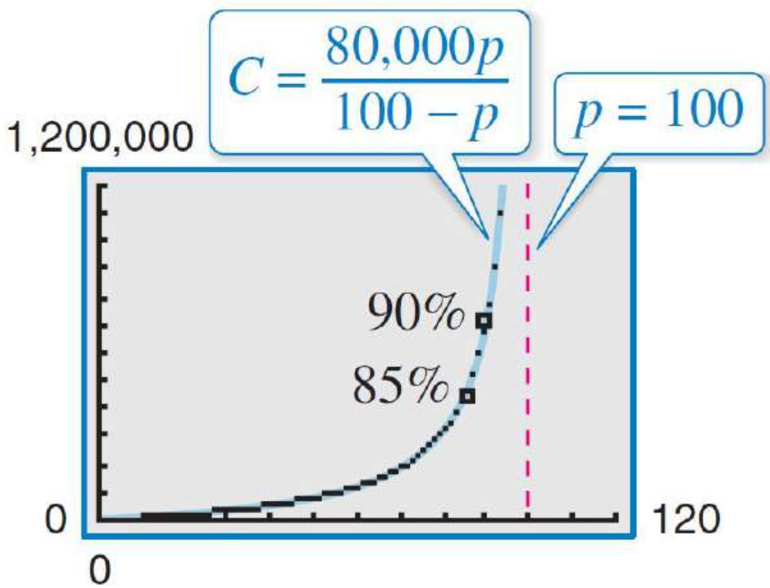

Figure 2.42

Example 5 – *Solution*

cont'd

Because the current law requires 85% removal, the current cost to the utility company is

$$C = \frac{80{,}000p}{100 - p}$$ Write original function.

$$C = \frac{80{,}000(85)}{100 - 85}$$ Substitute 85 for *p*.

$$\approx \$453{,}333.$$ Simplify.

Example 5 – *Solution*

cont'd

If the new law increases the percent removal to 90%, the cost will be

$$C = \frac{80{,}000p}{100 - p}$$ Write original function.

$$C = \frac{80{,}000(90)}{100 - 90}$$ Substitute 90 for *p*

$$= \$720{,}000.$$ Simplify.

Example 5 – *Solution*

cont'd

So, the new law would require the utility company to spend an additional

720,000 – 453,333 = $266,667.

Subtract 85% removal cost from 90% removal cost.