Pre-AP Precalculus Math Team

"Summer Reading" for
2021-2022

Just a little love from Mr. Hurry to you.
☺

You are responsible for the knowing the content of this packet by the second day of class when you will be given your first of many quizzes over this material. No one, in my career spanning four different decades, has done well on these quizzes if they waited until school begins to begin studying. I suggest you read over the material a couple of times, then make flashcards, quizlets, etc. to help you learn whatever you are having trouble learning. I also suggest you begin this process early in the summer it will have time to marinate. Good luck!

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FORMULAS and DEFINITIONS for MATH TEAM

Random Misc. Math Team Formulas

1. Sum of first *n* numbers $$1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

Ex) Find the sum of $1+2+3+\ldots+1000$

Solution: $\frac{1000(1000+1)}{2}=\boxed{500500}$

2. Sum of first *n* squares $$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

Ex) Find the sum of $1^2+2^2+3^2+\ldots+20^2$

Solution: $\frac{20(20+1)(2\cdot 20+1)}{6}=\boxed{2870}$

3. Sum of first *n* cubes $$1^3+2^3+3^3+\ldots+n^3=\frac{n^2(n+1)^2}{4}$$

Ex) Simplify: $1+8+27+\ldots+1000$

Solution: Since $1+8+27+\ldots+1000=1^3+2^3+3^2+...+10^3$

Then, $\frac{10^2(10+1)^2}{4}=\boxed{3025}$

4. Sum of first n fourth powers. $$1^4+2^4+3^4+\ldots+n^4=\frac{n(n+1)(2n+1)\left(3n^2+3n-1\right)}{30}$$

Ex) Simplify: $\sum_{n=1}^{10} n^4 = 1^4+2^4+3^4+\ldots+10^4=\frac{10(10+1)(2\cdot 10+1)\left(3\cdot 10^2+3\cdot 10-1\right)}{30}=\boxed{25333}$

5. Sum of first n fifth powers. $$1^5+2^5+3^5+\ldots+n^5=\frac{n^2(n+1)^2\left(2n^2+2n-1\right)}{12}$$

6. Classifying a Conic from its General Equation $Ax^2+Cy^2+Dx+Ey+F=0$

1. Circle: $A=C$ Ex) $3x^2+3y^2-12x+18y-20=0$
2. Parabola: $AC=0$ *A = 0, C = 0 but not both* Ex) $3x^2-12x+18y-20=0$
3. Ellipse: $AC>0$ Ex) $3x^2+9y^2-12x+18y-20=0$
4. Hyperbola: $AC<0$ Ex) $3x^2-9y^2-12x+18y-20=0$

7. Classification of Conics by the Discriminant (*xy* term) $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

1. Ellipse $B^2-4AC<0$

   Ex) $3x^2+2xy+9y^2-12x+18y-20=0$

2. Parabola $B^2-4AC=0$

   Ex) $3x^2+6xy+3y^2-11x+18y-7=0$

3. Hyperbola $B^2-4AC>0$

   Ex) $3x^2+2xy-9y^2-12x+18y-20=0$

* This is not always accurate, because some equations will be degenerate or another type of graph. However, if it is a conic, then it would abide by these rules.

Here are some examples of types of exceptions (all of these have NO Dx or Ey terms).

a) $x^2+2xy+y^2-9=0$ This is two parallel lines.

Why? The variable part of the equation can be factored as a perfect square thus giving us a difference of two squares.

$$(x+y)^2-9=0$$

$$(x+y-3)(x+y+3)=0$$

$$\therefore y=-x+3 \text{ and } y=-x-3$$

b) $x^2+4xy+y^2=0$ This is two intersecting lines.

Why? At first glance this looks like a hyperbola because of the B^2-4AC check. However, because there is NO constant term this becomes a degenerate hyperbola. So the intersecting lines are actually the asymptotes. Notice, also, that the problem can NOT be factored with real numbers.

c) $x^2-4xy+4y^2=0$ This is one line.

Why? This looks a lot like the prior problem, but this time the variable part of the equation can be factored as a perfect square. However unlike the first example, you get the same factor twice.

$$(x-2y)^2=0$$

$$x-2y=0$$

$$\therefore y=\frac{1}{2}x$$

9. Sum of the roots of a polynomial . . . $ax^n+bx^{n-1}+cx^{n-2}+\cdots+yx+z=0$

$$r_1+r_2+\cdots=-\frac{b}{a} \quad \text{or} \quad -\frac{2nd\ \ coefficient}{1st\ coefficient}$$

10. Product of the roots of a polynomial . . . $ax^n+bx^{n-1}+cx^{n-2}+\cdots+yx+z=0$

$$r_1\cdot r_2\cdot\ldots=\begin{cases}\dfrac{z}{a}\ \text{ or }\ \dfrac{\text{last coeff.}}{\text{first coeff.}}, & \text{if } n \text{ is even}\\ -\dfrac{z}{a}\ \text{ or }\ -\dfrac{\text{last coeff}}{\text{firstcoeff}}, & \text{if } n \text{ is odd}\end{cases}$$

11. Sum of the reciprocals of the roots of . . . $ax^n+bx^{n-1}+cx^{n-2}+\cdots+yx+z=0$

$$\frac{1}{r_1}+\frac{1}{r_2}+\cdots=-\frac{y}{\text{z}} \quad \text{or} \quad -\frac{\text{2nd to last coeff.}}{\text{last coeff.}}$$

12. Sum of the squares of the roots of . . . $ax^n+bx^{n-1}+cx^{n-2}+\cdots+yx+z=0$

$$r_1^{\ 2}+r_2^{\ 2}+r_3^{\ 2}+\cdots=\frac{b^2-2ac}{a^2}$$

★Examples of **# 9-12 above...**

For the following polynomial, $f(x)=3x^5-7x^4+5x^3+12x^2-6x+9$, find the . . .

Sum of the roots:

$$-\frac{b}{a}=-\frac{-7}{3}=\boxed{\frac{7}{3}}$$

Product of the roots:

odd power, so . . . $-\dfrac{last}{first}=-\dfrac{9}{3}=\boxed{-3}$

Sum of the reciprocals

$$-\frac{\text{2nd to last}}{\text{last}}=-\frac{-6}{9}=\boxed{\frac{2}{3}}$$

Sum of the squares of the roots:

If $ax^n+bx^{n-1}+cx^{n-2}+dx^{n-3}+\cdots=0$, then $\dfrac{b^2-2ac}{a^2}=\dfrac{(-7)^2-2(3)(5)}{(3)^2}=\boxed{\dfrac{19}{9}}$

13. Sum of the measures of interior angles of polygon in degrees:

$$S=(n-2)180$$

Ex) Find the sum of the interior angles of a 20 sided polygon.

Solution: $S_{20}=(20-2)180=\boxed{3240}$

Ex) If the sum of the interior angles of a polygon is $8640°$, then determine the number of sides.

$8640=(n-2)180$

Solution: $\boxed{48}=n-2$

$50=n$

14. Sum of the measures of "exterior angles" of polygon in degrees:

$$\text{Always} = 360°$$

15. Number of diagonals of a convex polygon with *n* vertices (or sides):

Number of diagonals = $\frac{n}{2}(n-3)$

Ex) Determine the number of diagonals of a 30-sided figure.

Solution: $\frac{30}{2}(30-3) = \boxed{405}$

Variations on the value of *e*

16. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$, $\lim_{n\to\infty}\left(1-\frac{a}{n}\right)^n = e^{-a}$, $\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n = e^{-2}$

17. To find the number of positive integral factors of a number like 720 . . .
Rather than actually write out all of the factors like this: 1,2,3,4,5,6,8,9,10,12, etc. . . .

1. factor a number into a product of prime factors.
   (ex. $720 \to 2^4 \cdot 3^2 \cdot 5$)
2. add the "1" to each of the exponents.
   (ex. $4, 2, 1 \to 5, 3, 2$)
3. multiply these "new" numbers together.
   (ex. $5 \cdot 3 \cdot 2 = 30$ Therefore, there are 30 positive integral factors of the number 720)

*note: these are the ***positive*** integral factors. If the problem asks for the integral factors, then you must multiply by 2 to account for positive and negative factors.

18. Sum of the ***positive*** integral factors of a number. Rather than actually write out all the factors and then add them, . . .

1. factor a number into a product of prime factors.

(ex. $72 \to 2^3 \cdot 3^2$)

2. for each factorization such as $x^n \cdot y^m \cdot z^p$

find $(x^n + x^{n-1} + \cdots + x + 1)(y^m + y^{m-1} + \cdots + y + 1)(z^p + z^{p-1} + \cdots + z + 1)$

Ex.) Because $72 \to 2^3 \cdot 3^2$, then $\left(2^3 + 2^2 + 2 + 1\right)(3^2 + 3 + 1) = 195$

Therefore, the sum of the positive integral factors of the number 72 is 195

*If you were asked to find the sum of the integral factors (positive & negative), then it would always = 0.

19. Sum of the coefficients (and constants) in the expansion of $\left(ax^n + by^m + cz^p + \ldots\right)^h$

Simply calculate $\left(a+b+c+\ldots\right)^h$. We call this rule "Will's Rule" named after Will Gardner, '99.

Ex) Determine the sum of the coefficients and constants in the expansion of $\left(2x^3 - 3y + 4z^3 - 1\right)^6$

Solution: $\left(2-3+4-1\right)^6 = 2^6 = \boxed{64}$

20. The number of terms in the expansion of $\left(a+b+c+...\right)^n$ = ${}_{x+n-1}C_n$

where: x is the number of different terms in the polynomial.
n is the power that the polynomial is to be raised.

Ex.) Q: Find the number of terms in the expansion of $\left(A+B+C-B+D+A\right)^{10}$

A: First of all, you don't want to attempt to actually multiply this out and then count the terms. It would be annoying at best, very painful at worst.
Therefore, try this method . . .

First, simplify the problem . . . $\left(2A+C+D\right)^{10}$

Then, since $x=3$ & $n=10$, ${}_{(3+10-1)}C_{10} = {}_{12}C_{10} = \dfrac{12!}{2!10!} = \dfrac{12\cdot 11}{2} = \boxed{66}$

* *Mr. Hurry actually came up with this rule on his own. If you ever find it in some book, don't tell him, because it took a lot of work.*

21. Finding the Unit's digit of a number

1. notice the pattern of the units place of the digit as the exponent grows from 1 to ∞

   Ex.) What is the unit's digit of 4^{367} ?

   $4^1 = \underline{4}$

   $4^2 = 1\underline{6}$ See how the unit's digit alternates between 4 & 6

   $4^3 = 6\underline{4}$

   $4^4 = 25\underline{6}$

2. Divide the exponent by the number of different numbers in the repetition, (since it alternated every two times with 4 & 6 then divide by 2)

   (ex. $2\overline{)367}$ with quotient 183

   $\underline{366}$

   1

   Use the remainder, which will be either $0,1,2,\ldots k-1$)

3. Take the remainder and use it as the exponent of the base number.

   (ex. $4^1 \rightarrow 4$ Therefore, the units digit of the number 4^{367} is 4.)

* In the case that there is no remainder-the remainder is 0-then use k (the divisor) as the exponent.

22. Harmonic mean:

The harmonic mean, H, of two numbers, x and y, is . . .

$$\frac{2}{H}=\frac{1}{x}+\frac{1}{y} \quad \text{or} \quad H=\frac{2}{\frac{1}{x}+\frac{1}{y}} \quad \text{or simplified . . .} \quad \boxed{H=\frac{2xy}{x+y}}$$

Ex) Find the harmonic mean of 40 and 100.

Solution: $\frac{2\cdot 40\cdot 100}{40+100}=\frac{\cancel{2}\cdot 40\cdot 100}{\cancel{140}_{70}}=\boxed{\frac{400}{7}}$

It is not necessary to know this for quizzes in the Fall...

The harmonic mean, H, of more than two numbers, can be calculated in a similar way . . .

$$\frac{3}{H}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \quad \text{or} \quad H=\frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} \qquad \frac{4}{H}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{w} \quad \text{or} \quad H=\frac{4}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{w}}$$

$$\frac{n}{H}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ .\ .\ . \quad \text{or} \quad H=\frac{n}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ .\ .\ .}$$

, where n is the number of terms

23. Finding the Average Speed for two sections of a trip when you are given the speed going and the speed coming back and you travel the ***same distance*** each section.

Average speed $=\frac{2\cdot v_1\cdot v_2}{v_1+v_2}$ *This is the same as finding the "***harmonic mean***."

Ex) If Biff ran to school at 20 mph and then ran home at 10 mph, what was his average speed?

Solution: $\frac{2\cdot 20\cdot 10}{20+10}=\frac{400}{30}=\boxed{\frac{40}{3}}=\boxed{13\frac{1}{3}\text{ mph}}$

24. Finding the Average Speed for two sections of a trip when you are given the speed going and the speed coming back and you spend the ***same time*** traveling each way.

Average speed $=\frac{v_1+v_2}{2}$ *This is the same as finding the "***arithmetic mean***."

Ex) If Biff drove 50 mph for 30 minutes and then drove 40 mph for the next 30 minutes, then what was his average speed for the entire trip?

Solution: $\frac{50+40}{2}=\boxed{45\text{ mph}}$

25. Sum of the coefficients/numbers in the n row of Pascal's Triangle?

Simply calculate 2^n
(Assume that the top of the "triangle" is the *0* row)

* Remember the 1st row is considered the 0 row, or row 0, because it represents $(a+b)^0$
the 2nd row is considered the 1 row, or row 1, because it represents $(a+b)^1$ etc. . .

Ex) Find the sum of the terms in the 12 row of Pascal's Triangle.

Solution: $2^{12}=\boxed{4096}$

Ex) Find the sum of the terms in the 12th row of Pascal's Triangle.

Solution: $2^{11}=\boxed{2048}$

26. Double Factorials – Either $n!!=n(n-2)(n-4)(n-6)\cdots(4)\cdot(2)$

or... $n!!=n(n-2)(n-4)(n-6)\cdots(3)\cdot(1)$

★*Don't confuse double factorials with a factorial of a factorial.*

For example... $6!!=(6)(4)(2)=48$ and $7!!=(7)(5)(3)(1)=105$

But... $(4!)!=(4\cdot3\cdot2\cdot1)!=24!=24\cdot23\cdot22\cdots3\cdot2\cdot1\approx6.204x10^{23}$

27. Triple Factorials – Either $n!!!=n(n-3)(n-6)(n-9)\cdots(4)\cdot(1)$

or... $n!!!=n(n-3)(n-6)(n-9)\cdots(5)\cdot(2)$

or... $n!!!=n(n-3)(n-6)(n-9)\cdots(6)\cdot(3)$

★*Don't confuse triple factorials with a factorial of a factorial of a factorial*

For example... $12!!!=(12)(9)(6)(3)=1944$

But... $((3!)!)!=((3\cdot2\cdot1)!)!=(6!)!=720!\approx2.601x10^{1746}$

28. Trailing Zeros of a factorial (base 10)

1. "Zeros" occur through the factor 10 which clearly comes from $2\cdot5$.
2. Therefore, I need to know the number of 2's and 5's which are in a factorial.
3. Since there are going to be plenty enough 2's to pair with the 5's, it is only necessary to find the number of factors of 5 in the factorial, and this can be found using the sequence formula $t_n=t_1+(n-1)d$ and some creative thinking.

ex. $127!=1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10\cdot\ldots\cdot15\cdot\ldots20\cdot\ldots24\cdot25\cdot26\cdot\ldots\cdot125\cdot\ldots127$

A) notice that

$5\rightarrow5^1$ $25\rightarrow5^2$

$10\rightarrow5^1$ $50\rightarrow5^2$ $125\rightarrow5^3$

$15\rightarrow5^1$ $75\rightarrow5^2$

etc. *etc*

calculate the number, n_1, of multiples of 5 in the sequence 5,10,15,…125

calculate the number, n_2, of multiples of 25 in the sequence 25,50,75,…125

calculate the number, n_3, of multiples of 125 in the sequence 125

*shortcut would be to find $\left[\!\left[\frac{127}{5}\right]\!\right]+\left[\!\left[\frac{127}{25}\right]\!\right]+\left[\!\left[\frac{127}{125}\right]\!\right]=25+5+1=31$

where x is the greatest integer function of x.

Or use a trick . . .

$$=\left[\!\left[\frac{127}{5}\right]\!\right]+\left[\!\left[\frac{127}{25}\right]\!\right]+\left[\!\left[\frac{127}{125}\right]\!\right]$$

Ty Johnson's ('08) rule →

$$=\left[\!\left[\frac{127}{5}\right]\!\right]\nearrow+\left[\!\left[\frac{25}{5}\right]\!\right]\nearrow+\left[\!\left[\frac{5}{5}\right]\!\right]$$

$$=\ 25\ +\ 5\ +\ 1\ =31$$

B) find the sum $n_1+n_2+n_3$= 31, Therefore there are 31 trailing zeros in $127_{10}!$

29. Trailing Zeros of a factorial in a different base, *n*.

If each trailing zero of a base 10 number come from the factors of 10 which the factorial would have when you wrote it out (both the numbers which have a 10 in them and the composition of other numbers which have 5's and 2's), then the trailing zero of a base *n* number come from the factors of *n* that the factorial would have when you wrote it out (both the numbers which have an *n* in them and the composition of other numbers which have factors which multiply to equal *n)*, in the same fashion.

Ex. Find the number of trailing zeros of the number125! if written in base 6.

A) Because there are more 2's than 3's in the factorization of125! then we are going to be concerned with 3's

B) Using the shortcut:

$$\left\llbracket \frac{125}{3} \right\rrbracket + \left\llbracket \frac{125}{9} \right\rrbracket + \left\llbracket \frac{125}{27} \right\rrbracket + \left\llbracket \frac{125}{81} \right\rrbracket =$$

$$\left\llbracket \frac{125}{3} \right\rrbracket \nearrow + \left\llbracket \frac{41}{3} \right\rrbracket \nearrow + \left\llbracket \frac{13}{3} \right\rrbracket \nearrow + \left\llbracket \frac{4}{3} \right\rrbracket =$$

$$= 41 + 13 + 4 + 1 = 59$$

C) Remember, I am finding the number of times that the factor "3" occurs

* Be careful with bases like 12 or 20 because they can often be tricky
(Make sure there are enough 2's to go with the 3's or 5's)

30. Geometric Mean of two numbers $= \sqrt{xy}$

Ex) Find the geometric mean of 3 and 12.

Solution) $\sqrt{3 \cdot 12} = \boxed{6}$

31. General Guide to working with Solutions & Mixtures:

$$\%_1 \left(Amt_1 \right) + \%_2 \left(Amt_2 \right) + \%_3 \left(Amt_3 \right) + \cdots = \%_F \left(Amt_T \right)$$

$$Amt_1 + Amt_2 + Amt_3 + \cdots = Amt_T$$

32. Triangular number:

...any of the numbers (as 1,3,6,10,15,21,. . .) that represent the number of dots in the figures formed starting with one dot and adding rows one after another to form triangles each row of which has one more dot than the one before. See the 3rd diagonal of Pascal's triangle.

33. $\cos\theta + i\sin\theta = e^{i\theta}$ (Euler's Theorem – pronounced "Oiler's")

Ex) Simplify: $e^{240° i}$

$$e^{240° i} = \cos 240° + i \sin 240°$$

$$= \boxed{-\frac{1}{2} - \frac{\sqrt{3}}{2} i}$$

34. A perfect number, *P*, is a number whose factors (other than itself) sum to equal itself.

$$P = \left(2^{n-1}\right)\left(2^n - 1\right)$$

where *n* is prime and $2^n - 1$ is prime. ***You don't need to know this formula.***

ex. The first four perfect numbers are... 6, 28, 496, 8128

I've only seen questions which require you to know the first 4 perfect numbers, so just memorize them.

35. A short cut to finding the fractional equivalent of a repeating decimal

(Lewis Lehe ('05) gets the credit from this one, although he said he stole it from some ASFA kid).

Looking at the repeating decimal, let X = the number without the decimal and without the bar over the repeating part and let Y = the number to the left of the repeating part. Find $X - Y$ and divide this value by a value that begins with a "9" for each number under the bar, and ends with a "0" for each number to the right of the decimal but not under the bar (don't forget to simplify the fraction).

Ex) $3.12\overline{43} \quad = \dfrac{31243-312}{9900} = \boxed{\dfrac{30931}{9900}}$

Ex) $1.01\overline{4} \quad = \dfrac{1014-101}{900} = \boxed{\dfrac{913}{900}}$

36. **Expected Value** – When you are asked what is the expected value, don't try to use your common sense. There is a definition involved for "expected value" – ***use it***. The best way to illustrate the definition is to show you two examples:

Ex) Q: Find the expected value if you roll a fair, 6-sided die.

A: $1\left(\dfrac{1}{6}\right)+2\left(\dfrac{1}{6}\right)+3\left(\dfrac{1}{6}\right)+4\left(\dfrac{1}{6}\right)+5\left(\dfrac{1}{6}\right)+6\left(\dfrac{1}{6}\right)=\dfrac{21}{6}=\dfrac{7}{2}=\boxed{3.5}$

As you can see, you can't roll a 3.5, yet it is the "expected value."

Ex) Q: Find the expected value if you roll a loaded, 6-sided die, where the side that has a one has a $\frac{1}{36}$ probability of occurring, the side that has a two has a $\frac{1}{9}$, the three has a $\frac{1}{12}$, the four has a $\frac{5}{18}$, the five has a $\frac{1}{3}$, and the side that has a six has a $\frac{1}{6}$ probability of occurring.

$$1\left(\frac{1}{36}\right)+2\left(\frac{1}{9}\right)+3\left(\frac{1}{12}\right)+4\left(\frac{5}{18}\right)+5\left(\frac{1}{3}\right)+6\left(\frac{1}{6}\right)=$$

A: $$1\left(\frac{1}{36}\right)+2\left(\frac{4}{36}\right)+3\left(\frac{3}{36}\right)+4\left(\frac{10}{36}\right)+5\left(\frac{12}{36}\right)+6\left(\frac{6}{36}\right)=$$

$$\frac{154}{36}=\boxed{\frac{77}{18}}=4.2\overline{7}$$

As you can see, even though a die can't yield a roll a $4.2\overline{7}$, it is the "*expected value*."

37. Surveyor's formula for the AREA of any *polygonal* region

Given a series of consecutive vertices. . . $(x_1, y_1),(x_2, y_2),(x_3, y_3),\ldots(x_n, y_n)$

$$Area = \frac{1}{2}\Big[\big[(x_1 \cdot y_2)+(x_2 \cdot y_3)+\cdots+(x_n \cdot y_1)\big]-\big[(y_1 \cdot x_2)+(y_2 \cdot x_3)+\cdots+(y_n \cdot x_1)\big]\Big]$$

Ex) Determine the Area enclosed in the region with the following points as vertices: $(-4,12),(3,8),(5,1),(4,-3),(-2,-6)$

Solution:

$$Area = \frac{1}{2}\Big|\big[(-4\cdot 8)+(3\cdot 1)+(5\cdot(-3))+(4\cdot(-6))+(-2\cdot 12)\big]-\big[(12\cdot 3)+(8\cdot 5)+(1\cdot 4)+(-3\cdot(-2))+(-6\cdot(-4))\big]\Big|$$

$$= \frac{1}{2}\big|(-32+3-15-24-24)-(36+40+4+6+24)\big| = \frac{1}{2}\big|(-92)-(110)\big| = \boxed{101}$$

38. The Proof of the following sums is pretty ugly and I don't think worth seeing in this summer reading packet, but I want you to know these misc sums:

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{6} \qquad \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}\ldots=\frac{\pi^2}{8} \qquad \frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}\ldots=\frac{\pi^2}{24}$$

They often will look like this in a problem...

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\ldots=\frac{\pi^2}{6} \quad \text{or} \quad 1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}+\ldots=\frac{\pi^2}{8} \qquad \text{etc...}$$

36. Sum of the roots, sum of the squares of the roots, sum of the cubes of the roots, etc. using **Newton's sums**.

***You will not need to know this unless we go over it later.**

Let $ax^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + \cdots = 0$ and

$s_1 =$ sum of the roots

$s_2 =$ sum of the squares of the roots

$s_3 =$ sum of the cubes of the roots

etc.

then.....

$$(a)s_1 + (b)1 = 0$$
$$(a)s_2 + (b)s_1 + (c)2 = 0$$
$$(a)s_3 + (b)s_2 + (c)s_1 + (d)3 = 0$$
$$(a)s_4 + (b)s_3 + (c)s_2 + (d)s_1 + (e)4 = 0$$

etc.

(i.e. if you want to find, for instance, the sum of the cubes of the roots of a polynomial, then first find s_1 by substituting the values for *a* and *b* into the 1st equation. Next, find s_2 by substituting the values for *a*, *b*, *c* and s_1 into the 2nd equation. Continue the process into the third equation, where you have found s_3.)

Algebra I and II rules and Information

37. Slope $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

38. Slope (of a line)

If $Ax + By = C,$ then slope, $m = -\dfrac{A}{B}$

If $y = mx + b,$ then slope $= m$

39. Point-slope form (of a line) $y - y_1 = m\left(x - x_1\right)$

40. Laws of Exponents (multiplication):

$$a^m \cdot a^n = a^{m+n} \qquad \left(ab\right)^m = a^m b^m \qquad \left(a^m\right)^n = a^{mn}$$

41. Laws of Exponents (division):

$$\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}} \qquad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

42. Laws of Exponents, continued . . .

$$a^0 = 1 \qquad a^{-n} = \frac{1}{a^n}$$

43. Special Products/Factors:

$$a^3 + b^3 = \left(a+b\right)\left(a^2 - ab + b^2\right) \qquad a^3 - b^3 = \left(a-b\right)\left(a^2 + ab + b^2\right)$$

44. Relatively prime - two integers are relatively prime if their greatest common factor is 1.
ex) even though 8 and 9 are not prime, they are considered relatively prime because their greatest common factor is 1.

45.

$$\left(\sqrt[n]{b}\right)^n = b$$

$\sqrt[n]{b^n} = b$ if n is odd

$\sqrt[n]{b^n} = |b|$ if n is even

Ex)

$$\sqrt[3]{16x^5} = \sqrt[3]{8 \cdot 2 \cdot x^3 \cdot x^2} = \boxed{2x\sqrt[3]{2x^2}}$$

$$\sqrt[4]{32x^5} = \sqrt[4]{16 \cdot 2 \cdot x^4 \cdot x} = |2x|\sqrt[4]{2x} = \boxed{2|x|\sqrt[4]{2x}}$$

46.

$\sqrt{-1} = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

Ex)

$$i^{27} = \cancel{\left(i^4\right)^6} \cdot \left(i^3\right) = \boxed{-i} \qquad i^{1230} = i^{30} = \cancel{\left(i^4\right)^7}\left(i^2\right) = \boxed{-1}$$

$$i^{35679810353} = i^{53} = \cancel{\left(i^4\right)^{13}}\left(i\right) = \boxed{i}$$

---

47. Complex Numbers $a+bi$

$a =$ real part

$b=$ imaginary part

---

48. Absolute value of a complex number

$|a+bi|=\sqrt{a^2+b^2}$ ex) $|-2+3i|=\sqrt{(-2)^2+(3)^2}=\sqrt{4+9}=\boxed{\sqrt{13}}$

note: not $\xcancel{\sqrt{a^2+(bi)^2}}$

---

49. If $z=a+bi$, then $\bar{z}=a-bi$

$a+bi$ and $a-bi$ are considered conjugates of each other, therefore $\bar{z}$ is the conjugate of z.

---

50. Quadratic Formula if $ax^2+bx+c=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

---

51. Discriminant b^2-4ac

If $b^2-4ac>0$ then there are two unequal **real** roots

If $b^2-4ac=0$ then there is a **real** double root

If $b^2-4ac<0$ then there are two conjugate **imaginary** roots

A **rational root** exists if *a, b and c* are real & the discriminant is equal to a positive perfect square, and there is nothing left under the radical.

Ex) Question: Determine what values of *k* will give the following equation imaginary roots:
$3x^2-4x-2k=0$

Ans:

$$\begin{aligned} b^2-4ac&<0\\ (-4)^2-4(3)(-2k)&<0\\ 24k&<-16\\ k&<-\frac{2}{3} \end{aligned}$$

---

52. Odd Function if $f(-x)=-f(x)$ for every x

Basically, a function is odd if you get opposite values for "y" when you plug in opposite values for "x."

If $f(x)=\boxed{\dfrac{3x^3-4x}{5x^4}}$,

then $f(-x)=\dfrac{3(-x)^3-4(-x)}{5(-x)^4}$

$$=\frac{-3x^3+4x}{5x^4}=\frac{-(3x^3-4x)}{5x^4}=\boxed{-\frac{3x^3-4x}{5x^4}}$$

53. Even Functions are defined as $f\left(-x\right)=f\left(x\right)$ for every x

Basically, a function is an even function if you get the same value for "y" whether you plug in a positive x or a negative x.

If $f\left(x\right)=\boxed{\dfrac{3x^3-4x}{5x^5}}$,

ex) then $f\left(-x\right)=\dfrac{3\left(-x\right)^3-4\left(-x\right)}{5\left(-x\right)^5}$

$$=\frac{-3x^3+4x}{-5x^5}=\frac{-\left(3x^3-4x\right)}{-\left(5x^5\right)}=\boxed{\frac{3x^3-4x}{5x^5}}$$

---

54. Even Functions are symmetric about the *y*-axis and Odd Functions symmetric about the origin. No ***functions*** are symmetric about the *x*-axis, because they wouldn't pass the vertical line test of functions if they did this.

55. Direct Variation $y=kx$ where k is the constant of variation/proportionality.
We usually set up the equation using the following:

$\dfrac{y_1}{x_1}=\dfrac{y_2}{x_2}$, if y is directly related to x.

56. Inverse Variation $xy=k$ where k is the constant of variation/proportionality.
We usually set up the equation using the following:

$x_1y_1=x_2y_2$, if x is inversely related to y.

57. Joint Variation $z=kxy$ where k is the constant of variation/proportionality.
We usually set up the equation using the following:

$\dfrac{z_1}{x_1y_1}=\dfrac{z_2}{x_2y_2}$, if z is jointly related to x and y.

This is the same as saying that z is directly related to both x and y.

58. Pythagorean Theorem $c^2=a^2+b^2$

59. Distance formula (plane) $D=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}$

60. Midpoint formula (plane) $\left(x_m,y_m\right)=\left(\dfrac{x_1+x_2}{2},\dfrac{y_1+y_2}{2}\right)$

61. If m and n are integres with $n>0$, and b is a positive real number, then

$$b^{\frac{m}{n}}=\sqrt[n]{b^m}=\left(\sqrt[n]{b}\right)^m$$

62. <u>Horizontal Line Test</u> -a function **has** an **inverse function** iff (if and only if) every horizontal line intersects the graph of the function in *at most* one point.

63. Inverse Functions - the functions *f* and *g* are inverse functions if $f\left(g\left(x\right)\right)=g\left(f\left(x\right)\right)=x$ for all x in the domain of *g* and *f* , respectively.
Basically, this means when I plug in a number into one function, and then plug the result into the other function, I will always end up with the original value.

---

64. Radian Measure is defined to be the ratio of arc length *s* to radius *r* :

$$\theta=\frac{s}{r}$$

s is the length of the arc

r is the radius

---

65. Arc length $s=r\theta$ θ must be in radians.

---

66. Area of a sector $A=\frac{1}{2}r^2\theta$ or $A=\frac{1}{2}rs$ θ must be in radians.

---

67. Permutations, ${}_nP_r$ $${}_nP_r=\frac{n!}{\left(n-r\right)!}$$

---

68. Combinations, ${}_nC_r$ $${}_nC_r=\frac{n!}{\left(n-r\right)!r!}$$

---

69. If a set of *n* elements has n_1 elements alike, n_2 alike, and so on, then the number of Permutations of the *n* elements taken *n* at a time is given by $P=\dfrac{n!}{n_1!n_2!n_3!\cdots}$

---

70. Transpose of a matrix – when you switch the $r\times c$ of each element in a matrix.

Ex) $\begin{bmatrix}2&-1&3\\4&5&-7\\0&1&6\end{bmatrix}^T=\begin{bmatrix}2&4&0\\-1&5&1\\3&-7&6\end{bmatrix}$ Ex) $\begin{bmatrix}2&3&-5\\-4&5&9\end{bmatrix}^T=\begin{bmatrix}2&-4\\3&5\\-5&9\end{bmatrix}$

---

71. Inverse of a 2x2 matrix…
If $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ and $\det A\neq 0$, then $A^{-1}=\dfrac{1}{\det A}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

Sequences and Series

72. Formula for the nth term of an Arithmetic Sequence

$t_n = t_1 + (n-1)d$ or $t_n = t_m + (n-m)d$

where t_n and t_m *are terms in the sequence, n and m are the places of these terms, and d is the common difference*

---

73. Formula for the nth term of a Geometric Sequence

$t_n = t_1 r^{n-1}$ or $t_n = t_m r^{n-m}$

where t_n and t_m *are terms in the sequence, n and m are the places of these terms, and r is the common ratio.*

---

74. Formula for the sum of an Arithmetic Sequence $S_n = \frac{n}{2}(t_1 + t_n)$

* there is **no sum** of an infinite arithmetic sequence

---

75. Formulas for the sum of a Finite Geometric Sequence $S_n = \frac{t_1(1-r^n)}{1-r}$ or $S_n = \frac{t_1 - t_n r}{1-r}$

---

76. Formula for the sum of an Infinite Geometric Sequence $S_\infty = \frac{t_1}{1-r}$ if $-1 < r < 1$

- there is no sum of an infinite geometric sequence whose ratio is not $-1 < r < 1$.

---

77. Factorial $n! = n(n-1)(n-2)(n-3)\ldots 3\cdot 2\cdot 1$

By definition, *n* should be a whole number. However, in some applications at the comprehensive level fractions are used, which would mean the progression would be infinite.

ex: $6! = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$

Tricky example: $\left(\frac{1}{3}\right)! = \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\cdots$

---

78. Arithmetic mean (of two numbers): $\frac{t_1 + t_2}{2}$

---

79. Arithmetic mean (of n numbers): $\frac{t_1 + t_2 + t_3 + \ldots t_n}{n}$

---

80. Geometric mean (of two numbers): $\sqrt{t_1 \cdot t_2}$

---

81. Geometric mean (of n numbers): $\sqrt[n]{t_1 \cdot t_2 \cdot t_3 \cdot \ldots \cdot t_n}$

82. Harmonic mean (of two numbers):

H, is the harmonic mean of x and y if $\frac{2}{H}=\frac{1}{x}+\frac{1}{y}$

The harmonic mean of two numbers x and y can be found by

$$H=\frac{2xy}{x+y} \quad \text{or} \quad H=\frac{2t_1t_3}{t_1+t_3}$$

ex.) the harmonic mean of 6 and 12 is . . .

$$\frac{2(6\cdot 12)}{6+12}=\frac{2\cdot 6\cdot 12}{18}=8$$

---

83. The harmonic mean, H, of more than two numbers, can be calculated in a similar way . . .

$$\frac{3}{H}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \quad \text{or} \quad H=\frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$$

$$\frac{4}{H}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{w} \quad \text{or} \quad H=\frac{4}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{w}}$$

$$\frac{n}{H}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ .\ .\ . \quad \text{or} \quad H=\frac{n}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ .\ .\ .}$$, where n is the number of terms

---

84. Harmonic sequences:
A harmonic sequence is one that can be written with a constant ***numerator*** and an arithmetic ***denominator***. You can always use the harmonic mean definition above, $t_2=\frac{2t_1t_3}{t_1+t_3}$, to find a term between two other terms in a harmonic sequence, but see below how nice it is when you consider our harmonic definition:

Ex) Find the next three terms in the harmonic sequence: 12, 8, 6, . . .

Solution: Rewrite the sequence as $\frac{24}{2},\frac{24}{3},\frac{24}{4},...$ and then continue the pattern $...,\frac{24}{5},\frac{24}{6},\frac{24}{7}$.

*** Using $t_2=\frac{2t_1t_3}{t_1+t_3}$ by plugging in 8 for t_1, 6 for t_2 and then solving for t_3 (again & again) would be cumbersome and tedious.

---

85. Binomial Theorem $\frac{n!}{(n-k)!k!}a^{n-k}b^k$ where k is one less than the term you are looking for.

Also, ${}_nC_k a^{n-k}b^k$

Ex.) Find the 4th term in the expansion of $(x-3y)^8$

$$\frac{8!}{5!3!}(x)^5(-3y)^3=\ .\ .\ .$$

---

86. $\phi(x)$ This is an expression used to signify "the number of positive terms, smaller than x that are relatively prime with x.

For example.

a) $\phi(13)=12$, because there are 12 positive terms smaller than 13 which are relatively prime with 13 (1,2,3,4,5,6,7,8,9,10,11,12)

b) $\phi(12)=4$, because there are 4 positive terms smaller than 12 which are relatively prime with 12 (1,5,7,11)

Trick for finding $\phi(x)$ without having to write out all the numbers and observe them.

Gabe's Rule, named after Gabe Quijano who brought me this nugget of goodness in the Fall of 2019.

For any number, x, which factors into $x_1^{\,n}\cdot x_2^{\,m}\cdot x_3^{\,p}\cdot\ldots$, $\phi(x)=x\cdot\left(\frac{x_1-1}{x_1}\right)\cdot\left(\frac{x_2-1}{x_2}\right)\cdot\left(\frac{x_3-1}{x_3}\right)\cdot\ldots$

For example…

a) $\phi(100)=?$ Because $100=2^2\cdot5^2$

$$\phi(100)=100\cdot\left(\frac{2-1}{2}\right)\cdot\left(\frac{5-1}{5}\right)=100\cdot\frac{1}{2}\cdot\frac{4}{5}=\boxed{40}$$

---

87. Trick for determining how to distribute "n" items into "b" containers.

$${}_{n+b-1}C_{b-1}$$

Ex) How many ways can Mr. Hurry arrange 20 identical pencils into 4 different pencil holders if each pencil holder must have at least 2 pencils when he's finished arranging them?

Solution: Technically, since each pencil holder will already have 2 pencils each, there will be only 12 pencils left to arrange in the 4 different pencil holders. Therefore, based on my formula above, $n=12$ and $b=4$, … ${}_{12+4-1}C_{4-1}={}_{15}C_3==455$

This also happens to be the 12th term in the tetrahedral numbers sequence (starting at 4)

If I had been arranging 12 numbers in 2 pencil holders it would have been the 12th term of the counting numbers (starting with 2).

Ex) 2,3,4,5,6,7,8,9,10,11,12, $\boxed{13}$ or ${}_{12+2-1}C_{2-1}={}_{13}C_1=13$

If I had been arranging 12 numbers in 3 pencil holders it would have been the 12th term of the triangular numbers (starting with 3).

Ex) 3,6,10,15,21,28,36,45,55,66,78, $\boxed{91}$ or ${}_{12+3-1}C_{3-1}={}_{14}C_2=91$

If I had been arranging 12 numbers in 4 pencil holders it would have been the 12th term of the etrahedral numbers (starting with 4).

Ex) 4,10,20,35,56,84,120,165,220,286,364, $\boxed{455}$ or ${}_{12+4-1}C_{4-1}={}_{15}C_3=455$

Also of note, the aforementioned number sequences can be found in the diagonals of Pascal's triangle!

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 26 28 8 1

Logarithmic Properties

Logarithm

If b and y are positive numbers ($b \neq 1$),

then $\log_b y = x$

iff $y = b^x$

Examples

$\log_6 y = 3$

$y = 6^3$

or

$\log_x 5 = 2$

$5 = x^2$

Laws of Logarithms

$$\log_b MN = \log_b M + \log_b N$$

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

$$\log_b M^k = k \log_b M$$

$$\log_{b^a} M^k = \frac{k}{a} \log_b M$$

Examples

$$\log_5 36 = \log_5 4 + \log_5 9 = \log_5 2 + \log_5 18 = etc$$

$$\log_5 4 = \log_5 8 - \log_5 2 = \log_5 12 - \log_5 3 = etc$$

$$\log_5 81 = \log_5 3^4 = 4 \log_5 3$$

$$\log_9 25 = \log_{3^2} 5^2 = \cancel{\frac{2}{2}} \log_3 5 = \log_3 5$$

Be careful, these are NOT TRUE

$$\log_b (M + N) \neq \log_b M \cdot \log_b N$$

$$\log_b (M - N) \neq \frac{\log_b M}{\log_b N}$$

$$(\log_b M)^k \neq k \log_b M$$

$$\log_b M + \log_b N \neq \log_b (M + N)$$

$$\log_b M - \log_b N \neq \frac{\log_b M}{\log_b N}$$

$$k \log_b M \neq (\log_b M)^k$$

Properties of Logarithms

$\log_b b = 1$

$\log_b 1 = 0$

If $\log_b a$, then $a > 0$, $b > 0$, and $b \neq 1$

$$\log_{b^x} b^y = \frac{y}{x} \log_b b = \frac{y}{x}$$

$$\log_{x^a} y^b = \frac{b}{a} \log_x y$$

$$b^{\log_b x} = x$$

$$\log_{x^a} y = \log_x y^{\frac{1}{a}}$$

$$a^{\log_b c} = c^{\log_b a}$$

$\log_{10} b = \log b$ is called the "common log"

$\log_e b = \ln b$ is called the "natural log"

Examples

$\log_6 6 = 1$

$\log_{12} 1 = 0$

$\log_5 (-2) \Rightarrow DNE$, $\log_1 12 \Rightarrow DNE$

$$\log_{81} 9 = \log_{3^4} 3^2 = \frac{2}{4} = \frac{1}{2}$$

$$\log_{81} 32 = \log_{3^4} 2^5 = \frac{5}{4} \log_3 2$$

$$7^{\log_7 12} = 12$$

$$\log_9 X = \log_{3^2} X = \log_3 X^{\frac{1}{2}} = \log_3 \sqrt{X}$$

$$15^{\log_3 10} = 10^{\log_3 15}$$

Change of base formula

$$\log_a x = \frac{\ln x}{\ln a} = \frac{\log x}{\log a} = \frac{\log_b x}{\log_b a} = \dots$$

Examples

$$\log_5 12 = \frac{\ln 12}{\ln 5} = \frac{\log 12}{\log 5} = \frac{\log_7 12}{\log_7 5} = \dots$$

Trigonometric Identities & Formulas

Fundamental Identities

Trigonometric Functions

$$\sin\theta = \frac{y}{r} = \frac{opp}{hyp}$$

$$\cos\theta = \frac{x}{r} = \frac{adj}{hyp}$$

$$\tan\theta = \frac{y}{x} = \frac{opp}{adj}$$

$$\cot\theta = \frac{x}{y} = \frac{adj}{opp}$$

$$\sec\theta = \frac{r}{x} = \frac{hyp}{adj}$$

$$\csc\theta = \frac{r}{y} = \frac{hyp}{opp}$$

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta}$$

$$\cos\theta = \frac{1}{\sec\theta}$$

$$\tan\theta = \frac{1}{\cot\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

Quotient identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Cofunction Identities

$$\sin\theta = \cos\left(90^\circ - \theta\right)$$

$$\cos\theta = \sin\left(90^\circ - \theta\right)$$

$$\tan\theta = \cot\left(90^\circ - \theta\right)$$

$$\cot\theta = \tan\left(90^\circ - \theta\right)$$

$$\sec\theta = \csc\left(90^\circ - \theta\right)$$

$$\csc\theta = \sec\left(90^\circ - \theta\right)$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

Even/Odd Identities

$$\sin\left(-u\right) = -\sin u$$

$$\cos\left(-u\right) = \cos u$$

$$\tan\left(-u\right) = -\tan u$$

$$\cot\left(-u\right) = -\cot u$$

$$\sec\left(-u\right) = \sec u$$

$$\csc\left(-u\right) = -\csc u$$

Sum & Difference Identities for sine, cosine, and tangent

$$\sin\left(\alpha + \beta\right) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos\left(\alpha + \beta\right) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\tan\left(\alpha + \beta\right) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\sin\left(\alpha - \beta\right) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos\left(\alpha - \beta\right) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan\left(\alpha - \beta\right) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Sine and Cosine of non-special angles (to memorize)

$$\sin 18^\circ = \sin\frac{\pi}{10} = \frac{\sqrt{5} - 1}{4}$$

$$\cos 18^\circ = \cos\frac{\pi}{10} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$$

Double-angle identity for sine, cosine, and tangent

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \qquad \cos 2\theta = 1 - 2\sin^2\theta \qquad \cos 2\theta = 2\cos^2\theta - 1$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Half-angle identity for sine, cosine, and tangent

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

This one is the best ↙

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \qquad \tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} \qquad \tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$$

Law of sines & cosines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Area formulas for Oblique Triangles

If SAS . . .

$$K = \frac{1}{2}ab\sin C \qquad K = \frac{1}{2}ac\sin B \qquad K = \frac{1}{2}bc\sin A$$

If ASA or AAS . . .

$$K = \frac{1}{2}a^2\frac{\sin B\,\sin C}{\sin A} \qquad K = \frac{1}{2}b^2\frac{\sin A\,\sin C}{\sin B} \qquad K = \frac{1}{2}c^2\frac{\sin A\,\sin B}{\sin C}$$

If SSS . . .

$$K = \sqrt{S(S-a)(S-b)(S-c)} \quad \text{where } S = \frac{1}{2}(a+b+c)$$

Conic Sections Notes

Circles

$$\boxed{(x-h)^2+(y-k)^2=r^2}$$

$$Center\left(h,k\right)$$
$$radius=r$$
$$Area=\pi r^2$$

Example . . .

$$3x^2+3y^2-30x+12y+39=0$$
$$x^2+y^2-10x+4y+13=0$$
$$x^2-10x\qquad+y^2+4y\qquad=-13$$
$$x^2-10x+25+y^2+4y+4=-13+25+4$$
$$\left(x^2-10x+25\right)+\left(y^2+4y+4\right)=16$$
$$(x-5)^2+(y+2)^2=16$$

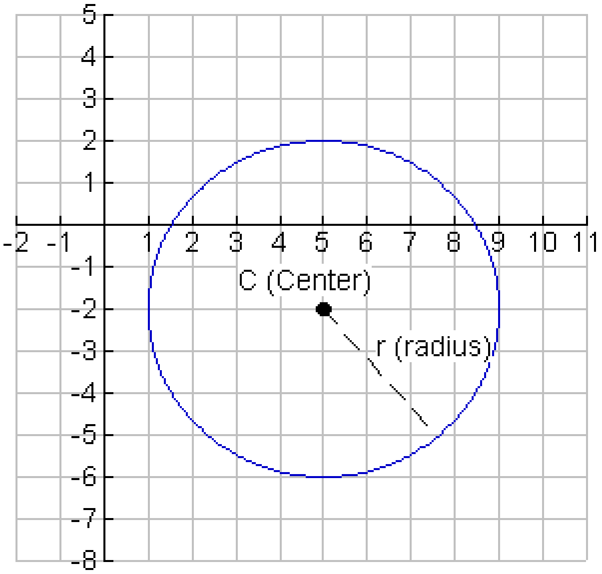
Therefore . . .

$$Center\left(5,-2\right)$$
$$radius=\sqrt{16}=4$$
$$Area=\pi\left(4\right)^2=16\pi$$

A degenerate circle will be in this form $\underline{(x-h)^2+(y-k)^2=0}$ and will be the "point" $\left(h,k\right)$.

How can you tell if an equation is a circle without completing the square?

- Both variables are squared, somewhere in the equation.
- Both x^2 *and* y^2 have exactly the same coefficient and the same sign.



Parabolas

$$\boxed{y-k=a(x-h)^2} \quad \text{or} \quad \boxed{x-h=a(y-k)^2}$$

$$Vertex(h,k) \qquad a=\frac{1}{4c} \ \text{ or } \ c=\frac{1}{4a} \qquad \text{length of lat. rect.}=\left|\frac{1}{a}\right|=|4c|$$

$$c=\text{dist. from Vertex to Focus or from Vertex to Directrix} \qquad eccentricity, e=1$$

example...

$$2x^2+16x-16y+80=0$$

$$2x^2+16x \qquad =16y-80$$

$$2(x^2+8x \qquad)=16y-80$$

$$2\left(x^2+8x+16\right)=16y-80+32$$

$$2\left(x+4\right)^2=16y-48$$

$$16y-48=2\left(x+4\right)^2$$

$$\left(\tfrac{1}{16}\right)\left(16y-48\right)=\left(\tfrac{1}{16}\right)2\left(x+4\right)^2$$

$$y-3=\tfrac{1}{8}(x+4)^2$$

Therefore . . .

$$Vertex(-4,3)$$

$$a=\frac{1}{8}$$

$$\text{length of L.R.}=\left|\frac{1}{a}\right|=\left|\frac{1}{\left(\frac{1}{8}\right)}\right|=8$$

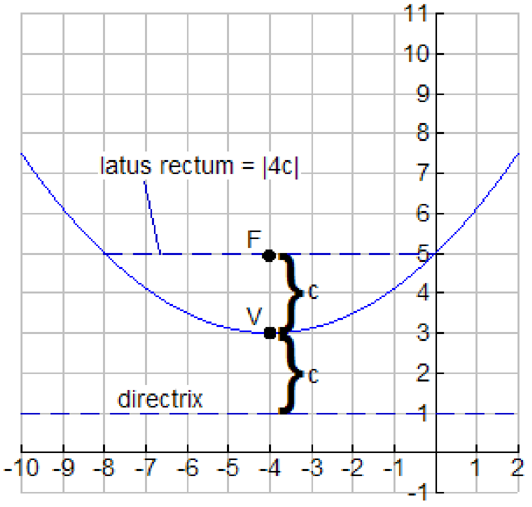
$$|c|=\tfrac{1}{4}(L.R.)=\tfrac{1}{4}\cdot 8=2$$

* This graph of this particular equation is directed upward, because the *x*-term is squared and the *a* is positive

A degenerate parabola will not exist in this form $\underline{y-k=a(x-h)^2}$, but would be a "line" in theory.

How can you tell if an equation is a parabola without completing the square?

- Only x or y is squared, but not both.
- If x is squared, then the graph goes up $(a>0)$ or down $(a<0)$.
- If y is squared, then the graph goes right $(a>0)$ or left $(a<0)$.



Ellipses

$$\boxed{\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1} \quad \text{or} \quad \boxed{\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1}$$

$a > b$ $\quad$ $Center\left(h,k\right)$ $\quad$ length of each latus rectum $=\frac{2b^2}{a}$

$a^2 = b^2 + c^2$ $\quad$ $c=$ distance from center to each of the foci $\quad$ $eccentricity, e=\frac{c}{a}$

Area of interior of ellipse = πab $\quad$ length of major axis $=2a$

Sum of focal radii = $2a$ $\quad$ length of minor axis $=2b$

Example . . .

$$16x^2+25y^2-96x+100y-156=0$$

$$16x^2-96x \qquad +25y^2+100y \qquad =156$$

$$16\left(x^2-6x \qquad\right)+25\left(y^2+4y \qquad\right)=156$$

$$16\left(x^2-6x+9\right)+25\left(y^2+4y+4\right)=156+144+100$$

$$16(x-3)^2+25(y+2)^2=400$$

$$\frac{16\left(x-3\right)^2}{400}+\frac{25\left(y+2\right)^2}{400}=\frac{400}{400}$$

$$\frac{\left(x-3\right)^2}{25}+\frac{\left(y+2\right)^2}{16}=1$$

Therefore . . .

$a=5 \quad b=4$

$c=\sqrt{a^2-b^2} \qquad \therefore c=\sqrt{5^2-4^2} \qquad \therefore c=3$

$Center\left(3,-2\right)$

length of LR$=\frac{2b^2}{a}=\frac{2\left(4^2\right)}{5}=\frac{32}{5}$

$eccentricity=\frac{c}{a}=\frac{3}{5}$ or 0.6

$Area=\pi ab=\pi\left(5\right)\left(4\right)=20\pi$

Sum of FR $=2a=10$

length of major axis $=2a=10$

length of minor axis $=2b=8$

DIRECTRIX:

If direction is left/right, then

directrix is . . . $x=h\pm\frac{a}{e}$

If direction is up/down, then

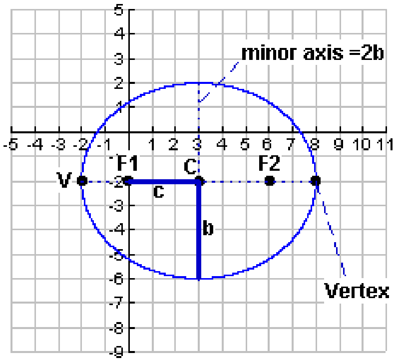
directrix is . . .$y=k\pm\frac{a}{e}$

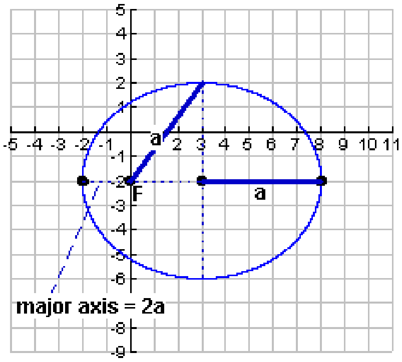
* This graph is stretched horizontally, because the “*a*” value is located beneath the *x*-term

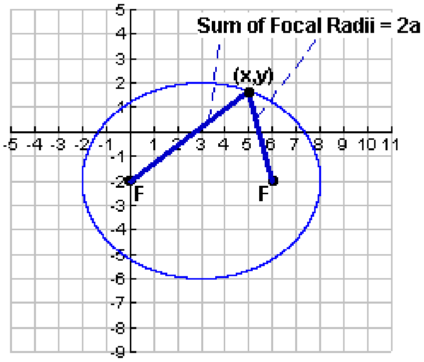
A degenerate ellipse will be in this form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=0$ and will be the “point” $\left(h,k\right)$.

How do you know if an equation is an ellipse without completing the square?

- Both of the variables are squared, somewhere in the equation.
- Coefficients of x^2 *and* y^2 are Different.
- Signs of the coefficients of x^2 *and* y^2 are the same.
- The square root of the number under the “*x*” term will indicate the horizontal stretch.
  The square root of the number under the “*y*” term will indicate the vertical stretch.







Hyperbolas

$$\boxed{\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1} \quad \text{or} \quad \boxed{\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1}$$

Center (h,k) Difference of focal radii = $2a$ length of each latus rectum = $\frac{2b^2}{a}$

$c^2=a^2+b^2$ c = Distance from center to each focus $eccentricity, e=\frac{c}{a}$

Equations of the Asymptotes:

Left-Right

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=0$$

Up-Down

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=0$$

Example . . .

$$16x^2-25y^2-96x-100y-356=0$$

$$16x^2-96x \qquad -25y^2-100y \qquad =356$$

$$16\left(x^2-6x \qquad\right)-25\left(y^2+4y \qquad\right)=356$$

$$16\left(x^2-6x+9\right)-25\left(y^2+4y+4\right)=356+144-100$$

$$16(x-3)^2-25(y+2)^2=400$$

$$\frac{16(x-3)^2}{400}-\frac{25(y+2)^2}{400}=\frac{400}{400}$$

$$\frac{(x-3)^2}{25}-\frac{(y+2)^2}{16}=1$$

Therefore . . .

Center $(3,-2)$

Difference of focal radii =10

length of latus rectum = $\frac{32}{5}$

$c=\sqrt{a^2+b^2}=\sqrt{41}\approx 6.4$

DIRECTRIX:

If direction is left/right, then

directrix is . . . $x=h\pm\frac{a}{e}$

If direction is up/down, then

directrix is . . .$y=k\pm\frac{a}{e}$

equations of asymptotes:

$$\frac{(x-3)^2}{25}-\frac{(y+2)^2}{16}=0$$

$$\sqrt{\frac{(y+2)^2}{16}}=\sqrt{\frac{(x-3)^2}{25}}$$

$$\frac{y+2}{4}=\pm\frac{x-3}{5}$$

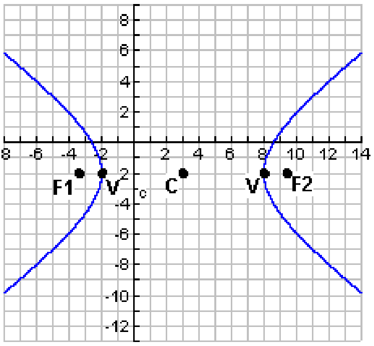
$$y+2=\pm\frac{4}{5}(x-3)$$

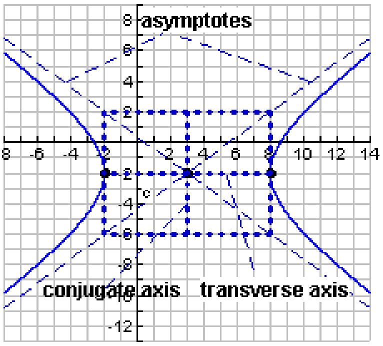
* The graph of this particular equation is directed left/right, because the positive/first term contains the *x*.

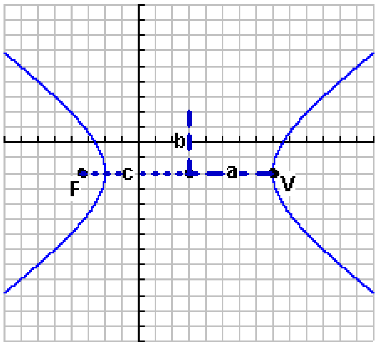
A degenerate hyperbola will be in this form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=0$ and will look like the actual "asymptotes."

How do you know if an equation is a hyperbola without completing the square?

- Both of variables are squared, somewhere in the equation.
- Coefficients of x^2 *and* y^2 may be the same or may be different.
- Signs of coefficients of x^2 *and* y^2 are different
- If the *x* term is positive/first, then the graph is "stretched" horizontally, if the *y* term is positive/first, then the graph is "stretched" vertically.







Name____________________________________

Practice Quiz #1 Summer Reading Packet

1. Using the formulas from the packet, find the sum of the first 20 squares.

2. Using the formulas from the packet, find the sum of $1^4 + 2^4 + 3^4 + \ldots + 15^4$.

3. What kind of conic is represented by this equation?

$$3x^2 - 4xy + 5y^2 + 6x - 7y + 8 = 0$$

Simplify the following trig identities using the identities from your packet.

4. $\cos^2\theta +$ __________$=1$

5. $\cos\alpha\cos\beta - \sin\alpha\sin\beta =$

6. $1 \quad + \quad \boxed{} = \csc^2\theta$

7. $\sin\alpha\cos\beta + \cos\alpha\sin\beta =$

8. $\cos^2\theta - \sin^2\theta =$

9. $\dfrac{1-\cos\theta}{\sin\theta} =$

10. $\cos\dfrac{\theta}{2} =$

11. $\sin\dfrac{\theta}{2} =$

12. $2\cos^2\theta - 1 =$

13. $\dfrac{2\tan\theta}{1-\tan^2\theta} =$

14. $\sin 2\theta =$

15. $\tan(\alpha - \beta) =$

16. $\dfrac{\sin\theta}{1-\cos\theta} =$

17. What is the equation of the Law of Sines?

18. What is the equation of the Law of Cosines?

19. Factor:

$8x^3 - 27y^3$

20. Simplify:

$|-4+5i|$

21. Solve using the Quadratic Formula.

$3x^2 - 5x + 7 = 0$

22. Determine what values of *k* will give the following equation real roots:

$2x^2 - 5x - 4k = 0$

---

23. The equation, $f(-x) = f(x)$, is true for ______________ functions.

24. The equation, $f(-x) = -f(x)$, is true for ______________ functions.

25. The equation, $f(g(x)) = g(f(x)) = x$, is true for _______________________ functions.

26. What is a formula used to find the nth term of an Arithmetic Sequence?

27. What is a formula used to find the nth term of an Geometric Sequence?

28. What is a formula for the sum of an Arithmetic Sequence?

29. What are the two formulas used to find the sum of a finite Geometric Sequence?

30. What is the formula used to find the sum of an <u>infinite</u> Geometric Sequence?

31. Find the 8th term in the expansion of $\left(2x^3-3y^2\right)^{10}$

32. Find the area of a sector if the radius of the sector is 10 units, and the angle is $120°$

Simplify.

33. ${}_{11}P_3 =$

34. ${}_{11}C_3 =$

35. the harmonic mean of 10 and 30 is . . .

36. What is the length of the latus rectum of the following parabola:

$y=-4x^2+2x+3$?

37. Given $\dfrac{(x-4)^2}{25}+\dfrac{(y+5)^2}{36}=1$, what is the area of the ellipse?

38. Solve.

$\log_5 y=4$

$5^{x-2}=3$

39. Simplify.

$\log_5 3 + \log_5 7 =$

$\log_5 36 - \log_5 4 =$

40. Simplify.

$\log_7 49 =$

$\log_{81} 9 =$

41. Simplify.

$\log_{10} 1 =$

$6^{\log_6 12} =$

42. Simplify.

$\log_5 \left(-2\right) =$

$\log_7 49^x =$

43-46. Given the following polynomial, $f\left(x\right) = 5x^5 - 3x^4 + 2x^3 + 4x^2 - 7x + 12$, find the . . .

Sum of the roots:

Product of the roots:

Sum of the reciprocals:

Sum of the squares of the roots:

47-50. Given the following polynomial, $f\left(x\right) = 2x^8 - 5x^7 + 3x^3 + 7x^2 - 6x + 13$, find the . . .

Sum of the roots:

Product of the roots:

Sum of the reciprocals:

Sum of the squares of the roots:

51. Sum of the measures of interior angles of a 12 sided polygon.

52. Number of diagonals of a convex 23 sided polygon.

53. $\lim\limits_{n \to \infty} \left(1 - \frac{3}{n}\right)^n =$

54-57. From the number 3780, find the …

…number of positive integral factors. … number of integral factors.

…sum of the positive integral factors. …sum of the integral factors.

58. Find the sum of the coefficients/numbers in the 12 row of Pascal's Triangle (this is technically the 13th row, but since it begins and ends with " 1 12 …12 1" we call it the "12 row").

59. Find the unit's digit of 2^{2363}?

60. Find the sum of the coefficients (and constants) in the expansion of

$$\left(2x^3+4y^5-3z^2+m^3\right)^6$$

61. Simplify

$e^{180°i} =$

62. What are the first 4 perfect numbers?

63. Determine the number of trailing zeros in 2013!

64. Write the repeating decimal as a simplified fraction:

$5.12\overline{36}$

65. Simplify.

a) $7!! =$

b) $10!!! =$

66. Simplify.

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\ldots=$$

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