Precalc takeaways Name:

Algebra 2 Unit 3: Number Systems and Operations

Learning targets

3.1 I can add, subtract, and multiply polynomials. (Algebra)
3.2 I can expand powers of binomials quickly using Pascal's triangle. (Algebra)
3.3 I can divide polynomials. (Algebra)
3.4 I can factor any quadratic polynomial. (Algebra)
3.5 I can add, subtract, multiply, and divide complex numbers. (Algebra)
3.6 I can explain which number systems are closed under which operations. (Reasoning)

Definition of polynomial

$\#x^n + \#x^{n-1} + \ldots + \#x^2 + \#x + \#$ ← standard form

the #s are coefficients

n is the degree of the polynomial. n can be $0, 1, 2, \ldots$

Adding and subtracting polynomials

add or subtract like terms

for example, add the x^2s together, add the x^5s together

when subtracting, remember to subtract every term

Multiplying polynomials

multiply two polynomials at a time

multiply each term in the first polynomial with each term in the second polynomial using the box method

add all the terms, combining like terms

example: Expand $(x+5)(x^2-x-3)$

	x^2	$-x$	-3
x	x^3	$-x^2$	$-3x$
5	$5x^2$	$-5x$	-15

$\rightarrow x^3 - x^2 - 3x + 5x^2 - 5x - 15$

$= x^3 + 4x^2 - 8x - 15$

Powers of binomials

$(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = (a+b)(a+b) = a^2+2ab+b^2$

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$

↑

coefficients in powers of binomials come from Pascal's triangle ↗

Pascal's triangle

not invented by Pascal

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

example: Expand $(x-3y)^4$

This is $(a+b)^4$ if $a=x$ and $b=-3y$

Raised to the 4th power, so there are 5 terms:

$$\underline{\underbrace{1}_{\substack{\text{coefficient,}\\\text{from Pascal's}\\\text{triangle}}}\cdot \underbrace{x^4}_{a^4}} + \underline{\underbrace{4}_{\text{coefficient}}\cdot \underbrace{x^3}_{a^3}\cdot \underbrace{-3y}_{b}} + \underline{\underbrace{6}_{\text{coeff}}\cdot \underbrace{x^2}_{a^2}\cdot \underbrace{(-3y)^2}_{b^2}} + \underline{4\cdot x\cdot(-3y)^3} + \underline{1\cdot(-3y)^4}$$

$(-3y)^2 = 9y^2$ $\qquad$ $(-3y)^3 = -27y^3$

simplify:

$$\underline{1\cdot x^4} + \underline{4\cdot x^3\cdot -3y} + \underline{6\cdot x^2\cdot 9y^2} + \underline{4\cdot x\cdot(-3y)^3} + \underline{1\cdot(-3y)^4}$$

$$= x^4 - 12x^3y + 54x^2y^2 - 108xy^3 + 81y^4$$

Polynomial division

example: $(6x^3-11x^2-4x+5) \div (2x-1)$

dividend — divisor

	② $3x^2$	⑤ $-4x$	⑧ -4	Rem
$2x$	① $6x^3$	④ $-8x^2$	⑦ $-8x$	⑩ 1
-1	③ $-3x^2$	⑥ $4x$	⑨ 4	

quotient: $3x^2-4x-4$ remainder: 1

⓪ Set up a multiplication box. Write the divisor on the left.

① Write the term of the dividend with the highest power.

② Un-multiply. $2x \cdot ?? = 6x^3$

③ Multiply.

④ Cells ③+④ add to the x^2 term of the dividend.

$-3x^2 + ?? = -11x^2$

⑤ Un-multiply.

...

⑩ If your box doesn't add to your dividend, then you have a remainder.

multiplication/division equations with remainders

multiplication:

$$\underbrace{6x^3-11x^2-4x+5}_{\text{dividend}} = \underbrace{(2x-1)}_{\text{divisor}}\underbrace{(3x^2-4x-4)}_{\text{quotient}} + \underbrace{1}_{\text{remainder}}$$

division:

$$\frac{6x^3-11x^2-4x+5}{2x-1} = 3x^2-4x-4 + \frac{1}{2x-1}$$

dividend, quotient, remainder, divisor

Remainder Theorem

If you divide a polynomial $f(x)$ by $(x-n)$, then the remainder is $f(n)$.

Also: If $f(n)=0$, then $f(x)$ is divisible by $(x-n)$.

Division example: $(2x^4-10x^3+40+29x^2) \div (x^2-2x+5)$

	$2x^2$	$-6x$	7	Rem
x^2	$2x^4$	$-6x^3$	$7x^2$	$44x$
$-2x$	$-4x^3$	$12x^2$	$-14x$	5
5	$10x^2$	$-30x$	35	

quotient: $2x^2-6x+7$

remainder: $44x+5$

multiplication equation: $2x^4-10x^3+40+29x^2 = (x^2-2x+5)(2x^2-6x+7)+44x+5$

division equation: $\dfrac{2x^4-10x^3+40+29x^2}{x^2-2x+5} = 2x^2-6x+7+\dfrac{44x+5}{x^2-2x+5}$

Roots of quadratic functions a.k.a. zeros a.k.a. x-intercepts

x-values that make the quadratic function = 0

if the function factors to $y = a(x-\#)(x-\#)$, ← factored form

then the #s are the roots

example: if the x-intercepts are 2 and −3 and $a=-2$, write the quadratic in factored form.

answer: $-2(x-2)(x+3)$

Quadratic formula → use this to find roots of quadratic functions!

if $ax^2+bx+c=0$, then $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

example: write $y=x^2-6x+13$ in factored form

use quadratic formula to find roots:

$a=1,\ b=-6,\ c=13$

$$x = \frac{+6 \pm \sqrt{(-6)^2-4\cdot 1\cdot 13}}{2\cdot 1}$$

$$= \frac{6 \pm \sqrt{36-52}}{2}$$

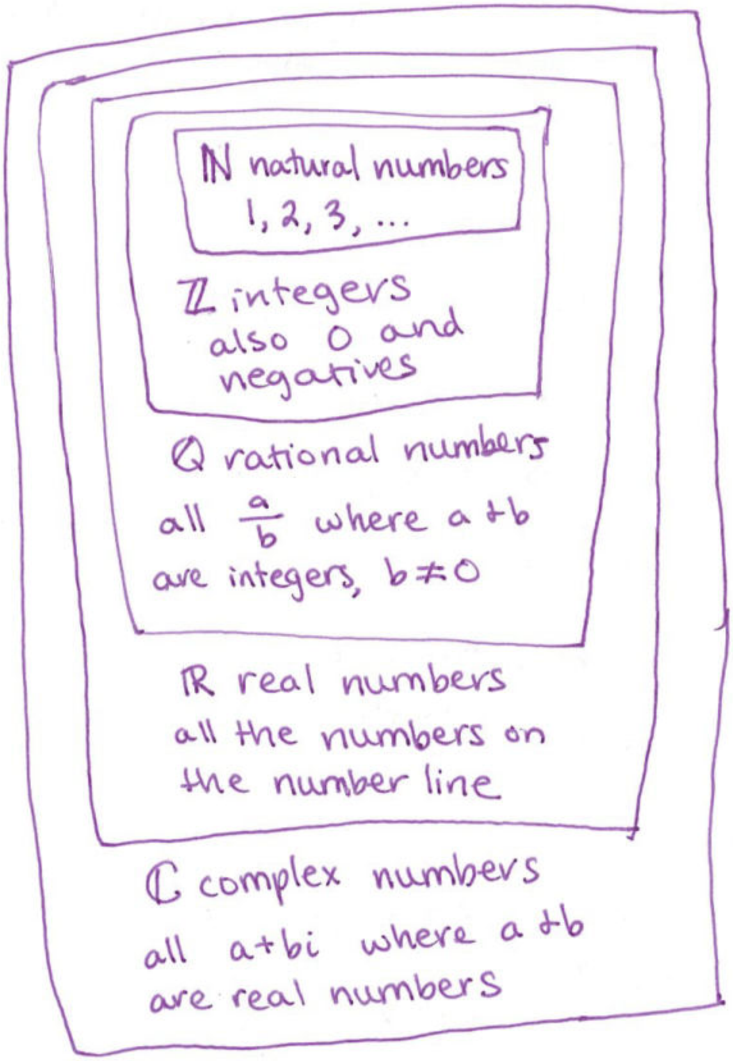
$$= \frac{6 \pm \sqrt{-16}}{2}$$

$$= \frac{6 \pm 4i}{2} = 3 \pm 2i$$

$$\boxed{y = (x-(3+2i))(x-(3-2i))}$$

$a=1$ because $y = 1x^2-6x+13$

Number systems



we can also think of the polynomials as a number system.

we can do operations with them

+ − • ÷

Adding and subtracting complex numbers

combine like terms

(add / subtract the real parts, add/subtract the imaginary parts)

Multiplying complex numbers

use the box,

remember that $i^2 = -1$

example: $(3+2i)(5-7i)$

$$\begin{array}{c|c|c|} & 3 & 2i \\ \hline 5 & 15 & 10i \\ \hline -7i & -21i & -14i^2 \\ \hline \end{array} \leftarrow -14i^2 = -14 \cdot -1 = 14$$

$15 + 10i - 21i + 14$

$= 29 - 11i$

Dividing complex numbers

the complex conjugate of $a+bi$ is $a-bi$

when you multiply a complex number by its conjugate, you get a real number!

when dividing two complex numbers, you want no i in the denominator.

fix the problem by multiplying by a clever form of 1:

$\frac{\text{conjugate of denominator}}{\text{conjugate of denomintor}}$ and then simplify

example: $(29-11i) \div (5-7i)$

$$\frac{29-11i}{5-7i} \cdot \frac{5+7i}{5+7i} = \frac{(29-11i)(5+7i)}{(5-7i)(5+7i)}$$

multiply by a clever form of 1 that uses the conjugate of the denominator

$$= \frac{222+148i}{74}$$

$$= \frac{222}{74} + \frac{148}{74}i$$

$$= \boxed{3+2i}$$

box for numerator:

$$\begin{array}{c|c|c|} & 29 & -11i \\ \hline 5 & 145 & -55i \\ \hline 7i & 203i & -77i^2 \\ \hline \end{array} \rightarrow 77$$

$145 + 77 + 203i - 55i$

$222 + 148i$

box for denominator:

$$\begin{array}{c|c|c|} & 5 & -7i \\ \hline 5 & 25 & -35i \\ \hline 7i & 35i & 49 \\ \hline \end{array}$$

$25 + 49 = 74$

Closure of a number system under an operation

a number system is closed under a particular operation if, when you combine numbers from the number system using that operation, you always get a number in that number system.

example: the positive numbers are closed under multiplication because positive • positive = positive

example: the odd numbers are not closed under addition because $3+5=8$ (an example of odd+odd = even)

to prove that a number system is closed under an operation, show that it always works.

to disprove that a number system is closed under an operation, you only need one counterexample.

example: the set of odd numbers is closed under multiplication.

True! Proof: Every odd # is $2n+1$, where n is an integer.

Take two integers, n and m.

$(2n+1)(2m+1)$

odd • odd

$$\begin{array}{c|c|c|} & 2m & 1 \\ \hline 2n & 4nm & 2n \\ \hline 1 & 2m & 1 \\ \hline \end{array} = 4nm+2n+2m+1$$

$$= 2\underbrace{(2nm+n+m)}_{\text{this is an integer}} + 1$$

So the product is $2 \cdot$ integer $+ 1$, which is odd! $\square$

example: the set of polynomials of degree 3 is closed under addition.

False!

Counterexample:

$p(x) = 2x^3 + 4x^2 - 1$

$q(x) = -2x^3 + 3x + 4$

$p(x) + q(x) = 4x^2 + 3x + 3$

$p(x) + q(x)$ is not a polynomial of degree 3 (because it is degree 2)