Notes: Polynomial division (mostly synthetic division)

Writing remainders

Polynomial division is division of two polynomials. There may or may not be a remainder. What does the remainder mean?

Consider division of numbers, such as $25 \div 7$. 25 is the **dividend** and 7 is the **divisor**. If you did this division, you would get a **quotient** of 3 and a **remainder** of 4. You could write any of these things:

$25 \div 7 = 3 + \frac{4}{7}$

$\frac{25}{7} = 3 + \frac{4}{7}$

$25 = 7 \cdot 3 + 4$

It works in a similar way for polynomials. Let's say you did $(2x^2 + 4x - 3) \div (x - 5)$ and you got as an answer $2x + 14$ with a remainder of 67. You could write any of these:

$$(2x^2 + 4x - 3) \div (x - 5) = 2x + 14 + \frac{67}{x-5}$$

$$\frac{2x^2 + 4x - 3}{x - 5} = 2x + 14 + \frac{67}{x-5}$$

$$2x^2 + 4x - 3 = (x - 5)(2x + 14) + 67$$

Synthetic division with a monic linear divisor

A monic polynomial has a leading coefficient of 1. A linear polynomial has x terms, but no x^2 terms or terms with a higher exponent on x. So monic linear polynomials are just $x + c$, where c is a number.

Synthetic division is easiest with a monic linear divisor. Let's do an example, $(x^3 + 8) \div (x + 2)$.

First, we need to set our divisor equal to 0 and solve for x:

$x + 2 = 0$

$x = -2$

Then, we need to find the coefficients of our dividend. We need to think of ALL the coefficients, including 0s, and we need them to be in order. So we can rewrite our dividend as $x^3 + 0x^2 + 0x + 8$. The coefficients are 1 0 0 8.

Now the setup. This kind of synthetic division problem involves three rows of writing. The -2 from setting our divisor equal to 0 is in a special corner. The coefficients go on the top row. We leave a row blank. Then we draw a line between the second and third rows, and a little wall below the last coefficient. It looks like this (turn the page):

Setup:

-2	1	0	0	8

Step 1: Drop the first coefficient down to the third row.

-2	1	0	0	8
	1			

Step 2: Multiply the new number in the third row by the corner. $1 \times -2 = -2$. Write it under the next coefficient.

-2	1	0	0	8
		-2		
	1			

Step 3: Add the two numbers in this column together. $0 + -2 = -2$. Write it in the bottom row.

-2	1	0	0	8
		-2		
	1	-2		

Step 4: Multiply the new bottom row number by the corner. $-2 \times -2 = 4$. Write it under the next coefficient.

-2	1	0	0	8
		-2	4	
	1	-2		

Step 5: Add this column. $0 + 4 = 4$. Write it in the bottom row.

-2	1	0	0	8
		-2	4	
	1	-2	4	

Step 6: Multiply the new number by the corner. $4 \times -2 = -8$. Write it under the next coefficient.

-2	1	0	0	8
		-2	4	-8
	1	-2	4	

Step 7: Add this column. Write it in the bottom row.

-2	1	0	0	8
		-2	4	-8
	1	-2	4	0

Interpret your answer: The walled-off number in the last column is the remainder. This has a remainder of zero, which means that x^3+8 is divisible by $x+2$. The other bottom-row numbers are coefficients of the answer. The 4 is the constant, the -2 is the coefficient of the x term, and the 1 is the coefficient of the x^2 term. Just start from the right of the wall, remember that the last one is the constant term, and add 1 to the exponent on x every time you step to the left.
Answer: x^2-2x+4.

This means that $x^3+8=(x+2)(x^2-2x+4)$. Factored!

Synthetic division with a more complicated divisor (non-monic)

Synthetic division (also called **synthetic substitution** or **division through Ruffini's rule**) is easiest with a monic linear divisor such as $x-5$. However, you can also do it with a non-monic linear divisor such as $2x+3$.

Imagine if we were to set the divisor equal to 0 and solve for x.

$2x+3=0$

$2x = -3$ (after subtracting 3 from both sides)

$x = \frac{-3}{2}$ (after dividing both sides by 2)

We will re-enact both of these steps while doing the synthetic division.

We will now divide $(2x^2+4x-3)\div(2x+3)$. There is one more step than with monic divisors: whenever we put something in the bottom row, we also need to divide it by 2. So we will add another row in which to do that. Note that we have the -3 in the corner and the /2 in the new bottom row.

New setup:

-3	2	4	-3
/2			

Steps 1 & 2: First, drop the leading coefficient of the dividend like normal. Then, divide it by 2.

-3	2	4	-3
	2		
/2	1		

Step 3: Multiply the 1 by the -3 in the corner and write it in the next column.

-3	2	4	-3
		-3	
	2		
/2	1		

Steps 4 & 5: Add the column: $4+-3=1$, then divide by 2 again.

-3	2	4	-3
		-3	
	2	1	
/2	1	$\frac{1}{2}$	

Steps 6 & 7: Multiply the $\frac{1}{2}$ by -3 and, write it in the last column, and then add. You don't need to divide the remainder by 2.

-3	2	4	-3
		-3	$\frac{-3}{2}$
	2	1	$\frac{-9}{2}$
/2	1	$\frac{1}{2}$	

Result: The quotient is from the very bottom row: $x+\frac{1}{2}$, and the remainder is $\frac{-9}{2}$.

Synthetic division with an even more complicated divisor (non-linear)

Another question is how to do synthetic division when the divisor is not linear, so degree 2 or higher. This is more complicated and requires yet more rows.

Let's divide $(x^3 - 12x^2 - 42) \div (x^2 + x - 3)$.

First, note that the leading coefficient of the divisor is 1, which means the divisor is monic. That means we don't have to do any kind of special dividing row like we did with the non-monic divisor on the previous page. Take the other coefficients, 1 and -3, and change the signs to get -1 and 3. (This is a lot like what you did with the simple, non-monic linear divisors.) Write them staggered in different rows like the setup below. Also, since we have a degree 2 divisor, we could have a degree 1 (linear) remainder. So we need to wall off two lines for our remainder instead of 1.

Setup:

		1	-12	0	-42
	3				
-1					

Step 1: Drop the first number down like usual.

		1	-12	0	-42
	3				
-1					
		1			

Step 2: Multiply the first number by the numbers on the left and write them in the next two columns, staggered.

		1	-12	0	-42
	3			3	
-1			-1		
		1			

Step 3: Add the next column, so $-12 + -1 = -13$.

		1	-12	0	-42
	3			3	
-1			-1		
		1	-13		

Turn the page...

Step 4: Multiply the -13 by both the numbers on the left, and write them staggered in the next two columns again.

		1	-12	0	-42
	3			3	-39
-1			-1	13	
		1	-13		

Step 5: Add the next column, so $0 + 3 + 13 = 16$.

		1	-12	0	-42
	3			3	-39
-1			-1	13	
		1	-13	16	

Step 6: If you multiply the new 16 by the numbers on the left and try to write them in the next two columns, then you run out of columns to write in. Don't do that. Instead, don't do the multiplication at all and just add up the last column to get the remainder.

		1	-12	0	-42
	3			3	-39
-1			-1	13	
		1	-13	16	-81

Result: The quotient is $x - 13$ with a remainder of $16x - 81$.