9.6

Graphs of Polar Equations

What You Should Learn

- Graph polar equations by point plotting
- Use symmetry and zeros as sketching aids
- Recognize special polar graphs

Introduction

Example 1 – *Graphing a Polar Equation by Point Plotting*

Sketch the graph of the polar equation $r = 4 \sin \theta$ by hand.

Solution:

The sine function is periodic, so you can get a full range of r-values by considering values of θ in the interval $0 \leq \theta \leq 2\pi$, as shown in the table.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
r	0	2	$2\sqrt{3}$	4	$2\sqrt{3}$	2	0	-2	-4	-2	0

Example 1 – *Solution*

cont'd

By plotting these points, as shown in Figure 9.70, it appears that the graph is a circle of radius 2 whose center is the point $(x, y) = (0, 2)$.

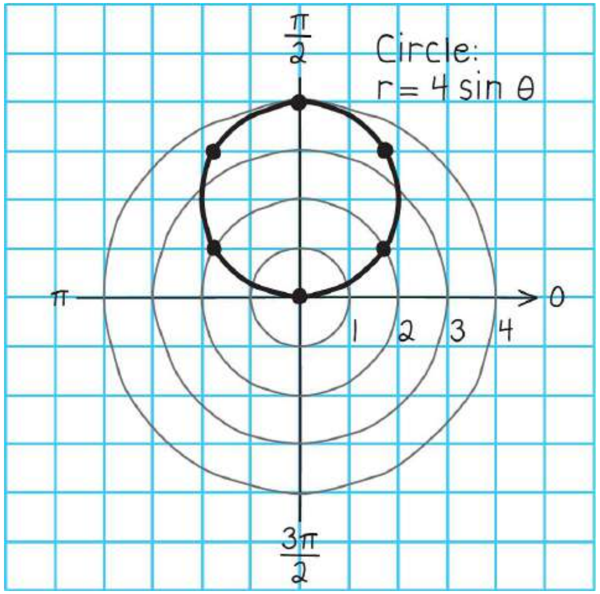


Figure 9.70

Symmetry and Zeros

Symmetry and Zeros

In Figure 9.70, note that θ as increases from 0 to 2π the graph is traced out twice. Moreover, note that the graph is *symmetric with respect to the line* $\theta = \pi/2$. Had you known about this symmetry and retracing ahead of time, you could have used fewer points.

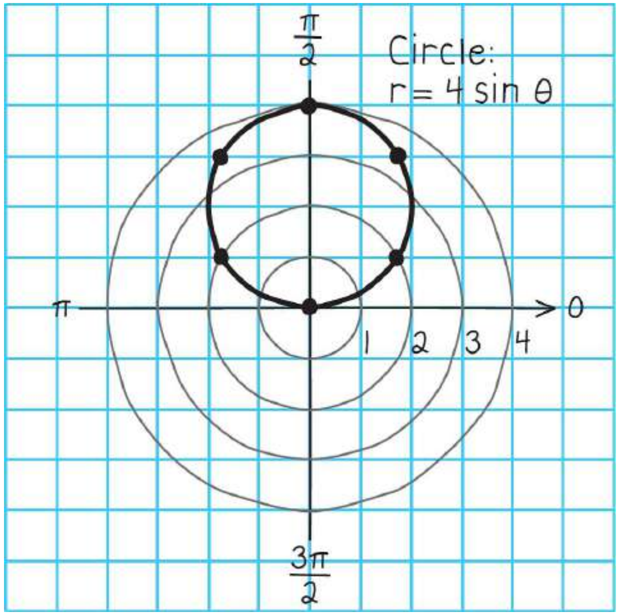
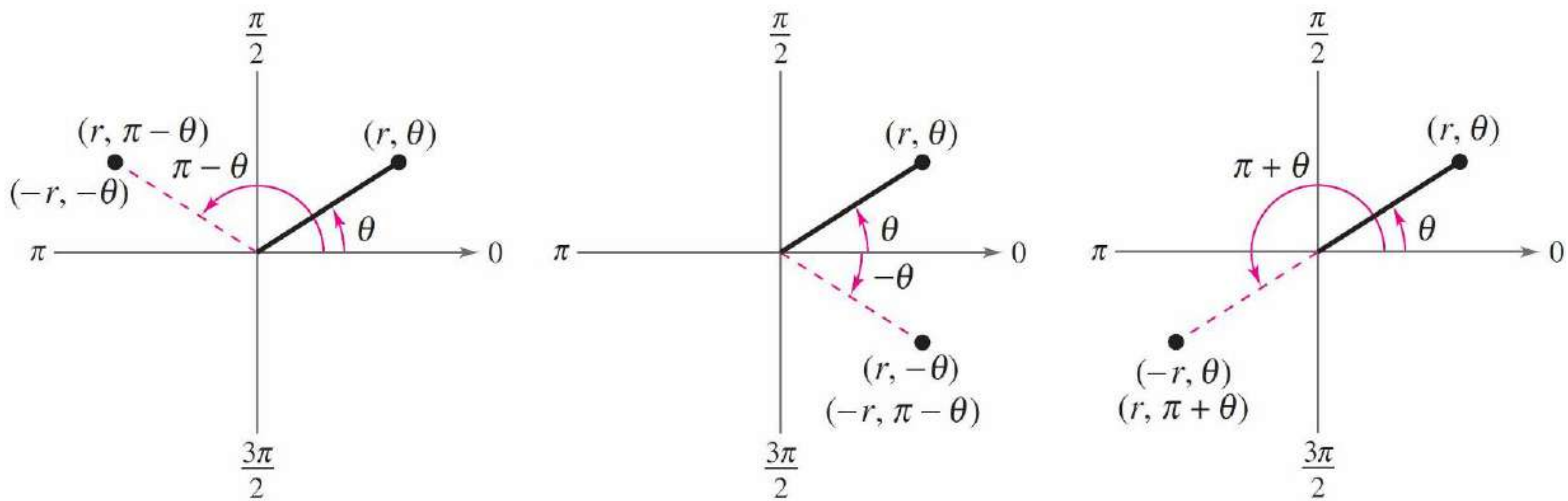


Figure 9.70

Symmetry and Zeros

The three important types of symmetry to consider in polar curve sketching are shown in Figure 9.71.



Symmetry with Respect $\theta = \frac{\pi}{2}$

Symmetry with Respect to the Polar Axis

Symmetry with Respect to the Pole

Figure 9.71

Symmetry and Zeros

Testing for Symmetry in Polar Coordinates

The graph of a polar equation is symmetric with respect to the following when the given substitution yields an equivalent equation.

1. The line $\theta = \frac{\pi}{2}$: Replace (r, θ) by $(r, \pi - \theta)$ or $(-r, -\theta)$.
2. The polar axis: Replace (r, θ) by $(r, -\theta)$ or $(-r, \pi - \theta)$.
3. The pole: Replace (r, θ) by $(r, \pi + \theta)$ or $(-r, \theta)$.

Example 2 – *Using Symmetry to Sketch a Polar Graph*

Use symmetry to sketch the graph of

$$r = 3 + 2 \cos \theta$$

by hand.

Solution:

Replacing (r, θ) by $(r, -\theta)$ produces

cos(–*u*) = cos *u*

$$r = 3 + 2 \cos (-\theta)$$
$$= 3 + 2 \cos \theta$$

So, by using the even trigonometric identity, you can conclude that the curve is symmetric with respect to the polar axis.

Example 2 – *Solution*

cont'd

Plotting the points in the table and using polar axis symmetry, you obtain the graph shown in Figure 9.72. This graph is called a **limaçon.**

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	5	$3+\sqrt{3}$	4	3	2	$3-\sqrt{3}$	1

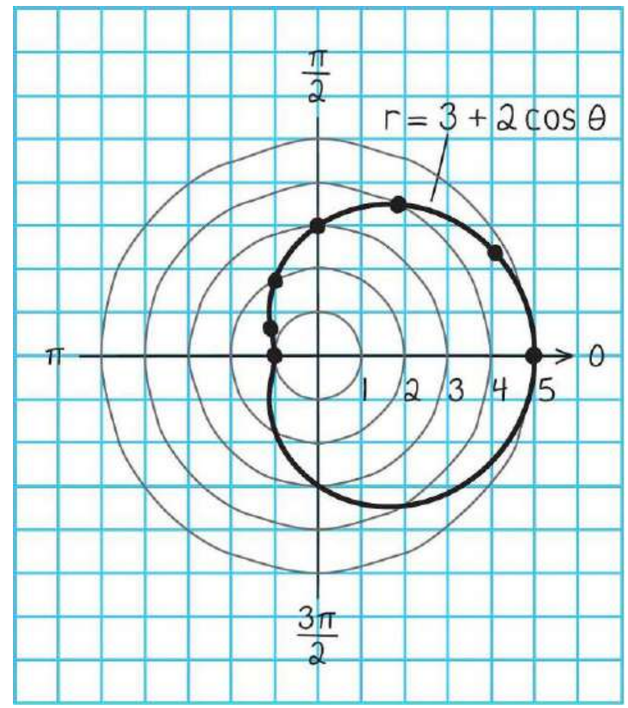


Figure 9.72

Use a graphing utility to confirm this graph.

Symmetry and Zeros

The following are the quick tests for symmetry.

Quick Tests for Symmetry in Polar Coordinates

1. The graph of $r = f(\sin \theta)$ is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.

2. The graph of $r = g(\cos \theta)$ is symmetric with respect to the polar axis.

Special Polar Graphs

Special Polar Graphs

Several important types of graphs have equations that are simpler in polar form than in rectangular form.

For example, the circle

$$r = 4 \sin \theta$$

in Example 1 has the more complicated rectangular equation

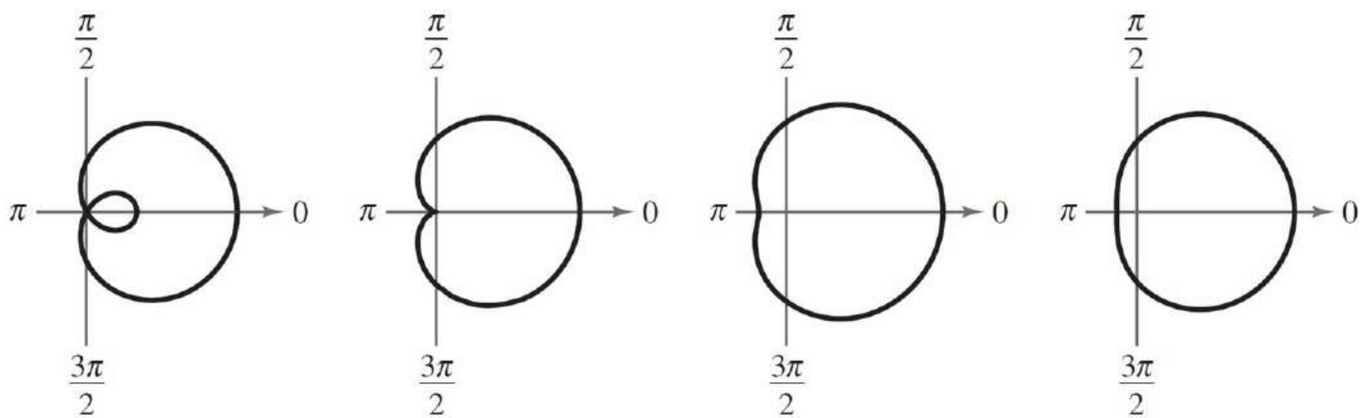
$$x^2 + (y - 2)^2 = 4.$$

Special Polar Graphs

Several other types of graphs that have simple polar equations are shown below

Limaçons

$$r = a \pm b \cos \theta,\ r = a \pm b \sin \theta\ (a > 0,\ b > 0)$$

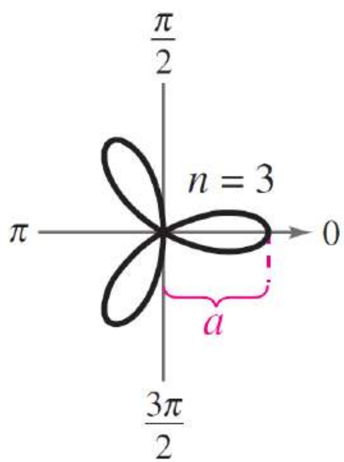


$\frac{a}{b} < 1$	$\frac{a}{b} = 1$	$1 < \frac{a}{b} < 2$	$\frac{a}{b} \geq 2$
Limaçon with inner loop	Cardioid (heart-shaped)	Dimpled limaçon	Convex limaçon

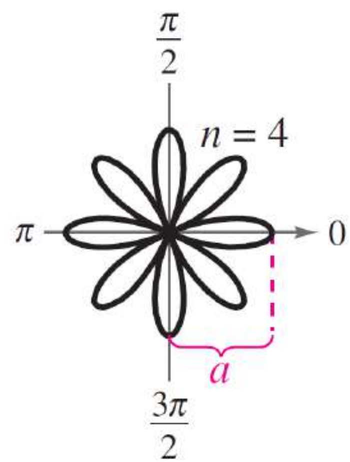
Special Polar Graphs

Rose Curves

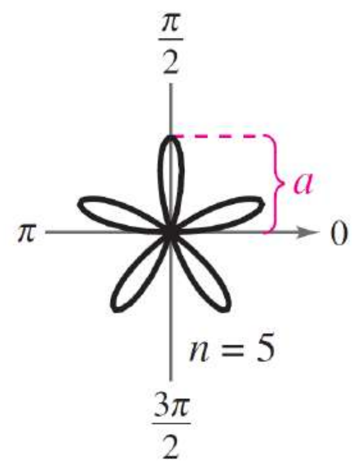
n petals when *n* is odd, 2*n* petals when *n* is even ($n \geq 2$)



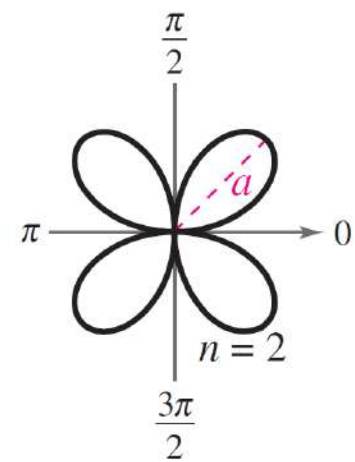
$r = a \cos n\theta$
Rose curve



$r = a \cos n\theta$
Rose curve



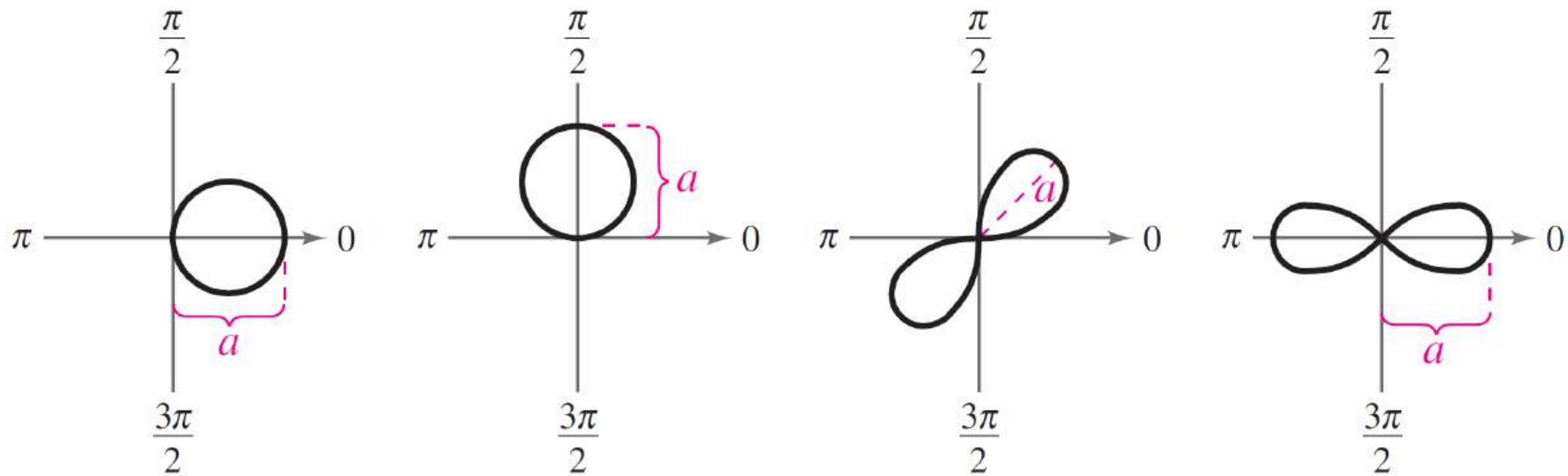
$r = a \sin n\theta$
Rose curve



$r = a \sin n\theta$
Rose curve

Special Polar Graphs

Circles and Lemniscates



$r = a \cos \theta$
Circle

$r = a \sin \theta$
Circle

$r^2 = a^2 \sin 2\theta$
Lemniscate

$r^2 = a^2 \cos 2\theta$
Lemniscate

Example 4 – *Analyzing a Rose Curve*

Analyze the graph of $r = 3 \cos 2\theta$.

Solution:

Type of curve: Rose curve with $2n = 4$ petals

Symmetry: With respect to the polar axis, the line $\theta = \frac{\pi}{2}$, and the pole

Zeros of r: $r = 0$ when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

Example 4 – *Solution*

cont'd

Using a graphing utility, enter the equation, as shown in Figure 9.75 (with $0 \le \theta \le 2\pi$).

You should obtain the graph shown in Figure 9.76.

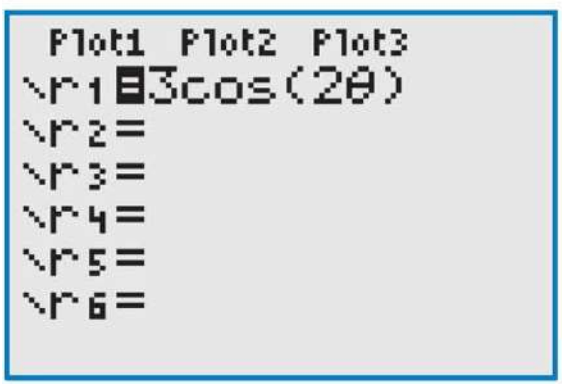


Figure 9.75

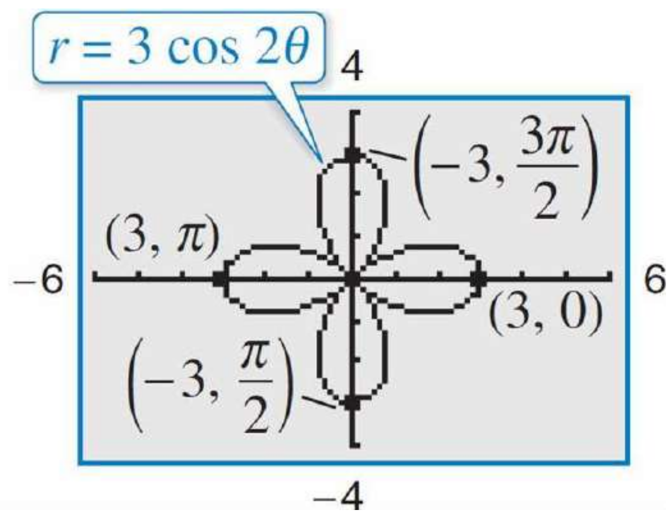


Figure 9.76