9.4 Parametric Equations

What You Should Learn

- Evaluate sets of parametric equations for given values of the parameter
- Graph curves that are represented by sets of parametric equations
- Rewrite sets of parametric equations as single rectangular equations by eliminating the parameter
- Find sets of parametric equations for graphs

Plane Curves

Plane Curves

Up to this point, you have been representing a graph by a single equation involving *two* variables such as *x* and *y*.

In this section, you will study situations in which it is useful to introduce a *third* variable to represent a curve in the plane.

To see the usefulness of this procedure, consider the path of an object that is propelled into the air at an angle of 45°.

Plane Curves

When the initial velocity of the object is 48 feet per second, it can be shown that the object follows the parabolic path

$$y = -\frac{x^2}{72} + x$$

Rectangular equation

as shown in Figure 9.42.

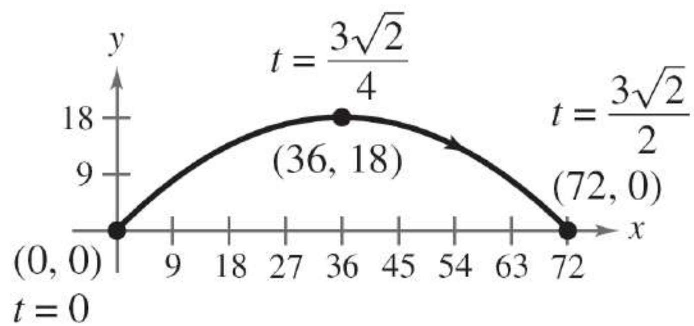
Rectangular equation:

$$y = -\frac{x^2}{72} + x$$

Parametric equations:

$$x = 24\sqrt{2}t$$

$$y = -16t^2 + 24\sqrt{2}t$$



Curvilinear motion: two variables for position, one variable for time

Figure 9.42

Plane Curves

However, this equation does not tell the whole story. Although it does tell you *where* the object has been, it does not tell you *when* the object was at a given point (x, y) on the path.

To determine this time, you can introduce a third variable t, called a **parameter.** It is possible to write both x and y as functions of t to obtain the **parametric equations**

$$x = 24\sqrt{2}t$$ Parametric equation for x

$$y = -16t^2 + 24\sqrt{2}t.$$ Parametric equation for y

Plane Curves

From this set of equations you can determine that at time t = 0, the object is at the point (0, 0). Similarly, at time t = 1, the object is at the point

$$\left(24\sqrt{2}, 24\sqrt{2} - 16\right)$$

and so on.

For this particular motion problem, x and y are continuous functions of t, and the resulting path is a **plane curve.** (Recall that a *continuous function* is one whose graph can be traced without lifting the pencil from the paper.)

Plane Curves

Definition of a Plane Curve

If f and g are continuous functions of t on an interval I, then the set of ordered pairs

$$(f(t), g(t))$$

is a **plane curve** C. The equations given by

$$x = f(t) \quad \text{and} \quad y = g(t)$$

are **parametric equations** for C, and t is the **parameter.**

Graphs of Plane Curves

Graphs of Plane Curves

One way to sketch a curve represented by a pair of parametric equations is to plot points in the *xy*-plane.

Each set of coordinates (x, y) is determined from a value chosen for the parameter t.

By plotting the resulting points in the order of *increasing* values of t, you trace the curve in a specific direction. This is called the **orientation** of the curve.

Example 1 – *Sketching a Plane Curve*

Sketch the curve given by the parametric equations

$$x = t^2 - 4 \quad \text{and} \quad y = \frac{t}{2}, \; -2 \le t \le 3.$$

Describe the orientation of the curve.

Example 1 – *Solution*

Using values of t in the interval, the parametric equations yield the points (x, y) shown in the table.

t	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$

Example 1 – *Solution*

cont'd

By plotting these points in the order of increasing *t*, you obtain the curve shown in Figure 9.43.

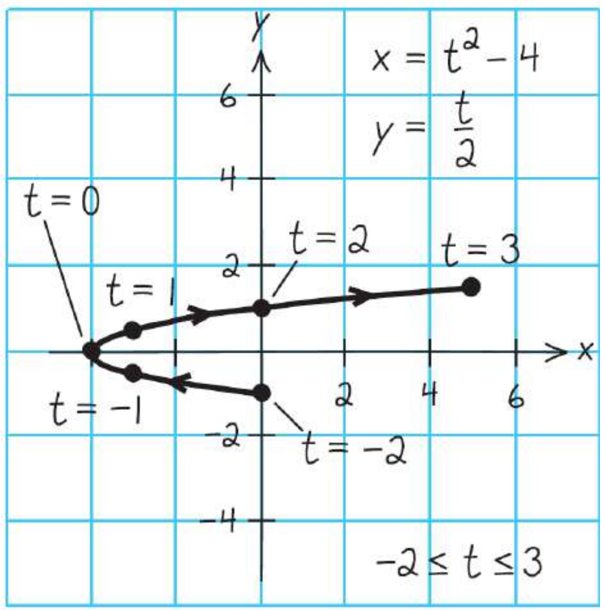


Figure 9.43

Example 1 – *Solution*

cont'd

The arrows on the curve indicate its orientation as *t* increases from –2 to 3.

So, when a particle moves on this curve, it would start at (0, –1) and then move along the curve to the point $\left(5, \frac{3}{2}\right)$.

Eliminating the Parameter

Eliminating the Parameter

Many curves that are represented by sets of parametric equations have graphs that can also be represented by rectangular equations (in *x* and *y*). The process of finding the rectangular equation is called **eliminating the parameter (using substitution)**.

Parametric equations → Solve for *t* in one equation. → Substitute in second equation. → Rectangular equation

$x = t^2 - 4$ $\quad$ $t = 2y$ $\quad$ $x = (2y)^2 - 4$ $\quad$ $x = 4y^2 - 4$

$y = \frac{1}{2}t$

Eliminating the Parameter

Now you can recognize that the equation $x = 4y^2 – 4$ represents a parabola with a horizontal axis and vertex at (–4, 0).

When converting equations from parametric to rectangular form, you may need to alter the domain of the rectangular equation so that its graph matches the graph of the parametric equations. This situation is demonstrated in Example 3.

Example 3 – *Eliminating the Parameter*

Identify the curve represented by the equations

$$x = \frac{1}{\sqrt{t + 1}} \quad \text{and} \quad y = \frac{t}{t + 1}.$$

Solution:

Solving for *t* in the equation for *x* produces

$$x^2 = \frac{1}{t + 1} \quad \Rightarrow \quad \frac{1}{x^2} = t + 1 \quad \Rightarrow \quad \frac{1}{x^2} - 1 = t.$$

Example 3 – *Solution*

cont'd

Substituting in the equation for *y*, you obtain the rectangular equation

$$y = \frac{t}{t+1} = \frac{\frac{1}{x^2} - 1}{\frac{1}{x^2} - 1 + 1} = \frac{\frac{1 - x^2}{x^2}}{\frac{1}{x^2}} \cdot \frac{x^2}{x^2} = 1 - x^2.$$

From the rectangular equation, you can recognize that the curve is a parabola that opens downward and has its vertex at (0, 1), as shown in Figure 9.54.

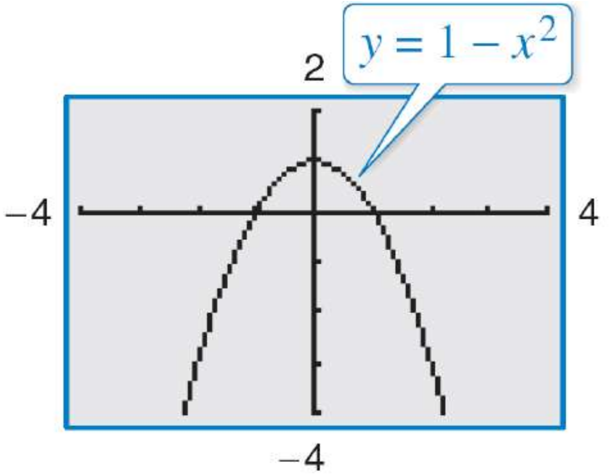


Figure 9.54

Example 3 – *Solution*

cont'd

The rectangular equation is defined for all values of *x*. The parametric equation for *x* however, is defined only when $t > -1$.

From the graph of the parametric equations, you can see that *x* is always positive, as shown in Figure 9.55.

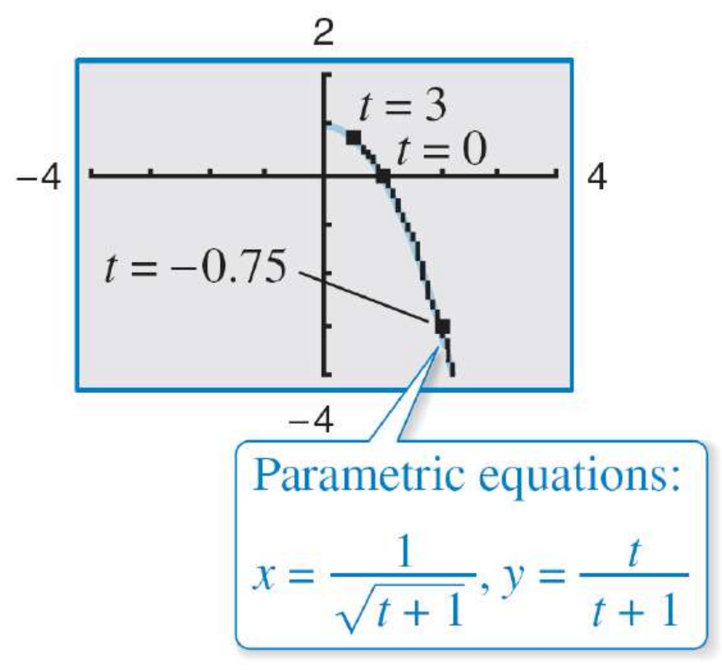


Figure 9.55

Example 3 – *Solution*

cont'd

So, you should restrict the domain of x to positive values, as shown in Figure 9.56.

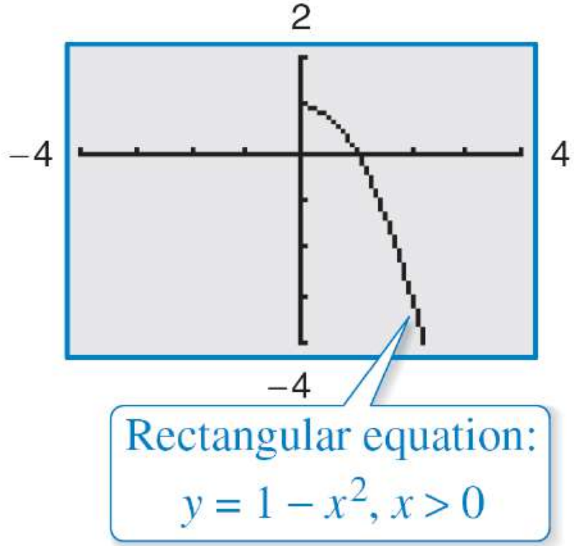


Figure 9.56

Finding Parametric Equations for a Graph

Finding Parametric Equations for a Graph

You have been studying techniques for sketching the graph represented by a set of parametric equations.

Now consider the *reverse* problem—that is, how can you find a set of parametric equations for a given graph or a given physical description?

From the discussion following Example 1, you know that such a representation is not unique.

Finding Parametric Equations for a Graph

That is, the equations

$$x = 4t^2 - 4 \quad \text{and} \quad y = t, \; -1 \le t \le \frac{3}{2}$$

produced the same graph as the equations

$$x = t^2 - 4 \quad \text{and} \quad y = \frac{t}{2}, \quad -2 \le t \le 3.$$

This is further demonstrated in Example 4.

Example 4 – *Finding Parametric Equations for a Given Graph*

Find a set of parametric equations to represent the graph of $y = 1 - x^2$ using the parameters

(a) $t = x$ and (b) $t = 1 - x$.

Solution:

a. Letting $t = x$, you obtain the following parametric equations.

$x = t$ Parametric equation for x

$y = 1 - t^2$ Parametric equation for y

Example 4 – *Solution*

cont'd

The graph of these equations is shown in Figure 9.57.

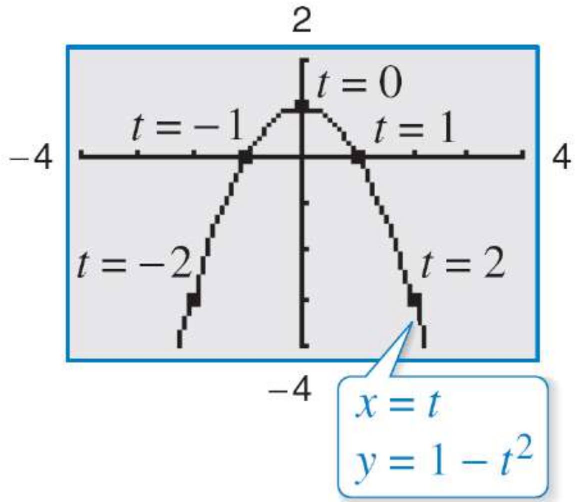


Figure 9.57

Example 4 – *Solution*

cont'd

b. Letting $t = 1 - x$, you obtain the following parametric equations.

$$x = 1 - t$$ Parametric equation for x

$$y = 1 - (1 - t)^2 = 2t - t^2$$ Parametric equation for y

Example 4 – *Solution*

cont'd

The graph of these equations is shown in Figure 9.58.

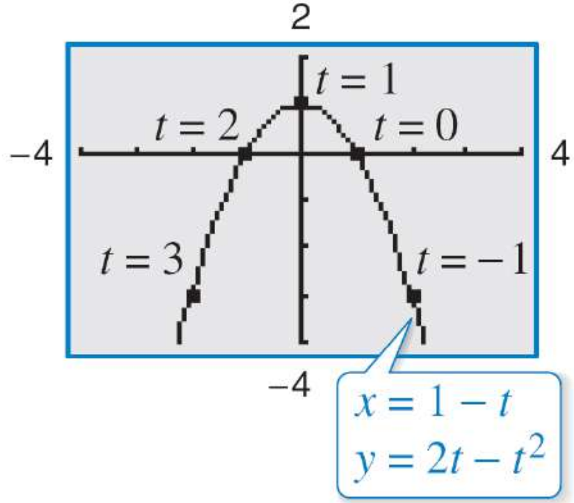


Figure 9.58

Example 4 – *Solution*

cont'd

Note that the graphs in Figures 9.57 and 9.58 have opposite orientations.

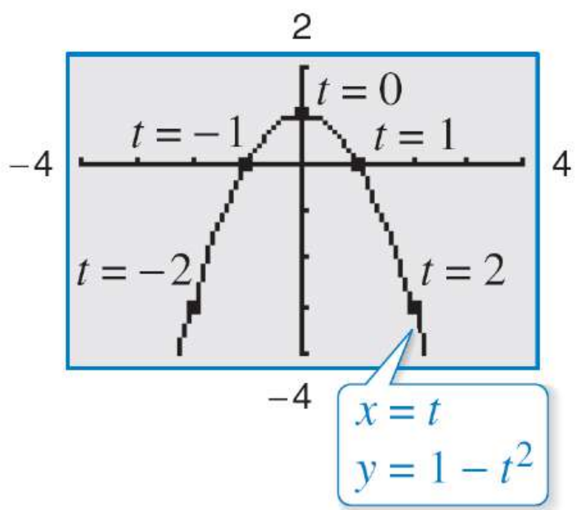


Figure 9.57

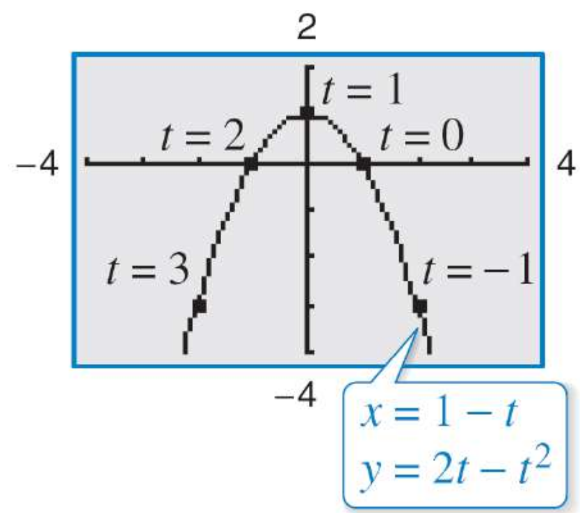


Figure 9.58