9.3

Hyperbolas and Rotation of Conics

What You Should Learn

- Write equations of hyperbolas in standard form.
- Find asymptotes of and graph hyperbolas.
- Use properties of hyperbolas to solve real-life problems.
- Classify conics from their general equations.
- Rotate the coordinate axes to eliminate the *xy*-term in equations of conics.

Introduction

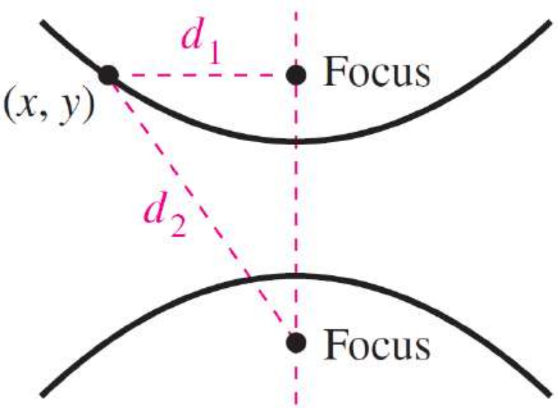
Introduction

The definition of a **hyperbola** is similar to that of an ellipse. The difference is that for an ellipse, the *sum* of the distances between the foci and a point on the ellipse is constant; whereas for a hyperbola, the *difference* of the distances between the foci and a point on the hyperbola is constant.

Introduction

Definition of a Hyperbola

A **hyperbola** is the set of all points (x, y) in a plane, the difference of whose distances from two distinct fixed points, the **foci,** is a positive constant. [See Figure 9.27(a).]



$d_2 - d_1$ is a positive constant.

Figure 9.27(a)

Introduction

The graph of a hyperbola has two disconnected parts called the **branches.** The line through the two foci intersects the hyperbola at two points called the **vertices.**

The line segment connecting the vertices is the **transverse axis,** and the midpoint of the transverse axis is the **center** of the hyperbola [see Figure 9.27(b)].

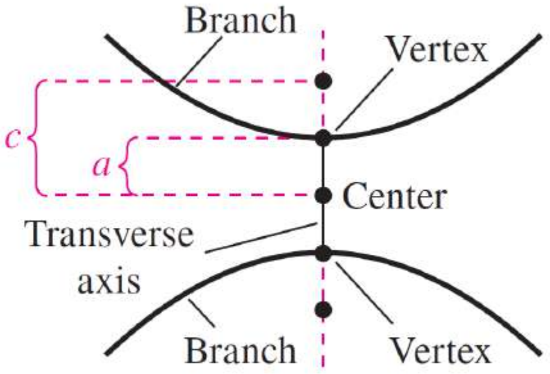


Figure 9.27(b)

Introduction

The development of the **standard form of the equation of a hyperbola** is similar to that of an ellipse.

Note, however, that *a*, *b* and *c* are related differently for hyperbolas than for ellipses.

For a hyperbola, the distance between the foci and the center is greater than the distance between the vertices and the center.

Introduction

Standard Equation of a Hyperbola

The **standard form of the equation of a hyperbola** with cen
(h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \qquad \text{Transverse axis is horizontal.}$$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \qquad \text{Transverse axis is vertical.}$$

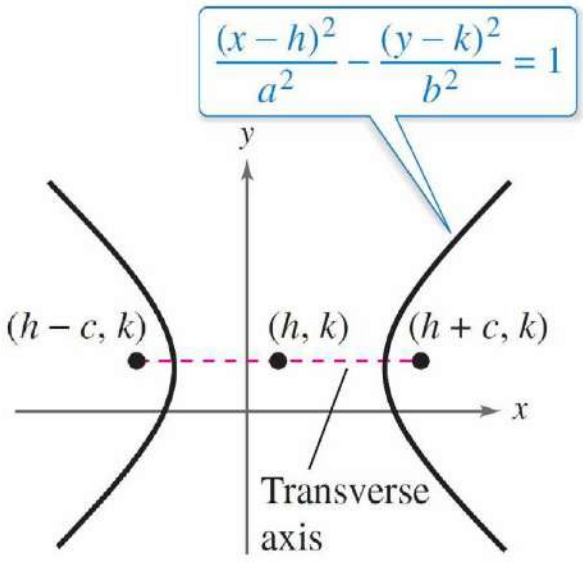
The vertices are a units from the center, and the foci are c units from the center. Moreover, $c^2 = a^2 + b^2$. If the center of the hyperbola is at the origin $(0, 0)$, then the equation takes one of the following forms.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{Transverse axis is horizontal.}$$

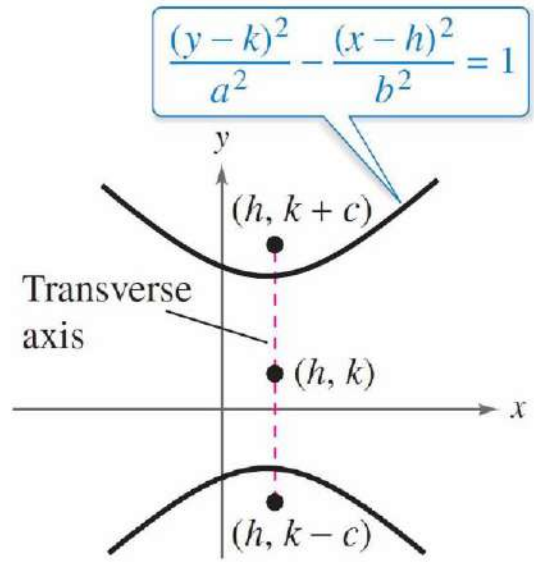
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad \text{Transverse axis is vertical.}$$

Introduction

Figure 9.28 shows both the horizontal and vertical orientations for a hyperbola.



Transverse axis is horizontal.



Transverse axis is vertical.

Figure 9.28

Example 1 – *Finding the Standard Equation of a Hyperbola*

Find the standard form of the equation of the hyperbola with foci (–1, 2) and (5, 2) and vertices (0, 2) and (4, 2).

Solution:

By the Midpoint Formula, the center of the hyperbola occurs at the point (2, 2). Furthermore, $c = 3$ and $a = 2$, and it follows that

$$\begin{aligned} b &= \sqrt{c^2 - a^2} \\ &= \sqrt{3^2 - 2^2} \\ &= \sqrt{9 - 4} \\ &= \sqrt{5}. \end{aligned}$$

Example 1 – *Solution*

cont'd

So, the hyperbola has a horizontal transverse axis, and the standard form of the equation of the hyperbola is

$$\frac{(x-2)^2}{2^2} - \frac{(y-2)^2}{\left(\sqrt{5}\right)^2} = 1.$$

This equation simplifies to

$$\frac{(x-2)^2}{4} - \frac{(y-2)^2}{5} = 1.$$

Example 1 – *Solution*

cont'd

Figure 9.29 shows the hyperbola.

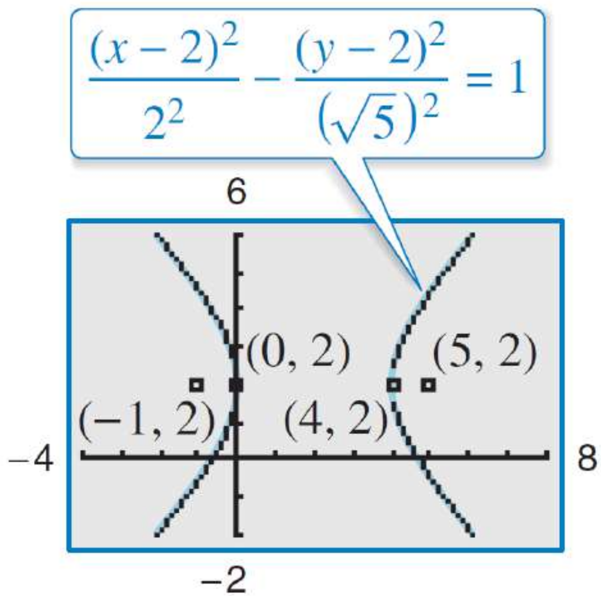


Figure 9.29

Asymptotes of a Hyperbola

Asymptotes of a Hyperbola

Each hyperbola has two **asymptotes** that intersect at the center of the hyperbola. The asymptotes pass through the corners of a rectangle of dimensions 2*a* by 2*b*, with its center at (*h*, *k*) as shown in Figure 9.30. Read the next slide, but do not copy.

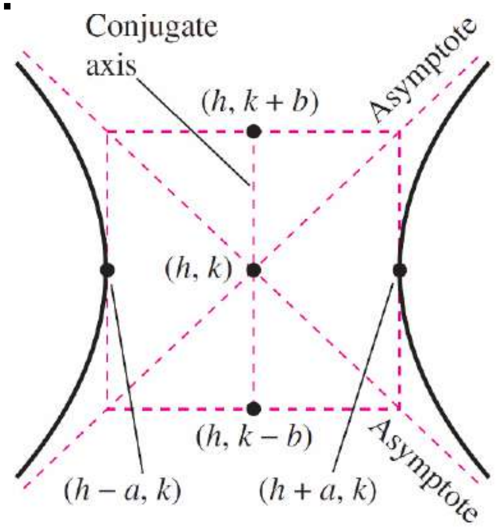


Figure 9.30

Asymptotes of a Hyperbola

Asymptotes of a Hyperbola

$$y = k \pm \frac{b}{a}(x - h)$$ Asymptotes for horizontal transverse axis

$$y = k \pm \frac{a}{b}(x - h)$$ Asymptotes for vertical transverse axis

The **conjugate axis** of a hyperbola is the line segment of length $2b$ joining $(h, k + b)$ and $(h, k - b)$ when the transverse axis is horizontal, and the line segment of length $2b$ joining $(h + b, k)$ and $(h - b, k)$ when the transverse axis is vertical.

Example 2 – *Sketching a Hyperbola*

Sketch the hyperbola whose equation is

$$4x^2 - y^2 = 16.$$

Solution:

$$4x^2 - y^2 = 16$$ Write original equation.

$$\frac{4x^2}{16} - \frac{y^2}{16} = \frac{16}{16}$$ Divide each side by 16.

$$\frac{x^2}{2^2} - \frac{y^2}{4^2} = 1$$ Write in standard form.

Example 2 – *Solution*

cont'd

Because the x^2-term is positive, you can conclude that the transverse axis is horizontal.

So, the vertices occur at (–2, 0) and (2, 0) the endpoints of the conjugate axis occur at (0, –4) and (0, 4), and you can sketch the rectangle shown in Figure 9.31.

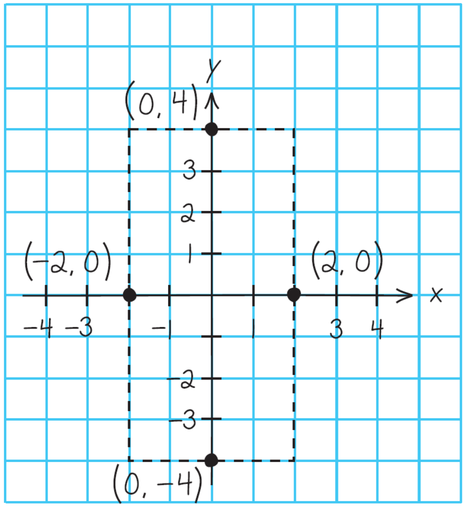


Figure 9.31

Example 2 – *Solution*

cont'd

Finally, by drawing the asymptotes

$$y = 2x \text{ and } y = -2x$$

through the corners of this rectangle, you can complete the sketch, as shown in Figure 9.32.

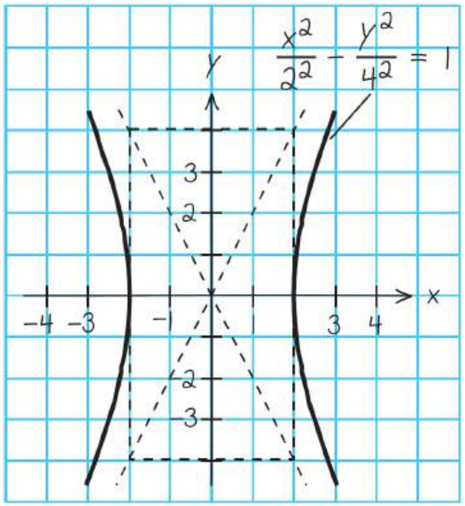


Figure 9.32

Example 3 – *Finding the Asymptotes of a Hyperbola*

Sketch the hyperbola given by

$$4x^2 - 3y^2 + 8x + 16 = 0$$

and find the equations of its asymptotes.

Solution:

$$4x^2 - 3y^2 + 8x + 16 = 0$$ Write original equation.

$$4(x^2 + 2x) - 3y^2 = -16$$ Subtract 16 from each side and factor.

Example 3 – *Solution*

cont'd

$$4(x^2 + 2x + 1) - 3y^2 = -16 + 4(1)$$ Complete the square.

$$4(x + 1)^2 - 3y^2 = -12$$ Write in completed square form.

$$\frac{y^2}{2^2} - \frac{(x + 1)^2}{(\sqrt{3})^2} = 1$$ Write in standard form.

From this equation you can conclude that the hyperbola has a vertical transverse axis, is centered at (–1, 0) has vertices (–1, 2) and (–1, –2), and has a conjugate axis with endpoints $(-1 - \sqrt{3}, 0)$ and $(-1 + \sqrt{3}, 0)$. To sketch the hyperbola, draw a rectangle through these four points.

Example 3 – *Solution*

cont'd

The asymptotes are the lines passing through the corners of the rectangle, as shown in Figure 9.34.

Finally, using $a = 2$ and $b = \sqrt{3}$, you can conclude that the equations of the asymptotes are

$$y = \frac{2}{\sqrt{3}}(x + 1) \text{ and } y = -\frac{2}{\sqrt{3}}(x + 1).$$

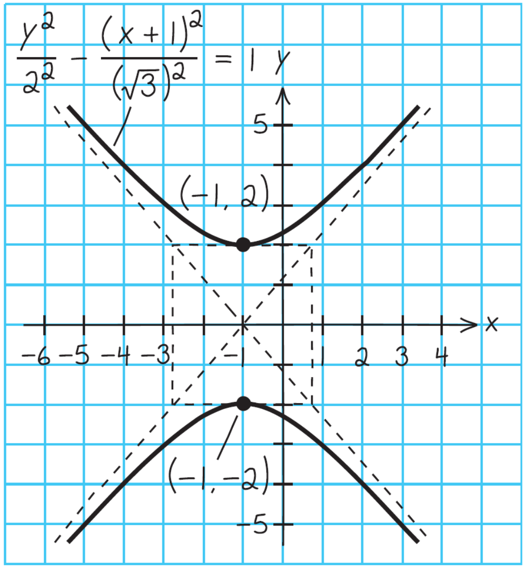


Figure 9.34

Example 3 – *Solution*

cont'd

You can verify your sketch using a graphing utility, as shown in Figure 9.35.

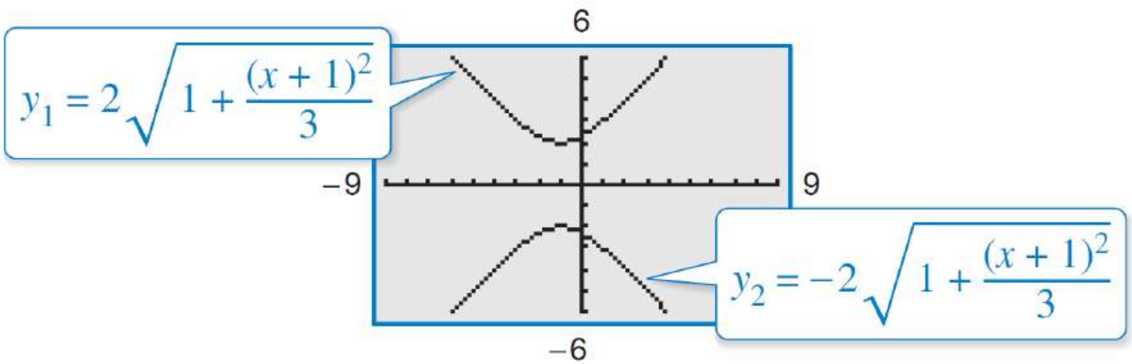


Figure 9.35

Notice that the graphing utility does not draw the asymptotes. When you trace along the branches, however, you will see that the values of the hyperbola approach the asymptotes.

Asymptotes of a Hyperbola

As with ellipses, the *eccentricity* of a hyperbola is

$$e = \frac{c}{a}$$ Eccentricity

and because $c > a$, it follows that $e > 1$.

Application

Example 5 – *An Application Involving Hyperbolas*

Two microphones, 1 mile apart, record an explosion. Microphone A receives the sound 2 seconds before microphone B. Where did the explosion occur?

Solution:

Assuming sound travels at 1100 feet per second, you know that the explosion took place 2200 feet farther from B than from A, as shown in Figure 9.38.

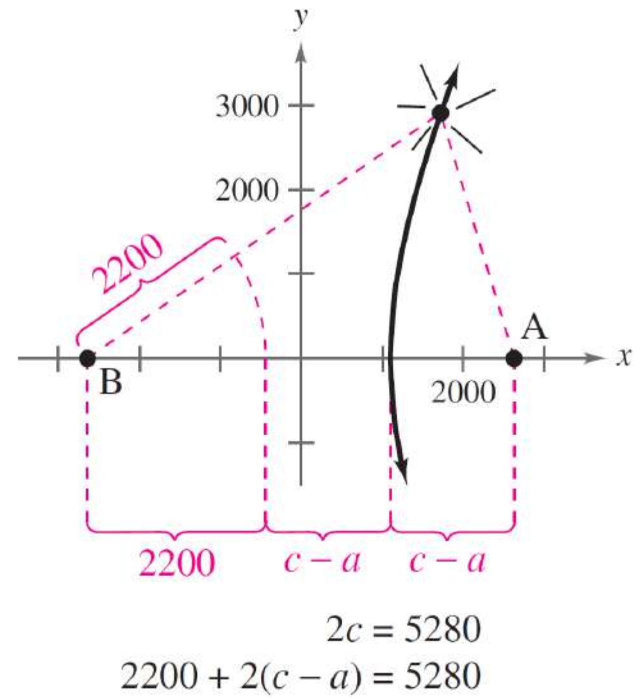


Figure 9.38

Example 5 – *Solution*

cont'd

The locus of all points that are 2200 feet closer to A than to B is one branch of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where

$$c = \frac{5280}{2} = 2640$$

and

$$a = \frac{2200}{2} = 1100.$$

Example 5 – *Solution*

cont'd

So,

$$b^2 = c^2 - a^2$$

$$= 2640^2 - 1100^2$$

$$= 5{,}759{,}600,$$

and you can conclude that the explosion occurred somewhere on the right branch of the hyperbola

$$\frac{x^2}{1{,}210{,}000} - \frac{y^2}{5{,}759{,}600} = 1.$$

General Equations of Conics

General Equations of Conics

Classifying a Conic from Its General Equation

The graph of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is one of the following.

1. Circle:	$A = C$	$A \neq 0$
2. Parabola:	$AC = 0$	$A = 0$ or $C = 0$, but not both.
3. Ellipse:	$AC > 0$	A and C have like signs.
4. Hyperbola:	$AC < 0$	A and C have unlike signs.

Example 6 – *Classifying Conics from General Equations*

Classify the graph of each equation.

a. $4x^2 - 9x + y - 5 = 0$

b. $4x^2 - y^2 + 8x - 6y + 4 = 0$

c. $2x^2 + 4y^2 - 4x + 12y = 0$

d. $2x^2 + 2y^2 - 8x + 12y + 2 = 0$

Solution:

Parabola

a. For the equation $4x^2 - 9x + y - 5 = 0$, you have

$$AC = 4(0) = 0.$$

Example 6 – *Solution*

cont'd

b. For the equation $4x^2 - y^2 + 8x - 6y + 4 = 0$, you have

$$AC = 4(-1) < 0.$$

Hyperbola

So, the graph is a hyperbola.

c. For the equation $2x^2 + 4y^2 - 4x + 12y = 0$, you have

$$AC = 2(4) > 0.$$

Ellipse

So, the graph is an ellipse.

Example 6 – *Solution*

cont'd

d. For the equation $2x^2 + 2y^2 - 8x + 12y + 2 = 0$, you have

$$A = C = 2.$$

Circle

So, the graph is a circle.

Rotation

Rotation

You have learned that the equation of a conic with axes parallel to one of the coordinates axes has a standard form that can be written in the general form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0.$$

Horizontal or vertical axis

You will now study the equations of conics whose axes are rotated so that they are not parallel to either the *x*-axis or the *y*-axis. The general equation for such conics contains an *xy*-term.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Equation in *xy*-plane

Rotation

To eliminate this *xy*-term, you can use a procedure called **rotation of axes.** The objective is to rotate the *x*- and *y*-axes until they are parallel to the axes of the conic.

The rotated axes are denoted as the *x*′-axis and the *y*′-axis, as shown in Figure 9.40.

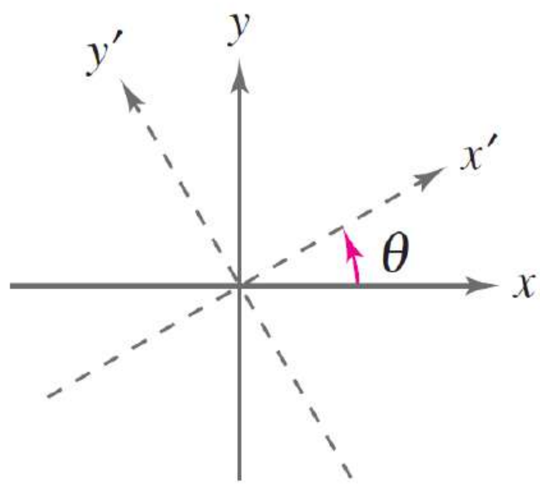


Figure 9.40

Rotation

After the rotation, the equation of the conic in the new $x'y'$-plane will have the form

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0.$$

Equation in $x'y'$-plane

Because this equation has no *xy*- term, you can obtain a standard form by completing the square.

Rotation

The following theorem identifies how much to rotate the axes to eliminate the *xy*-term and also the equations for determining the new coefficients *A*′, *C*′, *D*′, *E*′, and *F*′.

Rotation of Axes to Eliminate an *xy*-Term

The general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

can be rewritten as

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

by rotating the coordinate axes through an angle θ, where

$$\cot 2\theta = \frac{A - C}{B}.$$

The coefficients of the new equation are obtained by making the substitutions

$$x = x' \cos \theta - y' \sin \theta \qquad \text{and} \qquad y = x' \sin \theta + y' \cos \theta.$$

Example 7 – *Rotation of Axes for a Hyperbola*

Rotate the axes to eliminate the *xy*-term in the equation

$xy - 1 = 0.$

Then write the equation in standard form and sketch its graph.

Solution:

Because $A = 0$, $B = 1$ and $C = 0$, you have

$$\cot 2\theta = \frac{A - C}{B} = 0$$

which implies that $2\theta = \frac{\pi}{2}$, and $\theta = \frac{\pi}{4}$.

Example 7 – *Solution*

cont'd

The equation in the *x*′*y*′-system is obtained by making the substitutions

$$x = x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4}$$

$$= x'\left(\frac{1}{\sqrt{2}}\right) - y'\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{x' - y'}{\sqrt{2}}$$

and

$$y = x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4}$$

Example 7 – *Solution*

cont'd

$$= x'\left(\frac{1}{\sqrt{2}}\right) + y'\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{x' + y'}{\sqrt{2}}.$$

The equation in the $x'y'$-system is obtained by substituting these expressions into the equation $xy - 1 = 0$

$$\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) - 1 = 0$$

$$\frac{(x')^2 - (y')^2}{2} - 1 = 0$$

$$\frac{(x')^2}{(\sqrt{2})^2} - \frac{(y')^2}{(\sqrt{2})^2} = 1$$

Write in standard form.

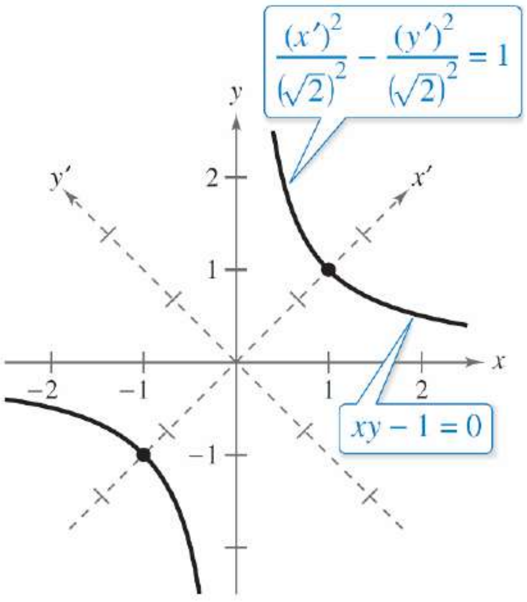
Example 7 – *Solution*

cont'd

In the $x'y'$-system. this is a hyperbola centered at the origin with vertices at $(\pm\sqrt{2}, 0)$, as shown in Figure 9.41.

To find the coordinates of the vertices in the xy-system, substitute the coordinates $(\pm\sqrt{2}, 0)$ into the equations

$$x = \frac{x' - y'}{\sqrt{2}} \quad \text{and} \quad y = \frac{x' + y'}{\sqrt{2}}.$$



Vertices:
In x′y′-system: $(\sqrt{2}, 0), (-\sqrt{2}, 0)$
In xy-system: $(1, 1), (-1, -1)$

Figure 9.41

Example 7 – *Solution*

cont'd

This substitution yields the vertices (1, 1) and (–1,–1) in the *xy*-system. Note also that the asymptotes of the hyperbola have equations

$$y' = \pm x'$$

which correspond to the original *x*-and *y*-axes.