9.2

Ellipses

What You Should Learn

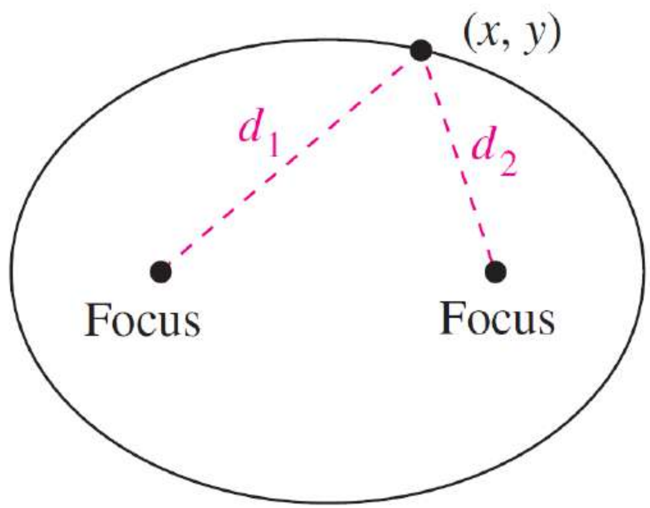
- Write equations of ellipses in standard form.
- Use properties of ellipses to model and solve real-life problems.
- Find eccentricities of ellipses.

Introduction

Introduction

Definition of an Ellipse

An **ellipse** is the set of all points (x, y) in a plane, the sum of whose distances from two distinct fixed points (**foci**) is constant. [See Figure 9.15(a).]



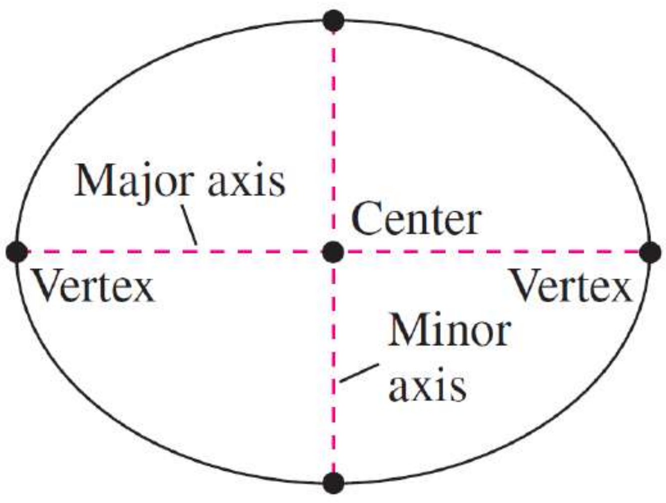
$d_1 + d_2$ is constant.

(a)

Figure 9.15

Introduction

The line through the foci intersects the ellipse at two points called **vertices.** The chord joining the vertices is the **major axis,** and its midpoint is the **center** of the ellipse. The chord perpendicular to the major axis at the center is the **minor axis.** [See Figure 9.15(b).]



(b)

Figure 9.15

Introduction

You can visualize the definition of an ellipse by imagining two thumbtacks placed at the foci, as shown in Figure 9.16.

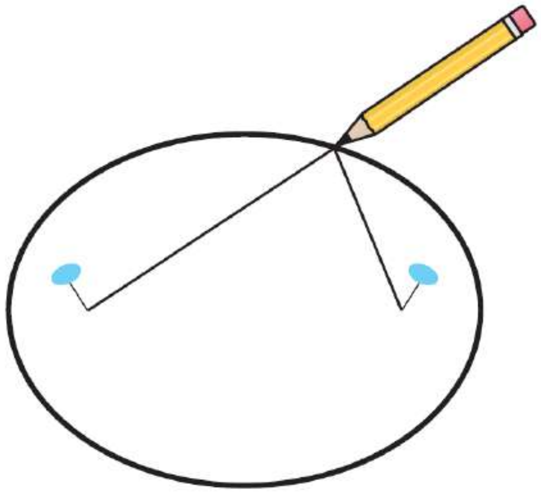


Figure 9.16

If the ends of a fixed length of string are fastened to the thumbtacks and the string is drawn taut with a pencil, then the path traced by the pencil will be an ellipse.

Introduction

Standard Equation of an Ellipse

The **standard form of the equation of an ellipse** with center (h, k) and major and minor axes of lengths $2a$ and $2b$, respectively, where $0 < b < a$, is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$ Major axis is horizontal.

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1.$$ Major axis is vertical.

The foci lie on the major axis, c units from the center, with

$$c^2 = a^2 - b^2.$$

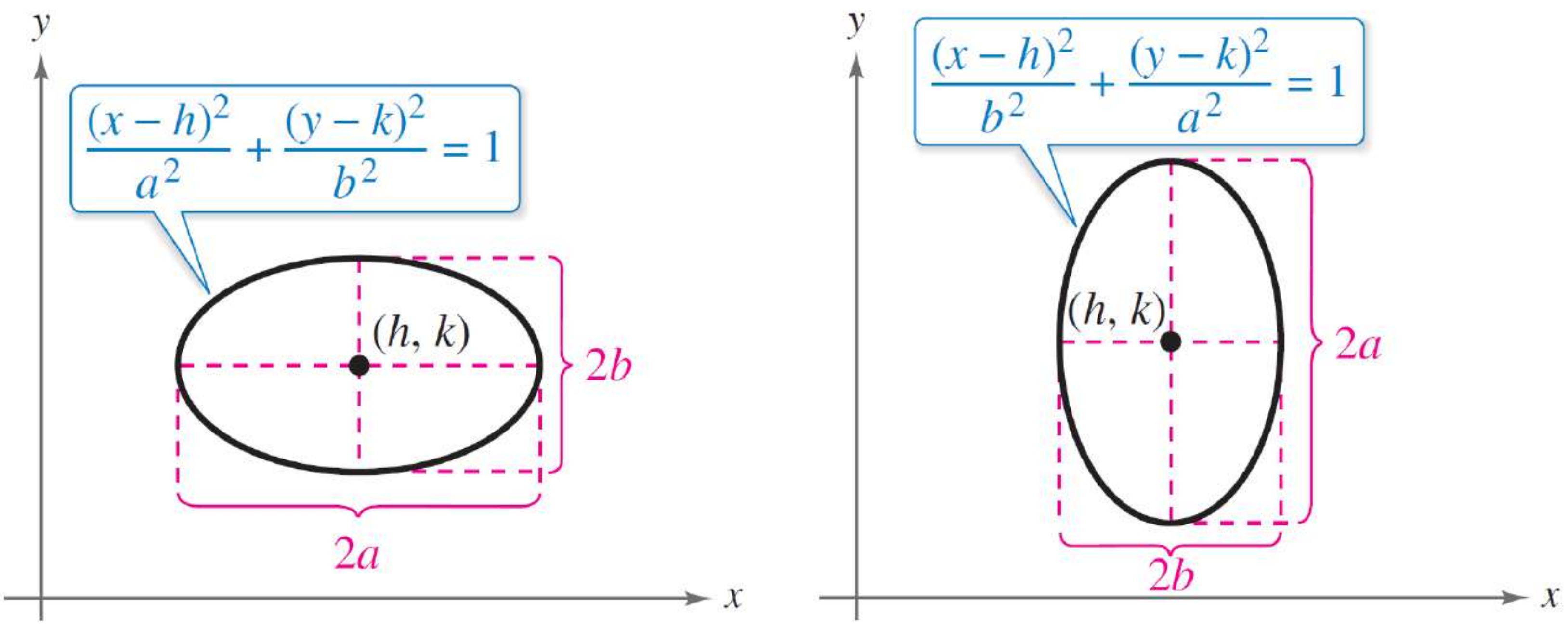
If the center is at the origin $(0, 0)$, then the equation takes one of the following forms.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ Major axis is horizontal.

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$ Major axis is vertical.

Introduction

Figure 9.18 shows both the vertical and horizontal orientations for an ellipse.



Major axis is horizontal.

Major axis is vertical.

Figure 9.18

Example 1 – *Finding the Standard Equation of an Ellipse*

Find the standard form of the equation of the ellipse having foci at

(0, 1) and (4, 1)

and a major axis of length 6, as shown in Figure 9.19.

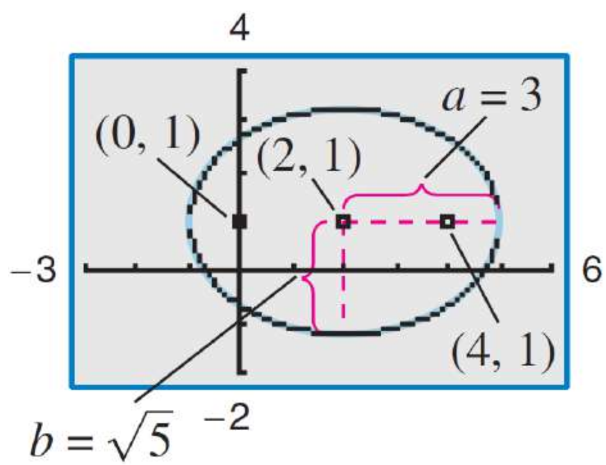


Figure 9.19

Example 1 – *Solution*

By the Midpoint Formula, the center of the ellipse is (2, 1) and the distance from the center to one of the foci is $c = 2$.

Because $2a = 6$, you know that $a = 3$. Now, from $c^2 = a^2 - b^2$, you have

$$b = \sqrt{a^2 - c^2} = \sqrt{9 - 4} = \sqrt{5}.$$

Because the major axis is horizontal, the standard equation is

$$\frac{(x - 2)^2}{3^2} + \frac{(y - 1)^2}{(\sqrt{5})^2} = 1.$$

Example 2 – *Sketching an Ellipse*

Sketch the ellipse given by

$$4x^2 + y^2 = 36$$

and identify the center and vertices.

Solution (start by writing the equation in standard form):

$$4x^2 + y^2 = 36$$ Write original equation.

$$\frac{4x^2}{36} + \frac{y^2}{36} = \frac{36}{36}$$ Divide each side by 36.

$$\frac{x^2}{3^2} + \frac{y^2}{6^2} = 1$$ Write in standard form.

Example 2 – *Solution*

cont'd

The center of the ellipse is (0, 0). Because the denominator of the y^2-term is larger than the denominator of the x^2-term, you can conclude that the major axis is vertical.

Moreover, because $a = 6$ the vertices are (0, –6) and (0, 6). Finally, because $b = 3$, the endpoints of the minor axis are (–3, 0) and (3, 0) as shown in Figure 9.20.

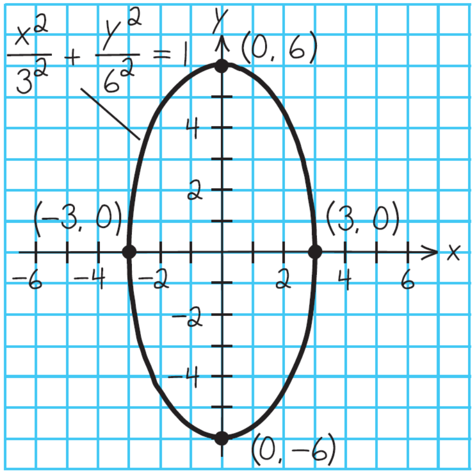


Figure 9.20

Example 5 – *An Application Involving an Elliptical Orbit*

The moon travels about Earth in an elliptical orbit with Earth at one focus, as shown in Figure 9.24. The major and minor axes of the orbit have lengths of 768,800 kilometers and 767,640 kilometers, respectively.

Find the greatest and least distances (the *apogee* and *perigee*) from Earth's center to the moon's center.

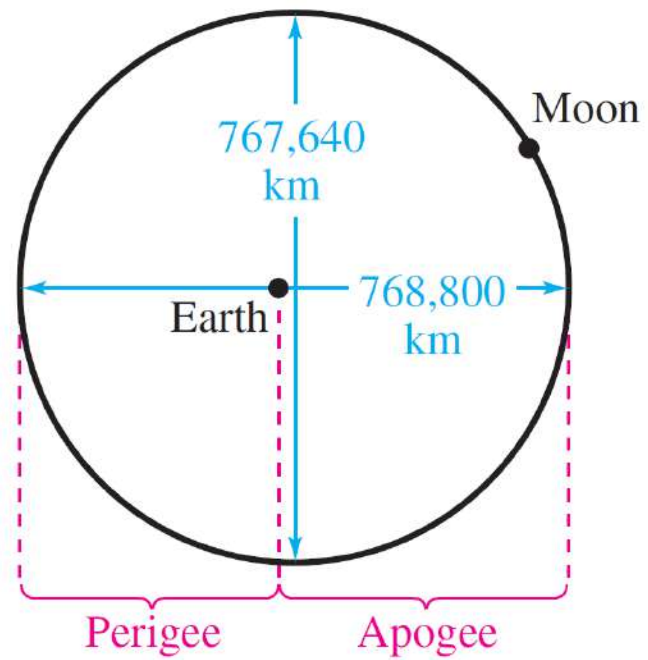


Figure 9.24

Example 5 – *Solution*

Because $2a$ = 768,800 and $2b$ = 767,640, you have

a = 384,400 and b = 383,820

which implies that

$$c = \sqrt{a^2 - b^2}$$

$$= \sqrt{384{,}400^2 - 383{,}820^2}$$

$$\approx 21{,}108$$

Example 5 – *Solution*

cont'd

So, the greatest distance between the center of Earth and the center of the moon is

$$a + c \approx 384{,}400 + 21{,}108$$

$$= 405{,}508 \text{ kilometers}$$

and the least distance is

$$a - c \approx 384{,}400 - 21{,}108$$

$$= 363{,}292 \text{ kilometers.}$$

Eccentricity

Eccentricity

One of the reasons it was difficult for early astronomers to detect that the orbits of the planets are ellipses is that the foci of the planetary orbits are relatively close to their centers, and so the orbits are nearly circular.

To measure the ovalness of an ellipse, you can use the concept of **eccentricity.**

Definition of Eccentricity

The **eccentricity** e of an ellipse is given by the ratio $e = \dfrac{c}{a}$.

Note that $0 < e < 1$ for *every* ellipse.