9.1 Circles and Parabolas

What You Should Learn

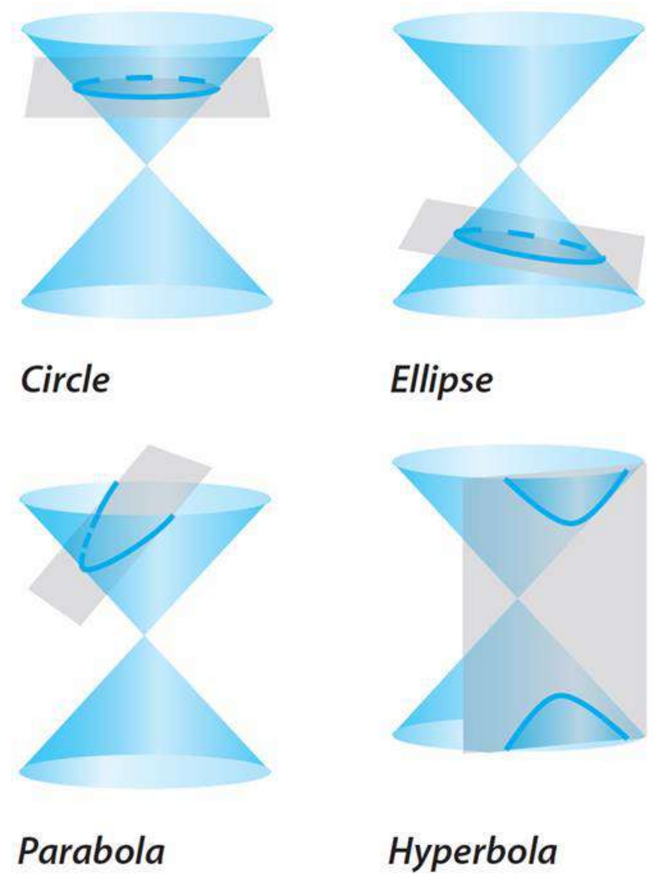
- Recognize a conic as the intersection of a plane and a double-napped cone.
- Write equations of circles in standard form.
- Write equations of parabolas in standard form.
- Use the reflective property of parabolas to solve real-life problems.

Conics

Conics

A **conic section** (or simply **conic**) is the intersection of a plane and a double-napped cone.

Notice in Figure 9.1 that in the formation of the four basic conics, the intersecting plane does not pass through the vertex of the cone.

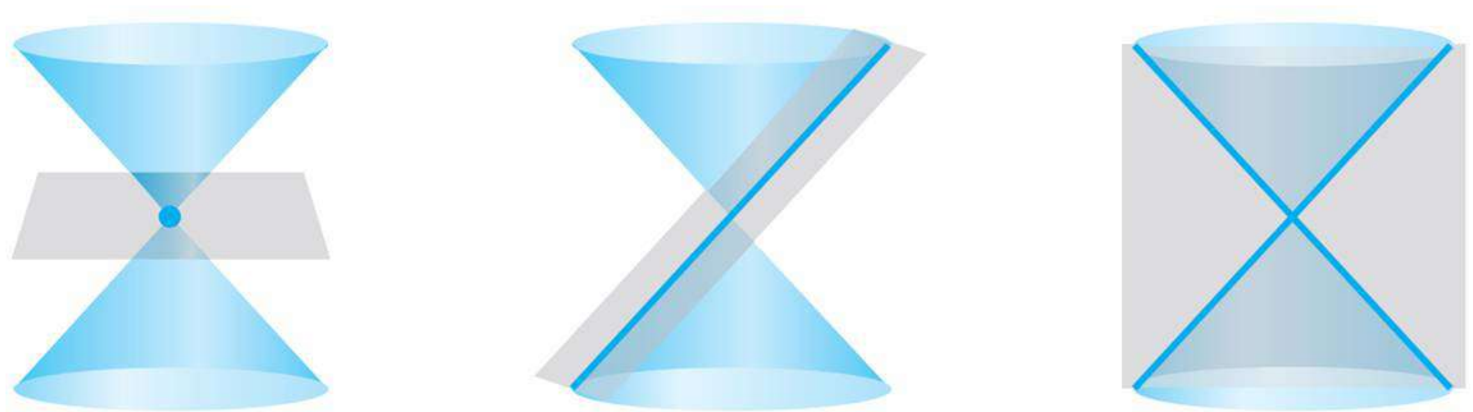


Basic Conics

Figure 9.1

Conics

When the plane does pass through the vertex, the resulting figure is a **degenerate conic,** as shown in Figure 9.2.



Point

Line

Two intersecting lines

Degenerate Conics

Figure 9.2

Conics

There are several ways to approach the study of conics. You could begin by defining conics in terms of the intersections of planes and cones, as the Greeks did, or you could define them algebraically, in terms of the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

However, you will study a third approach, in which each of the conics is defined as a **locus** (collection) of points satisfying a certain geometric property.

Conics

For example, the definition of a circle as *the collection of all points* (x, y) *that are equidistant from a fixed point* (h, k) leads to the standard equation of a circle

$$(x - h)^2 + (y - k)^2 = r^2. \qquad \text{Equation of circle}$$

Circles

Circles

Definition of a Circle

A **circle** is the set of all points (x, y) in a plane that are equidistant from a fixed point (h, k), called the **center** of the circle. (See Figure 9.3.) The distance r between the center and any point (x, y) on the circle is the **radius.**

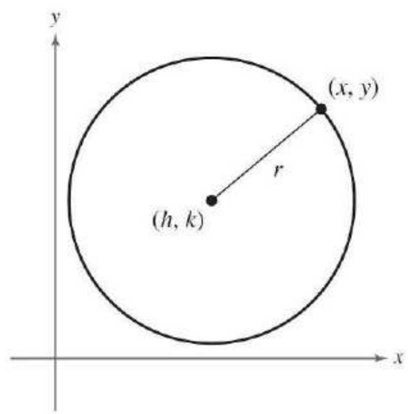


Figure 9.3

Circles

The Distance Formula can be used to obtain an equation of a circle whose center is (h, k) and whose radius is r.

$$\sqrt{(x-h)^2+(y-k)^2}=r$$ Distance Formula

$$(x-h)^2+(y-k)^2=r^2$$ Square each side.

Standard Form of the Equation of a Circle

The **standard form of the equation of a circle** is

$$(x-h)^2+(y-k)^2=r^2.$$

The point (h, k) is the center of the circle, and the positive number r is the radius of the circle. The standard form of the equation of a circle whose center is the origin, $(h, k) = (0, 0)$, is

$$x^2+y^2=r^2.$$

Example 1 – *Finding the Standard Equation of a Circle*

The point (1, 4) is on a circle whose center is at (–2, –3), as shown in Figure 9.4. Write the standard form of the equation of the circle. (standard form on the previous page is important)

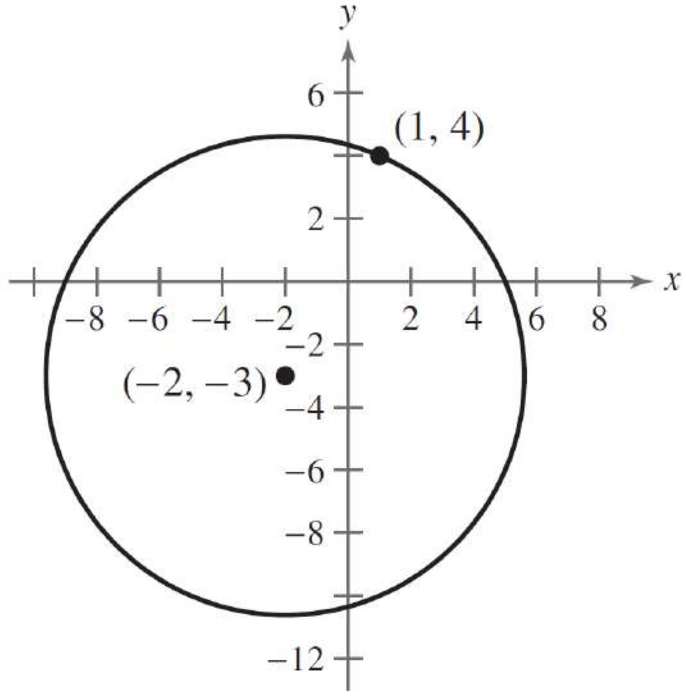


Figure 9.4

Example 1 – *Solution*

The radius of the circle is the distance between (–2, –3) and (1, 4).

$$r = \sqrt{[1 - (-2)]^2 + [4 - (-3)]^2} \qquad \text{Use Distance Formula}$$

$$= \sqrt{3^2 + 7^2} \qquad \text{Simplify}$$

$$= \sqrt{58} \qquad \text{Radius}$$

Example 1 – *Solution*

The equation of the circle with center $(h, k) = (-2, -3)$ and radius $r = \sqrt{58}$

$$(x - h)^2 + (y - k)^2 = r^2$$ Standard form

$$[x - (-2)]^2 + [y - (-3)]^2 = \left(\sqrt{58}\right)^2$$ Substitute for h, k, and r

$$(x + 2)^2 + (y + 3)^2 = 58.$$ Simplify

Example 2 – *Sketching a Circle*

Sketch the circle given by the equation

$x^2 - 6x + y^2 - 2y + 6 = 0$

and identify its center and radius.

Solution:

Begin by writing the equation in standard form which you find by completing the square for each variable.

$x^2 - 6x + y^2 - 2y + 6 = 0$ Write original equation

$(x^2 - 6x + 9) + (y^2 - 2y + 1) = -6 + 9 + 1$ Complete the squares

Example 2 – *Solution*

cont'd

$(x - 3)^2 + (y - 1)^2 = 4$ Write in standard form

In this form, you can see that the graph is a circle whose center is the point (3, 1) and whose radius is

$$r = \sqrt{4}$$

$$= 2$$

Parabolas

Parabolas

Definition of a Parabola

A **parabola** is the set of all points (x, y) in a plane that are equidistant from a fixed line, the **directrix,** and a fixed point, the **focus,** not on the line. (See Figure 9.7.) The midpoint between the focus and the directrix is the **vertex,** and the line passing through the focus and the vertex is the **axis** of the parabola.

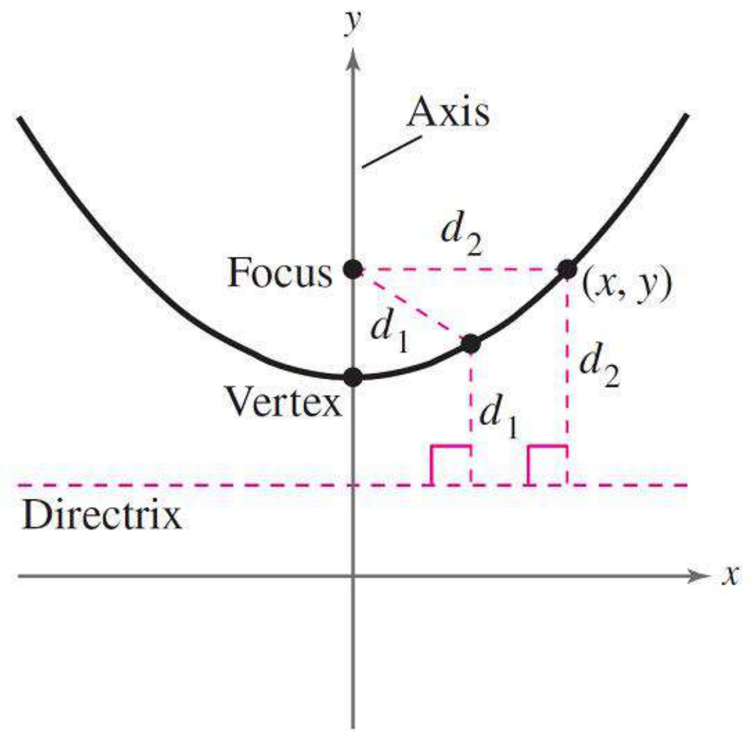


Figure 9.7

Parabolas

Standard Equation of a Parabola

The **standard form of the equation of a parabola** with vertex at (h, k) is as follows.

$(x - h)^2 = 4p(y - k), \ p \neq 0$ — Vertical axis; directrix: $y = k - p$

$(y - k)^2 = 4p(x - h), \ p \neq 0$ — Horizontal axis; directrix: $x = h - p$

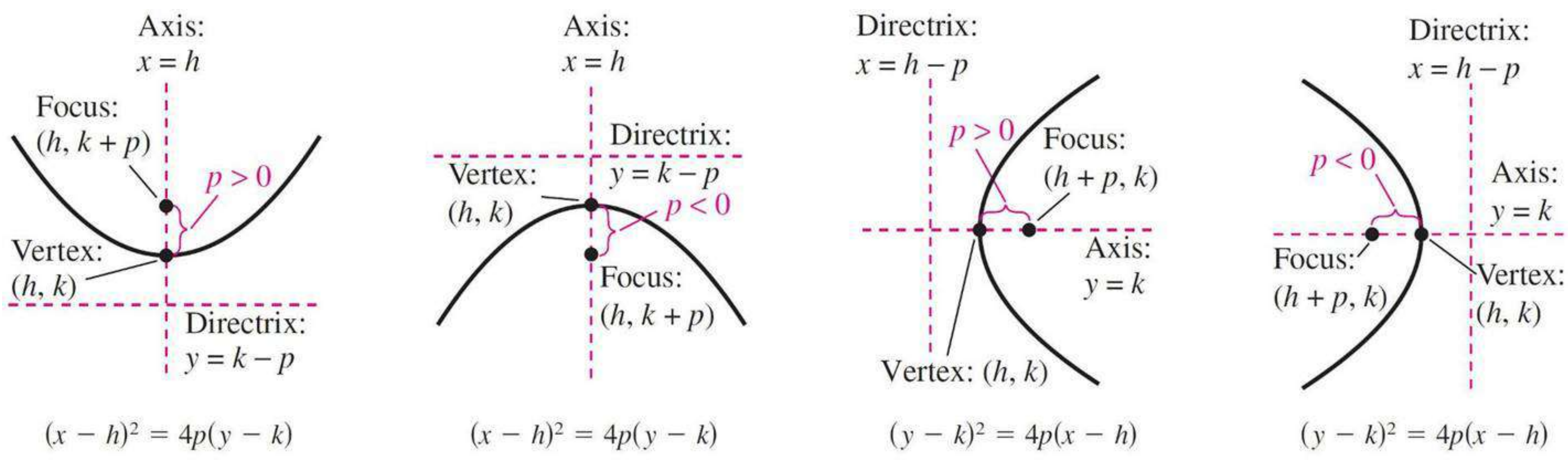
The focus lies on the axis p units (*directed distance*) from the vertex. If the vertex is at the origin $(0, 0)$, then the equation takes one of the following forms.

$x^2 = 4py$ — Vertical axis

$y^2 = 4px$ — Horizontal axis

See Figure 9.8.

Parabolas



(a) Vertical axis: $p > 0$

(b) Vertical axis: $p < 0$

(c) Horizontal axis: $p > 0$

(d) Horizontal axis: $p < 0$

Figure 9.8

Example 4 – *Finding the Standard Equation of a Parabola*

Find the standard form of the equation of the parabola with vertex at the origin and focus (0, 4). (I prefer the form

$y = a(x-h)^2 + k$ where $a=(1/4)p$.)

Solution:

The axis of the parabola is vertical, passing through (0, 0) and (0, 4), as shown in Figure 9.9.

The standard form is $x^2 = 4py$ (*I* prefer to use a = 1/4p) where

$p = 4$. So, the equation

is $y = \frac{1}{16}x^2$.

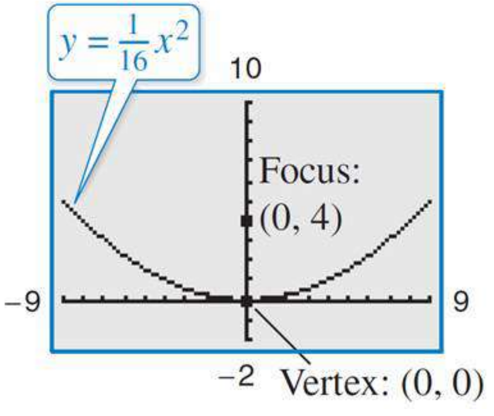


Figure 9.9

Example 5 – *Finding the Focus of a Parabola*

Find the focus of the parabola given by

$$y = -\frac{1}{2}x^2 - x + \frac{1}{2}.$$

Solution:

To find the focus, convert to standard form by completing the square or by using x=-b/2a to find the x of the vertex.

$$y = -\frac{1}{2}x^2 - x + \frac{1}{2}$$ Write original equation

$-2y = x^2 + 2x - 1$ Multiply each side by –2

$1 - 2y = x^2 + 2x$ Add 1 to each side

Example 5 – *Solution*

cont'd

$1 + 1 - 2y = x^2 + 2x + 1$ Complete the square

$2 - 2y = x^2 + 2x + 1$ Combine like terms

$-2(y - 1) = (x + 1)^2$ Write in standard form

or $y = -.5(x + 1)^2 + 1$

Comparing this equation with

$y = a(x-h)^2 + k$

you can conclude that $h = -1$, $k = 1$, and $p = -\frac{1}{2}$.

Example 5 – *Solution*

cont'd

Because *p* is negative, the parabola opens downward, as shown in Figure 9.10. Therefore, the focus of the parabola is below the vertex ½ of a unit at:

$$(h, k + p) = \left(-1, \tfrac{1}{2}\right).$$

Focus

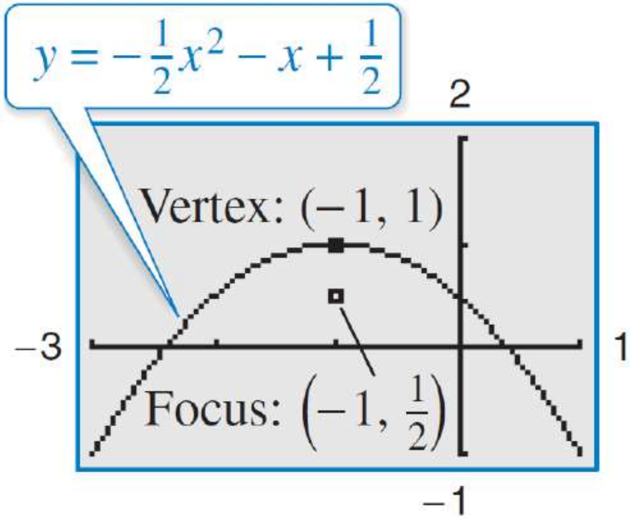


Figure 9.10

Reflective Property of Parabolas

Reflective Property of Parabolas

A line segment that passes through the focus of a parabola and has endpoints on the parabola is called a **focal chord.**

The specific focal chord perpendicular to the axis of the parabola is called the **latus rectum.**

Parabolas occur in a wide variety of applications. For instance, a parabolic reflector can be formed by revolving a parabola about its axis.

The resulting surface has the property that all incoming rays parallel to the axis are reflected through the focus of the parabola. This is the principle behind the construction of the parabolic mirrors used in reflecting telescopes.

Reflective Property of Parabolas

Conversely, the light rays emanating from the focus of a parabolic reflector used in a flashlight are all parallel to one another, as shown in Figure 9.12.

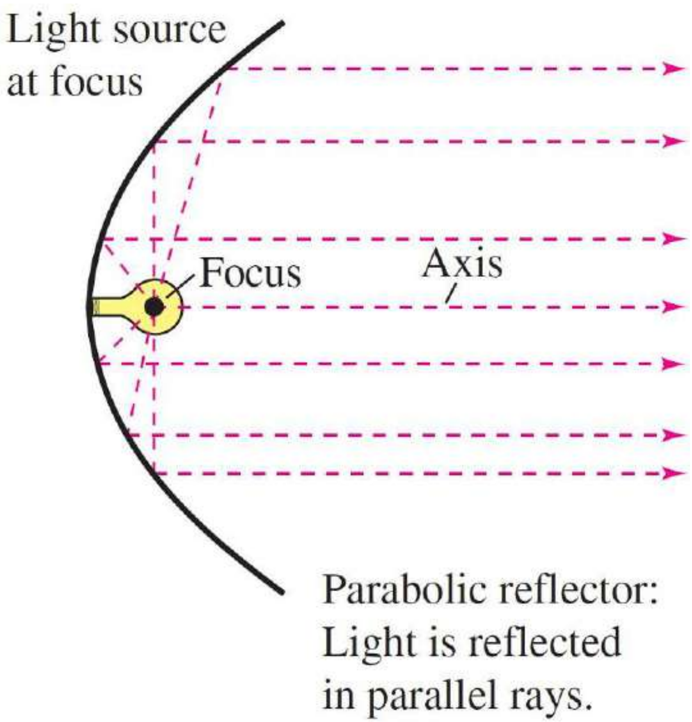


Figure 9.12

Reflective Property of Parabolas

A line is **tangent** to a parabola at a point on the parabola when the line intersects, but does not cross, the parabola at the point.

Tangent lines to parabolas have special properties related to the use of parabolas in constructing reflective surfaces.

Example 7 – *Finding the Tangent Line at a Point on a Parabola*

Find the equation of the tangent line to the parabola given by $y = x^2$ at the point (1, 1).

Solution:

For this parabola, $p = \frac{1}{4}$ and the focus is $\left(0, \frac{1}{4}\right)$ because it is above the vertex ¼ of a unit as shown in Figure 9.14.

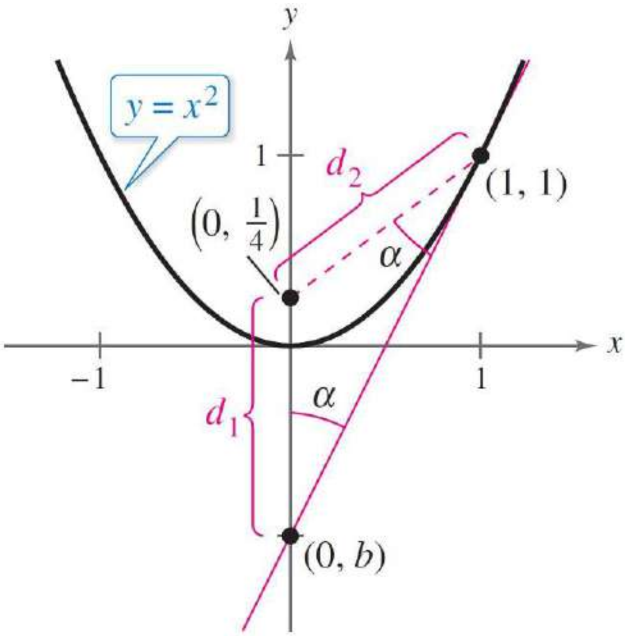


Figure 9.14

Example 7 – *Solution*

cont'd

You can find the y-intercept of $(0, b)$ the tangent line by equating the lengths of the two sides of the isosceles triangle shown in Figure 9.14:

$$d_1 = \frac{1}{4} - b$$

and

$$d_2 = \sqrt{(1 - 0)^2 + \left(1 - \frac{1}{4}\right)^2}$$

$$= \frac{5}{4}.$$

Example 7 – *Solution*

cont'd

Note that $d_1 = \frac{1}{4} - b$ n The o$b - \frac{1}{4}$ of subtraction for the distance is important because the distance must be positive. Setting $d_1 = d_2$ produces

$$\frac{1}{4} - b = \frac{5}{4}$$

$$b = -1.$$

So, the slope of the tangent line is

$$m = \frac{1 - (-1)}{1 - 0} = 2$$

and the equation of the tangent line in slope-intercept form is

$$y = 2x - 1.$$