8.6

Probability

What You Should Learn

- Find probabilities of events.
- Find probabilities of mutually exclusive events.
- Find probabilities of independent events.

The Probability of an Event

The Probability of an Event

Any happening whose result is uncertain is called an **experiment.**

The possible results of the experiment are **outcomes,** the set of all possible outcomes of the experiment is the **sample space** of the experiment, and any subcollection of a sample space is an **event.**

For instance, when a six-sided die is tossed, the sample space can be represented by the numbers 1 through 6. For the experiment to be fair, each of the outcomes is *equally likely*.

The Probability of an Event

To describe a sample space in such a way that each outcome is equally likely, you must sometimes distinguish between or among various outcomes in ways that appear artificial.

Example 1 illustrates such a situation.

Example 1 – *Finding the Sample Space*

Find the sample space for each of the following.

a. One coin is tossed.

b. Two coins are tossed.

c. Three coins are tossed.

Solution:

a. Because the coin will land either heads up (denoted by H) or tails up (denoted by T), the sample space S is

$S = \{H, T\}$.

Example 1 – *Solution*

cont'd

b. Because either coin can land heads up or tails up, the possible outcomes are as follows.

HH = heads up on both coins

HT = heads up on first coin and tails up on second coin

TH = tails up on first coin and heads up on second coin

TT = tails up on both coins

So, the sample space is

$$S = \{HH, HT, TH, TT\}.$$

Example 1 – *Solution*

cont'd

Note that this list distinguishes between the two cases

HT and *TH*

even though these two outcomes appear to be similar.

c. Following the notation in part (b), the sample space is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Note that this list distinguishes among the cases and

HHT, *HTH*, and *THH*

and among the cases

HTT, *THT*, and *TTH*.

The Probability of an Event

The Probability of an Event

If an event E has $n(E)$ equally likely outcomes and its sample space S has $n(S)$ equally likely outcomes, then the **probability** of event E is given by

$$P(E) = \frac{n(E)}{n(S)}.$$

Because the number of outcomes in an event must be less than or equal to the number of outcomes in the sample space, the probability of an event must be a number from 0 to 1, inclusive.

The Probability of an Event

That is,

$$0 \leq P(E) \leq 1$$

as indicated in Figure 8.13.

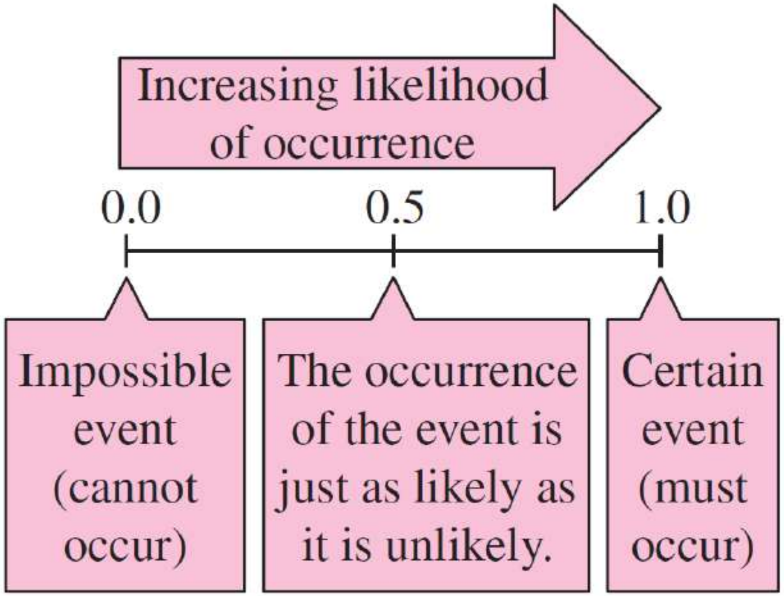

Figure 8.13

If $P(E) = 0$, then event E *cannot occur,* and is called an **impossible event.**

If $P(E) = 1$, then event *must occur,* and E is called a **certain event.**

Example 2 – *Finding the Probability of an Event*

a. Two coins are tossed. What is the probability that both land heads up?

b. A card is drawn at random from a standard deck of playing cards. What is the probability that it is an ace?

Solution:

a. Following the procedure in Example 1(b), let

$$E = \{HH\}$$

and

$S = \{HH, HT, TH, TT\}$.

Example 2 – *Solution*

cont'd

The probability of getting two heads is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}.$$

b. Because there are 52 cards in a standard deck of playing cards and there are four aces (one of each suit), the probability of drawing an ace is

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{52} = \frac{1}{13}.$$

Mutually Exclusive Events

Mutually Exclusive Events

Two events A and B (from the same sample space) are **mutually exclusive** when A and B have no outcomes in common.

In the terminology of sets, the intersection of A and B is the empty set, which implies that

$$P(A \cap B) = 0.$$

For instance, when two dice are tossed, the event A of rolling a total of 6 and the event B of rolling a total of 9 are mutually exclusive.

Mutually Exclusive Events

To find the probability that one or the other of two mutually exclusive events will occur, you can *add* their individual probabilities.

Probability of the Union of Two Events

If A and B are events in the same sample space, then the probability of A *or* B occurring is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

Example 5 – *The Probability of a Union*

One card is selected at random from a standard deck of 52 playing cards. What is the probability that the card is either a heart or a face card?

Solution:

Because the deck has 13 hearts, the probability of selecting a heart (event *A*) is

$$P(A) = \frac{13}{52}.$$

Example 5 – *Solution*

cont'd

Similarly, because the deck has 12 face cards, the probability of selecting a face card (event B) is

$$P(B) = \frac{12}{52}.$$

Because three of the cards are hearts and face cards (see Figure 8.16), it follows that

$$P(A \cap B) = \frac{3}{52}.$$

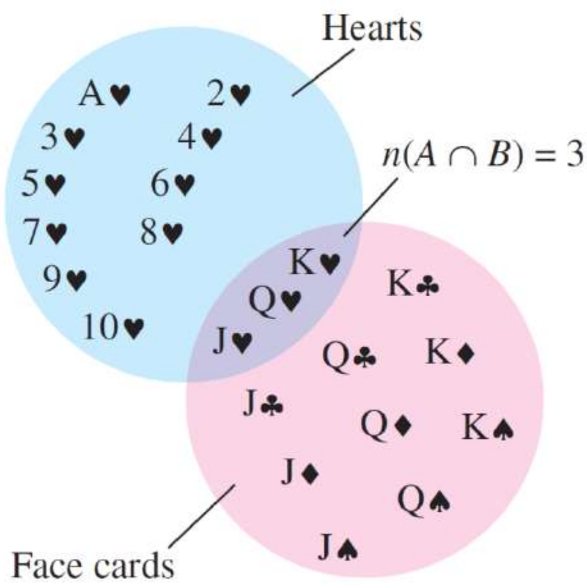

Figure 8.16

Example 5 – *Solution*

cont'd

Finally, applying the formula for the probability of the union of two events, you can conclude that the probability of selecting a heart or a face card is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} + \frac{12}{52} - \frac{3}{52}$$

$$= \frac{22}{52}$$

$$\approx 0.42.$$

Independent Events

Independent Events

Two events are **independent** when the occurrence of one has no effect on the occurrence of the other.

For instance, rolling a total of 12 with two six-sided dice has no effect on the outcome of future rolls of the dice. (See Figure 8.17.)

To find the probability that two independent events will occur, *multiply* the probabilities of each.

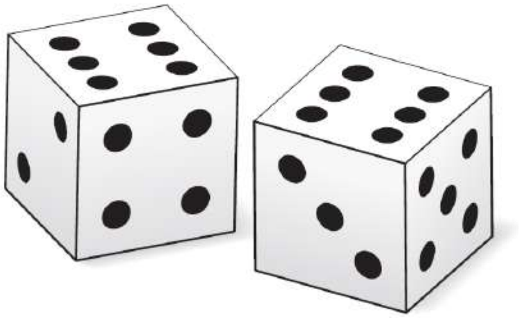

Figure 8.17

Independent Events

Probability of Independent Events

If A and B are **independent events,** then the probability that both A and B will occur is given by

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

Example 7 – *Probability of Independent Events*

A random number generator on a computer selects three integers from 1 to 20. What is the probability that all three numbers are less than or equal to 5?

Solution:

The probability of selecting a number from 1 to 5 is

$$P(A) = \frac{5}{20} = \frac{1}{4}.$$

So, the probability that all three numbers are less than or equal to 5 is

$$P(A) \cdot P(A) \cdot P(A) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{64}.$$