8.4 The Binomial Theorem

What You Should Learn

- Use the Binomial Theorem to calculate binomial coefficients.
- Use binomial coefficients to write binomial expansions.
- Use Pascal's Triangle to calculate binomial coefficients.

Binomial Coefficients

Binomial Coefficients

We know that a *binomial* is a polynomial that has two terms. In this section, you will study a formula that provides a quick method of raising a binomial to a power.

To begin, look at the expansion of

$(x + y)^n$

for several values of n

$(x + y)^0 = 1$

$(x + y)^1 = (x + y)$

$(x + y)^2 = x^2 + 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Binomial Coefficients

$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

There are several observations you can make about these expansions.

1. In each expansion, there are n + 1 terms.

2. In each expansion, x and y have symmetric roles. The powers of x decrease by 1 in successive terms, whereas the powers of y increase by 1.

Binomial Coefficients

3. The sum of the powers of each term is n. For instance, in the expansion of

$(x + y)^5$

the sum of the powers of each term is 5.

$$\overbrace{\quad}^{4+1=5} \qquad \overbrace{\quad}^{3+2=5}$$

$$(x + y)^5 = x^5 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + y^5$$

4. The coefficients increase and then decrease in a symmetric pattern.

The coefficients of a binomial expansion are called **binomial coefficients.**

Binomial Coefficients

To find them, you can use the **Binomial Theorem**.

The Binomial Theorem

In the expansion of $(x + y)^n$

$$(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_nC_r x^{n-r} y^r + \cdots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r} y^r$ is

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

The symbol

$$\binom{n}{r}$$

is often used in place of ${}_nC_r$ to denote binomial coefficients.

Example 1 – *Finding Binomial Coefficients*

Find each binomial coefficient.

a. ${}_8C_2$

b. $\binom{10}{3}$

c. ${}_7C_0$

d. $\binom{8}{8}$

Solution:

a. $${}_8C_2 = \frac{8!}{6! \cdot 2!} = \frac{(8 \cdot 7) \cdot \cancel{6!}}{\cancel{6!} \cdot 2!} = \frac{8 \cdot 7}{2 \cdot 1} = 28$$

b. $$\binom{10}{3} = \frac{10!}{7! \cdot 3!} = \frac{(10 \cdot 9 \cdot 8) \cdot \cancel{7!}}{\cancel{7!} \cdot 3!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

Example 1 – *Solution*

cont'd

c. $${}_7C_0 = \frac{\cancel{7!}}{\cancel{7!} \cdot 0!} = 1$$

d. $$\binom{8}{8} = \frac{\cancel{8!}}{0! \cdot \cancel{8!}} = 1$$

Binomial Expansions

Binomial Expansions

When you write out the coefficients for a binomial that is raised to a power, you are **expanding a binomial.**

The formulas for binomial coefficients give you an easy way to expand binomials, as demonstrated in the next example.

Example 3 – *Expanding a Binomial*

Write the expansion of the expression $(x + 1)^3$.

Solution:

The binomial coefficients are

${}_3C_0 = 1$, ${}_3C_1 = 3$, ${}_3C_2 = 3$, and ${}_3C_3 = 1$.

Therefore, the expansion is as follows.

$(x + 1)^3 = (1)x^3 + (3)x^2(1) + (3)x(1^2) + (1)(1^3)$

$= x^3 + 3x^2 + 3x + 1$

Binomial Expansions

Sometimes you will need to find a specific term in a binomial expansion.

Instead of writing out the entire expansion, you can use the fact that, from the Binomial Theorem, the $(r + 1)$th term is

$${}_nC_r\, x^{n-r}y^r.$$

Example 7 – *Finding a Term or Coefficient in a Binomial Expansion*

a. Find the sixth term of $(a + 2b)^8$.

b. Find the coefficient of the term a^6b^5 in the expansion of $(2a - 5b)^{11}$.

Solution:

a. Because the formula is for the $(r + 1)$th term, r is one less than the number of the term you need. So, to find the sixth term in this binomial expansion, use $r = 5$, $n = 8$, $x = a$ and $y = 2b$.

$$_nC_r\, x^{n-r}y^r = {}_8C_5 a^{8-5}(2b)^5$$

Example 7 – *Solution*

cont'd

$$= 56 \cdot a^3 \cdot (2b)^5$$

$$= 56(2^5)a^3b^5$$

$$= 1792a^3b^5$$

b. Note that

$$(2a - 5b)^{11} = [2a + (-5b)]^{11}.$$

So,

$n = 11$, $r = 5$, $x = 2a$, and $y = -5b$.

Example 7 – *Solution*

cont'd

Substitute these values to obtain

$${}_nC_r\, x^{n-r}y^r = {}_{11}C_5(2a)^6(-5b)^5$$

$$= (462)(64a^6)(-3125b^5)$$

$$= -92{,}400{,}000a^6b^5.$$

So, the coefficient is –92,400,000.

Pascal’s Triangle

Pascal's Triangle

There is a convenient way to remember the pattern for binomial coefficients. By arranging the coefficients in a triangular pattern, you obtain the following array, which is called **Pascal's Triangle.** This triangle is named after the famous French mathematician Blaise Pascal (1623–1662).

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1 — 4 + 6 = 10

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1 — 15 + 6 = 21

Pascal's Triangle

The first and last number in each row of Pascal's Triangle is 1. Every other number in each row is formed by adding the two numbers immediately above the number. Pascal noticed that the numbers in this triangle are precisely the same numbers as the coefficients of binomial expansions, as follows.

$$(x + y)^0 = 1 \qquad \text{0th row}$$

$$(x + y)^1 = 1x + 1y \qquad \text{1st row}$$

$$(x + y)^2 = 1x^2 + 2xy + 1y^2 \qquad \text{2nd row}$$

$$(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3 \qquad \text{3rd row}$$

$$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 \qquad \vdots$$

$$(x + y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$

$$(x + y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6$$

$$(x + y)^7 = 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7$$

Pascal's Triangle

The top row of Pascal's Triangle is called the *zeroth row* because it corresponds to the binomial expansion

$(x + y)^0 = 1.$

Similarly, the next row is called the *first row* because it corresponds to the binomial expansion

$(x + y)^1 = 1(x) + 1(y).$

In general, the *nth row* of Pascal's Triangle gives the coefficients of $(x + y)^n$.

Example 8 – *Using Pascal's Triangle*

Use the seventh row of Pascal's Triangle to find the binomial coefficients.

${}_8C_0 \quad {}_8C_1 \quad {}_8C_2 \quad {}_8C_3 \quad {}_8C_4 \quad {}_8C_5 \quad {}_8C_6 \quad {}_8C_7 \quad {}_8C_8$

Solution: