7.7 The Determinant of a Square Matrix

What You Should Learn

- Find the determinants of 2 × 2 matrices.
- Find minors and cofactors of square matrices.
- Find the determinants of square matrices.

The Determinant of a 2 × 2 Matrix

The Determinant of a 2 $\times$ 2 Matrix

Every *square* matrix can be associated with a real number called its **determinant.** Determinants have many uses, and several will be discussed in this section.

Historically, the use of determinants arose from special number patterns that occur when systems of linear equations are solved.

The Determinant of a 2 × 2 Matrix

Coefficient Matrix *Determinant*

det $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ a_2b_1

Definition of the Determinant of a 2 × 2 Matrix

The **determinant** of the matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

is given by

$$\det(A) = |A|$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$= a_1b_2 - a_2b_1.$$

Example 1 – *The Determinant of a 2 × 2 Matrix*

Find the determinant of each matrix.

a. $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{bmatrix}$

Solution:

a. $\det(A) = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}$

$= 2(2) - 1(-3)$

$= 4 + 3$

$= 7$

Example 1 – *Solution*

cont'd

b. $\det(B) = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix}$

$= 2(2) – 4(1)$

$= 4 – 4$

$= 0$

c. $\det(C) = \begin{vmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{vmatrix}$

$= 0(4) – 2\left(\frac{3}{2}\right)$

$= 0 – 3$

$= –3$

Minors and Cofactors

Minors and Cofactors

To define the determinant of a square matrix of dimension 3 × 3 or higher, it is helpful to introduce the concepts of **minors** and **cofactors**.

Minors and Cofactors of a Square Matrix

If A is a square matrix, then the **minor** M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The **cofactor** C_{ij} of the entry a_{ij} is given by

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

Example 2 – *Finding the Minors and Cofactors of a Matrix*

Find all the minors and cofactors of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

Solution:

To find the minor

M_{11}

delete the first row and first column of *A* and evaluate the determinant of the resulting matrix.

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix},$$

Example 2 – *Solution*

cont'd

$$M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix}$$

= –1(1) – 0(2)

= –1

Similarly, to find the minor

M_{12}

delete the first row and second column.

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix},$$

Example 2 – *Solution*

cont'd

$$M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$$

$= 3(1) – 4(2)$

$= –5$

Continuing this pattern, you obtain all the minors.

$M_{11} = –1$ $M_{12} = –5$ $M_{13} = 4$

$M_{21} = 2$ $M_{22} = –4$ $M_{23} = –8$

$M_{31} = 5$ $M_{32} = –3$ $M_{33} = –6$

Example 2 – *Solution*

cont'd

Now, find the cofactors, using the definition of Minors and Cofactors of a Square Matrix for a 3 × 3 matrix.

$C_{11} = -1$ $C_{12} = 5$ $C_{13} = 4$

$C_{21} = -2$ $C_{22} = -4$ $C_{23} = 8$

$C_{31} = 5$ $C_{32} = 3$ $C_{33} = -6$

The Determinant of a Square Matrix

The Determinant of a Square Matrix

The following definition is called *inductive* because it uses determinants of matrices of dimension $(n – 1) \times (n – 1)$ to define determinants of matrices of dimension $n \times n$.

Determinant of a Square Matrix

If A is a square matrix (of dimension 2×2 or greater), then the determinant of A is the sum of the entries in any row (or column) of A multiplied by their respective cofactors. For instance, expanding along the first row yields

$$|A| = a_{11}C_{11} + a_{12}C_{12} + \cdot \cdot \cdot + a_{1n}C_{1n}.$$

Applying this definition to find a determinant is called **expanding by cofactors.**

Example 3 – *The Determinant of a Matrix of Dimension 3 × 3*

Find the determinant of $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$.

Solution:

Note that this is the same matrix that was in Example 2. There you found the cofactors of the entries in the first row to be

$C_{11} = -1$, $C_{12} = 5$, and $C_{13} = 4$.

Example 3 – *Solution*

cont'd

So, by the definition of the determinant of a square matrix, you have

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$ First-row expansion

$$= 0(-1) + 2(5) + 1(4)$$

$$= 14.$$

If slides #12-17 were confusing, you may prefer the method on slide #18.

An Alternative Computational Approach

The determinant of a 3x3 matrix can also be found using the following method: repeat the first two columns of the matrix to the right of the last column.

a b c a b

d e f d e

g h k g h

Now, the determinant is the sum of the products of the upper left to lower right diagonals minus the sum of the product of the upper right to lower left diagonals: $|A|=(aek+bfg+cdh)-(bdk+afh+ceg)$.