6.3

Vectors in the Plane

What You Should Learn

- Represent vectors as directed line segments.
- Write the component forms of vectors.
- Perform basic vector operations and represent vectors graphically.
- Write vectors as linear combinations of unit vectors.

What You Should Learn

- Find the direction angles of vectors.
- Use vectors to model and solve real-life problems.

Introduction

Introduction

Many quantities in geometry and physics, such as area, time, and temperature, can be represented by a single real number. Other quantities, such as force and velocity, involve both *magnitude* and *direction* and cannot be completely characterized by a single real number. To represent such a quantity, you can use a **directed line segment,** as shown in Figure 6.17.

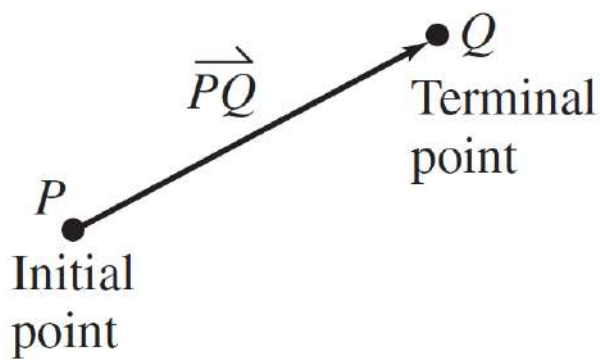

Figure 6.17

Introduction

The set of all directed line segments that are equivalent to a given directed line segment $\overrightarrow{PQ}$ **vector v in the plane**, written (meaning start at P and go towards/through Q):

$$\mathbf{v} = \overrightarrow{PQ}.$$

Vectors are denoted by lowercase, boldface letters such as **u**, **v** and **w**.

Example 1 – *Equivalent Directed Line Segments*

Let **u** be represented by the directed line segment from

P(0, 0) to *Q*(3, 2)

and let **v** be represented by the directed line segment from

R(1, 2) to *S*(4, 4)

as shown in Figure 6.19.
Show that **u** = **v**.

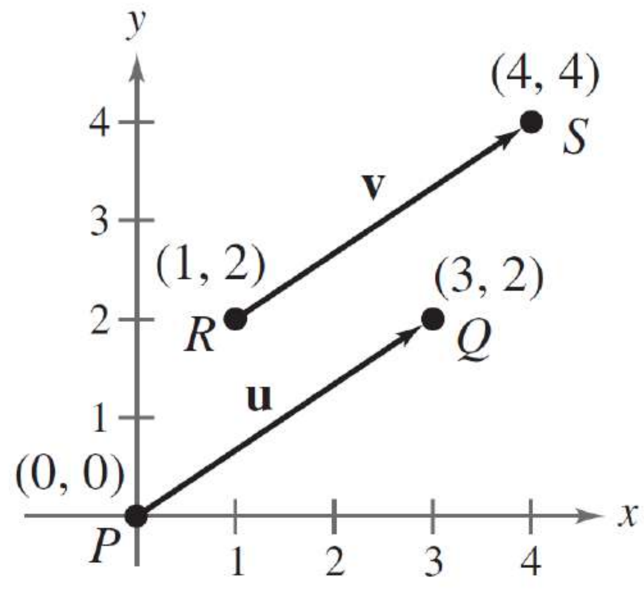

Figure 6.19

Example 1 – *Solution*

From the Distance Formula, it follows that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the *same magnitude.*

$$\|\overrightarrow{PQ}\| = \sqrt{(3-0)^2 + (2-0)^2} = \sqrt{13}$$

$$\|\overrightarrow{RS}\| = \sqrt{(4-1)^2 + (4-2)^2} = \sqrt{13}$$

Moreover, both line segments have the *same direction,* because they are both directed toward the upper right on lines having the same slope.

$$\text{Slope of } \overrightarrow{PQ} = \frac{2-0}{3-0} = \frac{2}{3}$$

Example 1 – *Solution*

cont'd

$$\text{Slope of } \overrightarrow{RS} = \frac{4-2}{4-1} = \frac{2}{3}$$

So, $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same magnitude and direction, and it follows that **u** = **v**.

Component Form of a Vector

Component Form of a Vector

The directed line segment whose initial point is the origin is often the most convenient representative of a set of equivalent directed line segments. This representative of the vector **v** is in **standard position.**

A vector whose initial point is at the origin (0, 0) can be uniquely represented by the coordinates of its terminal point (v_1, v_2). This is the **component form of a vector v**, written as

$$\mathbf{v} = \langle v_1, v_2 \rangle$$

.

Component Form of a Vector

The coordinates v_1 and v_2 are the *components* of **v**. If both the initial point and the terminal point lie at the origin, then **v** is the **zero vector** and is denoted by $\mathbf{0} = \langle 0, 0 \rangle$.

Component Form of a Vector

The component form of the vector with initial point $P(p_1, p_2)$ and terminal point $Q(q_1, q_2)$ is given by

$$\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle v_1, v_2 \rangle = \mathbf{v}.$$

The **magnitude** (or length) of **v** is given by

$$\|\mathbf{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{v_1^2 + v_2^2}.$$

If $\|\mathbf{v}\| = 1$, then **v** is a **unit vector.** Moreover, $\|\mathbf{v}\| = 0$ if and only if **v** is the zero vector **0**.

Component Form of a Vector

Two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ e *equ* $\mathbf{v} = \langle v_1, v_2 \rangle$ ly if $u_1 = v_1$ and $u_2 = v_2$.

For instance, in Example 1, the vector **u** from *P*(0, 0) to *Q*(3, 2) is

$$\mathbf{u} = \overrightarrow{PQ} = \langle 3 - 0, 2 - 0 \rangle = \langle 3, 2 \rangle$$

and the vector **v** from *R*(1, 2) to *S*(4, 4) is

$$\mathbf{v} = \overrightarrow{RS} = \langle 4 - 1, 4 - 2 \rangle = \langle 3, 2 \rangle.$$

Example 1 – *Finding the Component Form of a Vector*

Find the component form and magnitude of the vector **v** that has initial point (4, –7) and terminal point (–1, 5).

Solution:

Let

$P(4, -7) = (p_1, p_2)$

and

$Q(-1, 5) = (q_1, q_2)$

as shown in Figure 6.20.

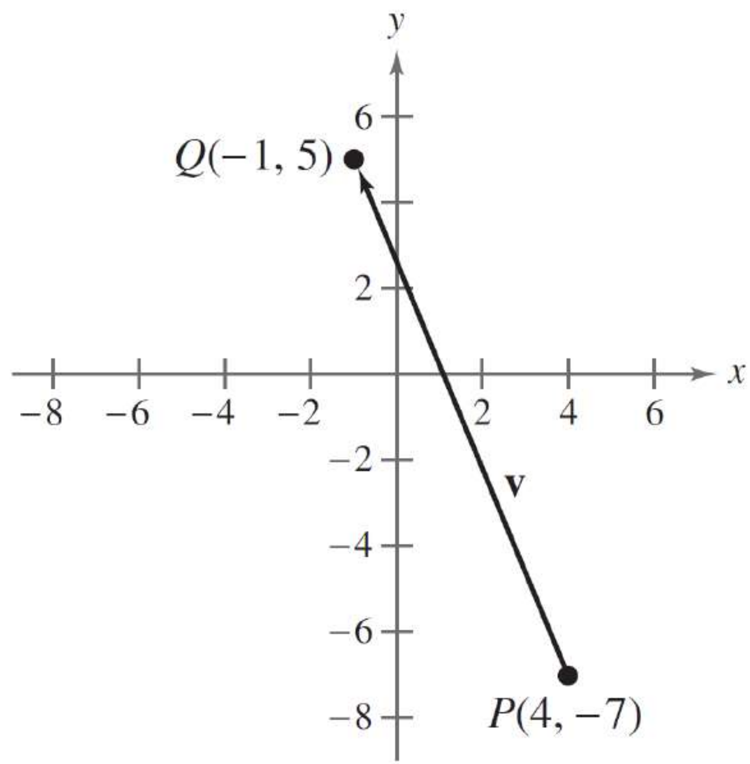

Figure 6.20

Example 1 – *Solution*

cont'd

Then, the components of $\mathbf{v} = \langle v_1, v_2 \rangle$

$v_1 = q_1 - p_1 = -1 - 4 = -5$

$v_2 = q_2 - p_2 = 5 - (-7) = 12.$

So, $\mathbf{v} = \langle -5, 12 \rangle$ and the magnitude of **v** is

$$\|\mathbf{v}\| = \sqrt{(-5)^2 + 12^2}$$

$$= \sqrt{169}$$

$$= 13.$$

Vector Operations

Vector Operations

The two basic vector operations are **scalar multiplication** and **vector addition.** Geometrically, the product of a vector **v** and a scalar *k* is the vector that is $|k|$ times as long as **v**. If *k* is positive, then *k***v** has the same direction as **v**, and if *k* is negative, then *k***v** has the opposite direction of **v**, as shown in Figure 6.21.

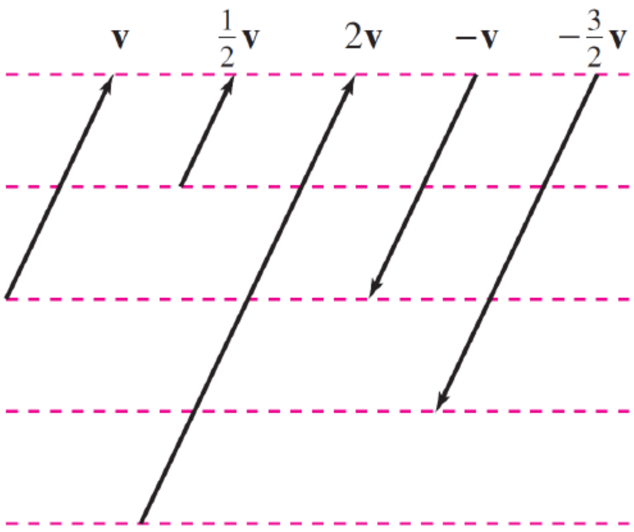

Figure 6.21

Vector Operations

To add two vectors **u** and **v** geometrically, first position them (without changing their lengths or directions) so that the initial point of the second vector **v** coincides with the terminal point of the first vector **u** (head to tail). The sum **u** + **v** is the vector formed by joining the initial point of the first vector **u** with the terminal point of the second vector **v**, as shown in Figure 6.22.

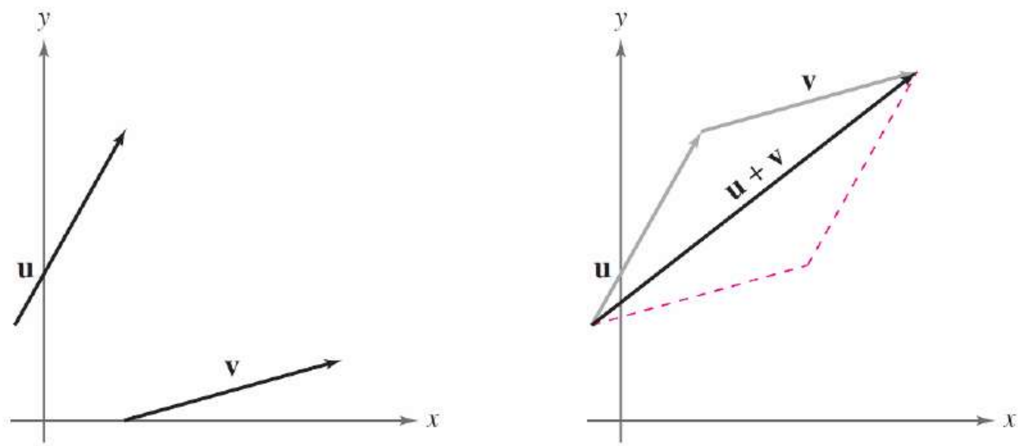

Figure 6.22

Vector Operations

The vector **u + v** is often called the **resultant** of vector addition

Definition of Vector Addition and Scalar Multiplication

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors and let k be a scalar (a real number). Then the **sum** of **u** and **v** is the vector

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle \qquad \text{Sum}$$

and the **scalar multiple** of k times **u** is the vector

$$k\mathbf{u} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle. \qquad \text{Scalar multiple}$$

Vector Operations

The **negative** of $\mathbf{v} = \langle v_1, v_2 \rangle$

$-\mathbf{v} = (-1)\mathbf{v}$

$$= \langle -v_1, -v_2 \rangle$$ Negative

and the **difference** of **u** and **v** is

$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$ Add (–v). See figure 6.23.

$$= \langle u_1 - v_1, u_2 - v_2 \rangle.$$ Difference

Example 3 – *Vector Operations*

Let $\mathbf{v} = \langle -2, 5 \rangle$ and $\mathbf{w} = \langle 3, 4 \rangle$, of the following vectors.

a. $2\mathbf{v}$ **b.** $\mathbf{w} - \mathbf{v}$ **c.** $\mathbf{v} + 2\mathbf{w}$

Solution:

a. Because $\mathbf{v} = \langle -2, 5 \rangle$ you have

$$2\mathbf{v} = 2\langle -2, 5 \rangle$$
$$= \langle 2(-2), 2(5) \rangle$$
$$= \langle -4, 10 \rangle.$$

A sketch of $2\mathbf{v}$ is shown in Figure 6.24.

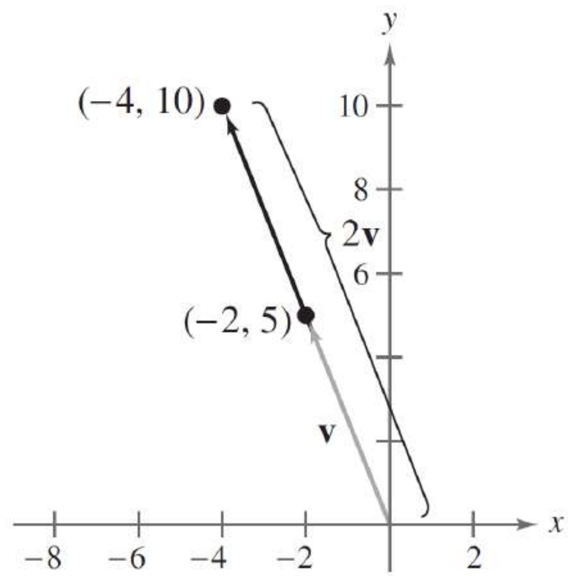

Figure 6.24

Example 3(b) – *Solution*

cont'd

The difference of **w** and **v** is

$$\begin{aligned}\mathbf{w} - \mathbf{v} &= \langle 3, 4\rangle - \langle -2, 5\rangle \\ &= \langle 3 - (-2), 4 - 5\rangle \\ &= \langle 5, -1\rangle.\end{aligned}$$

A sketch of **w** – **v** is shown in Figure 6.25. Note that the figure shows the vector difference **w** – **v** as the sum **w** + (– **v**).

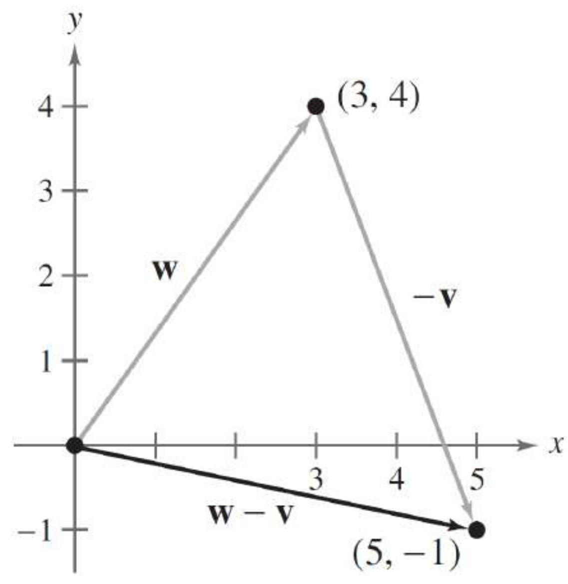

Figure 6.25

Example 3(c) – *Solution*

cont'd

The sum of **v** and 2**w** is

$$\begin{aligned}\mathbf{v} + 2\mathbf{w} &= \langle -2, 5\rangle + 2\langle 3, 4\rangle \\ &= \langle -2, 5\rangle + \langle 2(3), 2(4)\rangle \\ &= \langle -2, 5\rangle + \langle 6, 8\rangle \\ &= \langle -2 + 6, 5 + 8\rangle \\ &= \langle 4, 13\rangle.\end{aligned}$$

A sketch of **v** + 2**w** is shown in Figure 6.26. Read slide 24, but do not copy it down.

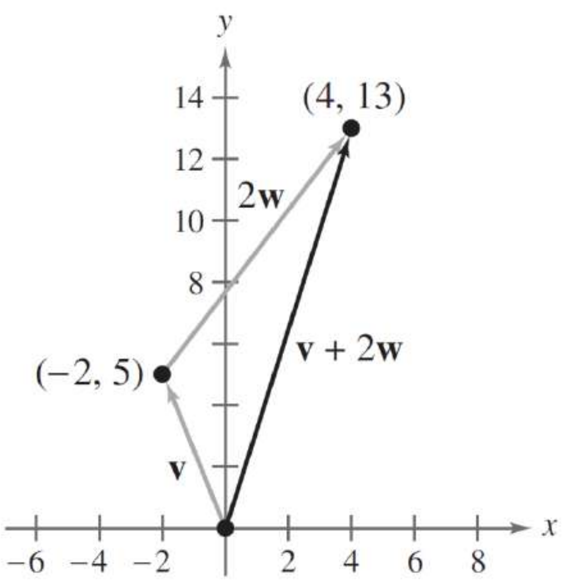

Figure 6.26

Vector Operations

Properties of Vector Addition and Scalar Multiplication

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors and let c and d be scalars. Then the following properties are true.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $c(d\mathbf{u}) = (cd)\mathbf{u}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $1(\mathbf{u}) = \mathbf{u}, 0(\mathbf{u}) = \mathbf{0}$
9. $\|c\mathbf{v}\| = |c| \, \|\mathbf{v}\|$

Unit Vectors

Unit Vectors

In many applications of vectors, it is useful to find a unit vector that has the same direction as a given nonzero vector **v**. To do this, you can divide **v** by its length to obtain

$$\mathbf{u} = \text{unit vector} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{1}{\|\mathbf{v}\|}\right)\mathbf{v}.$$

Unit vector in direction of **v**

Note that **u** is a scalar multiple of **v**. The vector **u** has a magnitude of 1 and the same direction as **v**. The vector **u** is called a **unit vector in the direction of v.**

Example 4 – *Finding a Unit Vector*

Find a unit vector in the direction of . $\mathbf{v} = \langle -2, 5 \rangle$

Solution:

The unit vector in the direction of **v** is

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -2, 5 \rangle}{\sqrt{(-2)^2 + 5^2}}$$

$$= \frac{1}{\sqrt{4 + 25}} \langle -2, 5 \rangle$$

$$= \frac{1}{\sqrt{29}} \langle -2, 5 \rangle$$

Example 4 – *Solution*

cont'd

$$= \left\langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$$

$$= \left\langle \frac{-2\sqrt{29}}{29}, \frac{5\sqrt{29}}{29} \right\rangle.$$

Unit Vectors

The unit vectors $\langle 1, 0 \rangle$d $\langle 0, 1 \rangle$ called the **standard unit vectors** and are denoted by

and $\mathbf{i} = \langle 1, 0 \rangle$ $\mathbf{j} = \langle 0, 1 \rangle$

as shown in Figure 6.27.(Note that the lowercase letter is written in boldface to distinguish it from the imaginary number i , $= \sqrt{-1}$.

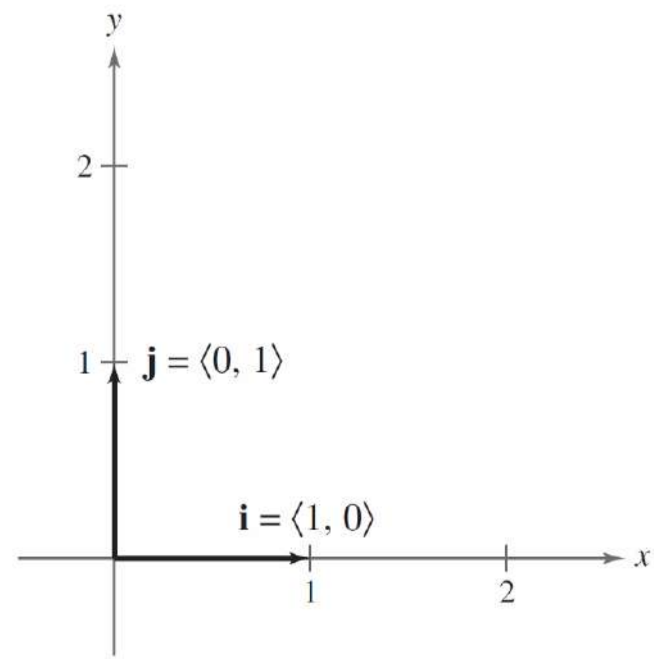

Figure 6.27

Unit Vectors

These vectors can be used to represent any vector $\mathbf{v} = \langle v_1, v_2 \rangle$ as follows.

$$\begin{aligned} \mathbf{v} &= \langle v_1, v_2 \rangle \\ &= v_1\langle 1, 0 \rangle + v_2\langle 0, 1 \rangle \\ &= v_1\mathbf{i} + v_2\mathbf{j} \end{aligned}$$

The scalars v_1 and v_2 are called the **horizontal and vertical components of v,** respectively. The vector sum $v_1\mathbf{i} + v_2\mathbf{j}$ a **linear combination** of the vectors **i** and **j**. Any vector in the plane can be written as a linear combination of the standard unit vectors **i** and **j**.

Example 5 – *Writing a Linear Combination of Unit Vectors*

Let **u** be the vector with initial point (2, –5) and terminal point (–1, 3). Write **u** as a linear combination of the standard unit vectors **i** and **j**.

Solution:

Begin by writing the component form of the vector **u**.

$$\mathbf{u} = \langle -1 - 2, 3 - (-5) \rangle$$

$$= \langle -3, 8 \rangle$$

$$= -3\mathbf{i} + 8\mathbf{j}$$

Direction Angles

Direction Angles

If **u** is a *unit vector* such that θ is the angle (measured counterclockwise) from the positive *x*-axis to **u**, then the terminal point of **u** lies on the unit circle and you have as

$$\mathbf{u} = \langle x, y \rangle = \langle \cos\theta, \sin\theta \rangle = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$$

shown in Figure 6.29. The angle θ is the **direction angle** of the vector **u**.

Read slides 33 & 34, but do not copy them down.

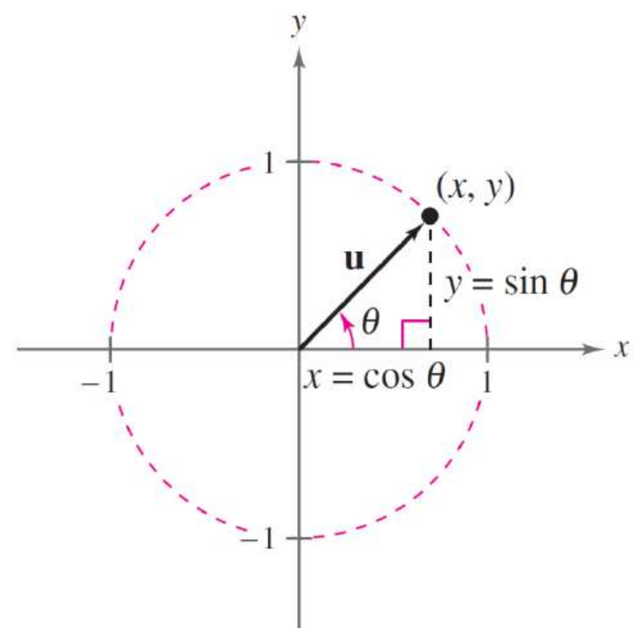

Figure 6.29

Direction Angles

Suppose that **u** is a unit vector with direction angle θ. If $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is any vector that makes an angle θ with the positive *x*-axis, then it has the same direction as **u** and you can write

$$\mathbf{v} = \|\mathbf{v}\|\langle\cos\theta, \sin\theta\rangle = \|\mathbf{v}\|(\cos\theta)\mathbf{i} + \|\mathbf{v}\|(\sin\theta)\mathbf{j}.$$

For instance, the vector **v** of length 3 that makes an angle of 30° with the positive *x*-axis is given by

$$\mathbf{v} = 3(\cos 30°)\mathbf{i} + 3(\sin 30°)\mathbf{j} = \frac{3\sqrt{3}}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}$$

where

$$\|\mathbf{v}\| = 3.$$

Direction Angles

Because $\mathbf{v} = a\mathbf{i} + b\mathbf{j} = \|\mathbf{v}\|(\cos\theta)\mathbf{i} + \|\mathbf{v}\|(\sin\theta)\mathbf{j}$, r **v** is determined from

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$ Quotient identity

$$= \frac{\|\mathbf{v}\|\sin\theta}{\|\mathbf{v}\|\cos\theta}$$ Multiply numerator and denominator by $\|\mathbf{v}\|$

$$= \frac{b}{a}.$$ Simplify.

Example 7 – *Finding Direction Angles of Vectors*

Find the direction angle of the vector.

$\mathbf{u} = 3\mathbf{i} + 3\mathbf{j}$

Solution:

The direction angle is

$$\tan \theta = \frac{b}{a} = \frac{3}{3} = 1.$$

So, $\theta = 45°$, as shown in Figure 6.30.

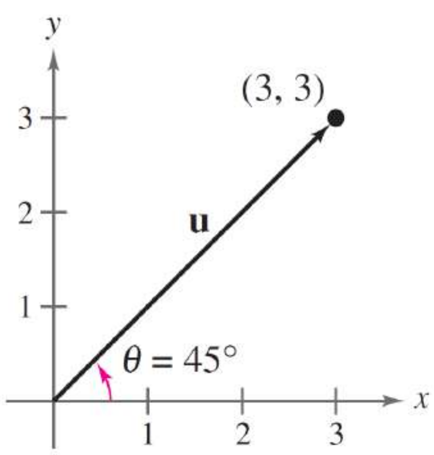

Figure 6.30