6.1

Vectors in the Plane

What You Should Learn

- Represent vectors as directed line segments.
- Write the component forms of vectors.
- Perform basic vector operations and represent vectors graphically.
- Write vectors as linear combinations of unit vectors.

What You Should Learn

- Find the direction angles of vectors.
- Use vectors to model and solve real-life problems.

Introduction

Introduction

Many quantities in geometry and physics, such as area, time, and temperature, can be represented by a single real number. Other quantities, such as force and velocity, involve both *magnitude* and *direction* and cannot be completely characterized by a single real number. To represent such a quantity, you can use a **directed line segment,** as shown in Figure 6.17.

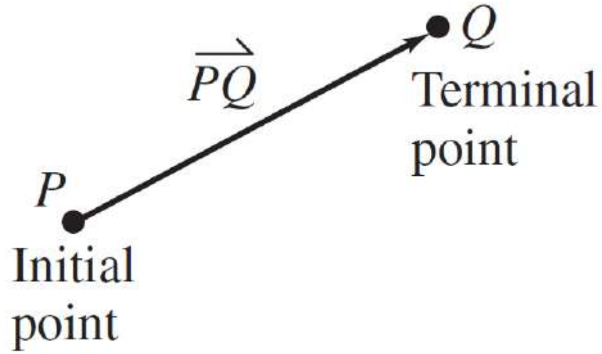

Figure 6.17

Introduction

The directed line segment $\overrightarrow{PQ}$ has **initial point** *P* and **terminal point** *Q*. Its **magnitude,** or **length,** is denoted by $\|\overrightarrow{PQ}\|$ can be found by using the Distance Formula. Two directed line segments that have the same magnitude and direction are *equivalent*. For example, the directed line segments in Figure 6.18 are all equivalent.

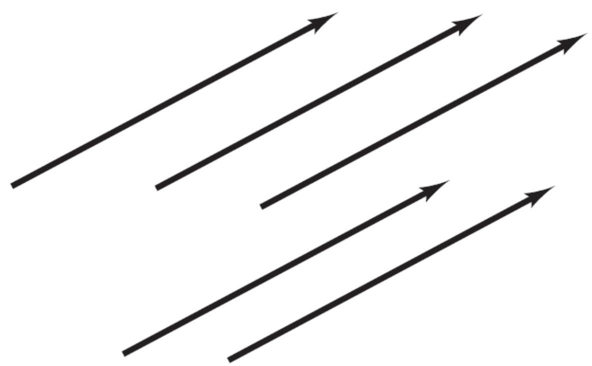

Figure 6.18

Introduction

The set of all directed line segments that are equivalent to a given directed line segment $\overrightarrow{PQ}$ **vector v in the plane**, written (meaning start at P and go towards/through Q):

$$\mathbf{v} = \overrightarrow{PQ}.$$

Vectors are denoted by lowercase, boldface letters such as **u**, **v** and **w**.

Example 1 – *Equivalent Directed Line Segments*

Let **u** be represented by the directed line segment from

P(0, 0) to *Q*(3, 2)

and let **v** be represented by the directed line segment from

R(1, 2) to *S*(4, 4)

as shown in Figure 6.19.
Show that **u** = **v**.

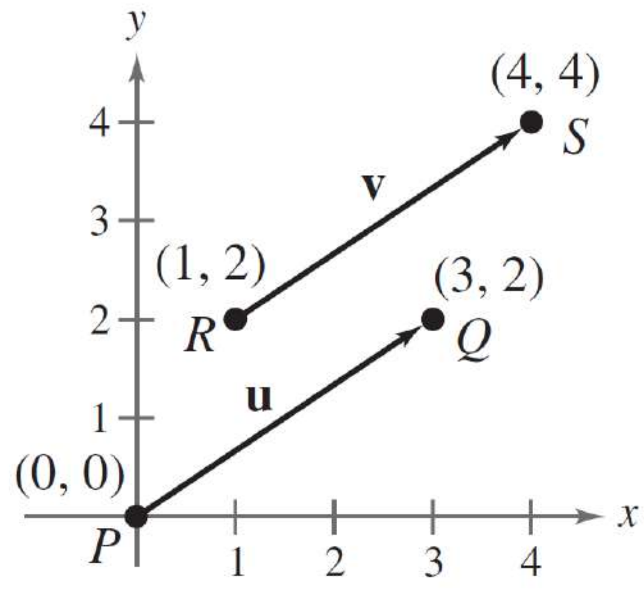

Figure 6.19

Example 1 – *Solution*

From the Distance Formula, it follows that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the *same magnitude.*

$$\|\overrightarrow{PQ}\| = \sqrt{(3-0)^2 + (2-0)^2} = \sqrt{13}$$

$$\|\overrightarrow{RS}\| = \sqrt{(4-1)^2 + (4-2)^2} = \sqrt{13}$$

Moreover, both line segments have the *same direction,* because they are both directed toward the upper right on lines having the same slope.

$$\text{Slope of } \overrightarrow{PQ} = \frac{2-0}{3-0} = \frac{2}{3}$$

Example 1 – *Solution*

cont'd

$$\text{Slope of } \overrightarrow{RS} = \frac{4-2}{4-1} = \frac{2}{3}$$

So, $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same magnitude and direction, and it follows that **u** = **v**.

Component Form of a Vector

Component Form of a Vector

The directed line segment whose initial point is the origin is often the most convenient representative of a set of equivalent directed line segments. This representative of the vector **v** is in **standard position.**

A vector whose initial point is at the origin (0, 0) can be uniquely represented by the coordinates of its terminal point (v_1, v_2). This is the **component form of a vector v**, written as

. $\mathbf{v} = \langle v_1, v_2 \rangle$

Component Form of a Vector

The coordinates v_1 and v_2 are the *components* of **v**. If both the initial point and the terminal point lie at the origin, then **v** is the **zero vector** and is denoted by $\mathbf{0} = \langle 0, 0 \rangle$.

Component Form of a Vector

The component form of the vector with initial point $P(p_1, p_2)$ and terminal point $Q(q_1, q_2)$ is given by

$$\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle v_1, v_2 \rangle = \mathbf{v}.$$

The **magnitude** (or length) of **v** is given by

$$\|\mathbf{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{v_1^2 + v_2^2}.$$

If $\|\mathbf{v}\| = 1$, then **v** is a **unit vector.** Moreover, $\|\mathbf{v}\| = 0$ if and only if **v** is the zero vector **0**.

Component Form of a Vector

Two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$e *equ*$\mathbf{v} = \langle v_1, v_2 \rangle$ly if $u_1 = v_1$ and $u_2 = v_2$.

For instance, in Example 1, the vector **u** from *P*(0, 0) to *Q*(3, 2) is

$$\mathbf{u} = \overrightarrow{PQ} = \langle 3 - 0, 2 - 0 \rangle = \langle 3, 2 \rangle$$

and the vector **v** from *R*(1, 2) to *S*(4, 4) is

$$\mathbf{v} = \overrightarrow{RS} = \langle 4 - 1, 4 - 2 \rangle = \langle 3, 2 \rangle.$$

Example 1 – *Finding the Component Form of a Vector*

Find the component form and magnitude of the vector **v** that has initial point (4, –7) and terminal point (–1, 5).

Solution:

Let

$P(4, -7) = (p_1, p_2)$

and

$Q(-1, 5) = (q_1, q_2)$

as shown in Figure 6.20.

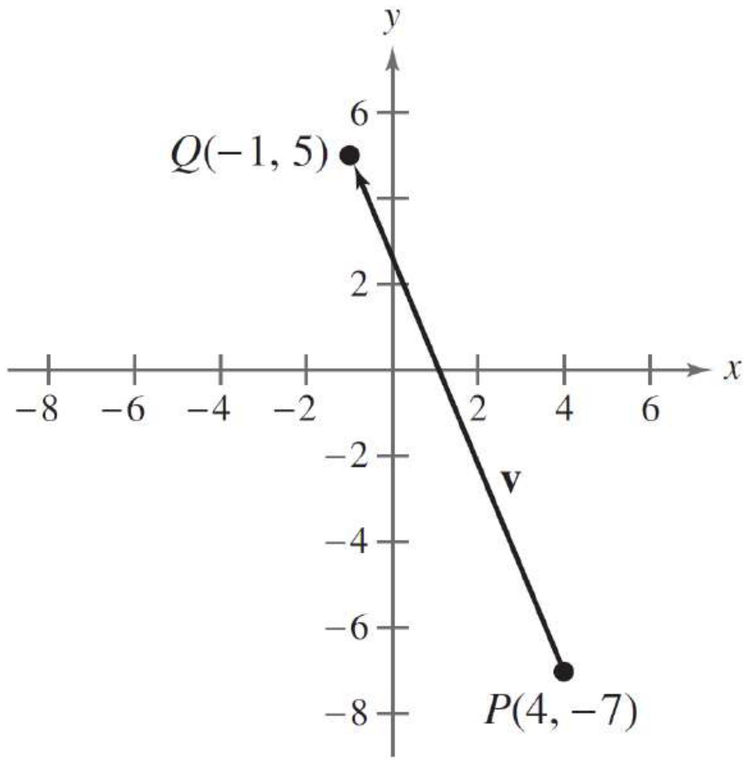

Figure 6.20

Example 1 – *Solution*

cont'd

Then, the components of $\mathbf{v} = \langle v_1, v_2 \rangle$

$v_1 = q_1 - p_1 = -1 - 4 = -5$

$v_2 = q_2 - p_2 = 5 - (-7) = 12.$

So, $\mathbf{v} = \langle -5, 12 \rangle$ and the magnitude of **v** is

$$\|\mathbf{v}\| = \sqrt{(-5)^2 + 12^2}$$

$$= \sqrt{169}$$

$$= 13.$$

Vector Operations

Vector Operations

The two basic vector operations are **scalar multiplication** and **vector addition.** Geometrically, the product of a vector **v** and a scalar *k* is the vector that is $|k|$ times as long as **v**. If *k* is positive, then *k***v** has the same direction as **v**, and if *k* is negative, then *k***v** has the opposite direction of **v**, as shown in Figure 6.21.

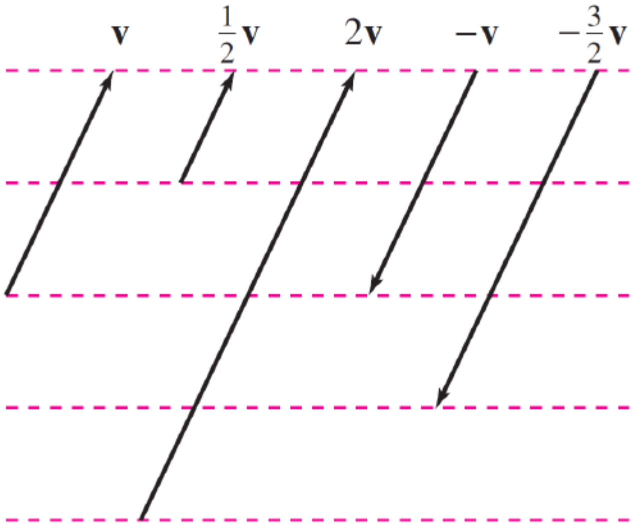

Figure 6.21

Vector Operations

To add two vectors **u** and **v** geometrically, first position them (without changing their lengths or directions) so that the initial point of the second vector **v** coincides with the terminal point of the first vector **u** (head to tail). The sum **u** + **v** is the vector formed by joining the initial point of the first vector **u** with the terminal point of the second vector **v**, as shown in Figure 6.22.

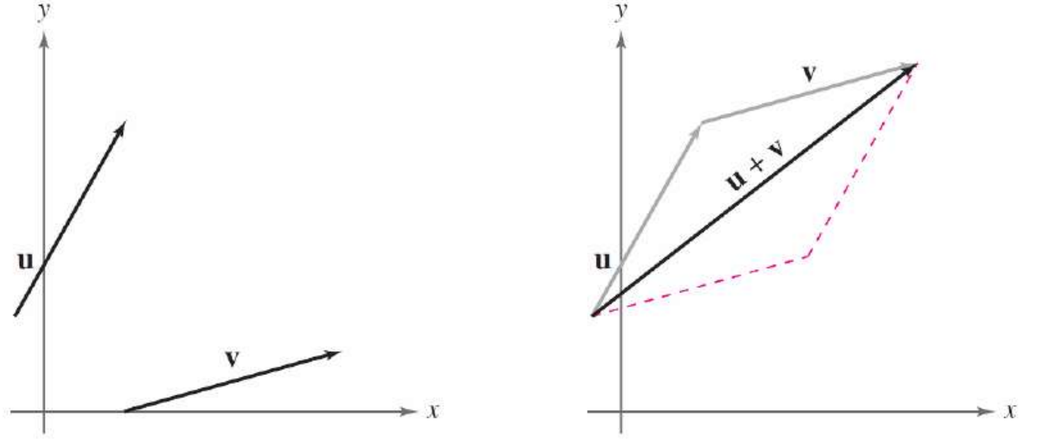

Figure 6.22

Vector Operations

This technique is called the **parallelogram law** for vector addition because the vector **u + v**, often called the **resultant** of vector addition, is the diagonal of a parallelogram having **u** and **v** as its adjacent sides.

Definition of Vector Addition and Scalar Multiplication

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors and let k be a scalar (a real number). Then the **sum** of **u** and **v** is the vector

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle \qquad \text{Sum}$$

and the **scalar multiple** of k times **u** is the vector

$$k\mathbf{u} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle. \qquad \text{Scalar multiple}$$

Vector Operations

The **negative** of $\mathbf{v} = \langle v_1, v_2 \rangle$

$-\mathbf{v} = (-1)\mathbf{v}$

$= \langle -v_1, -v_2 \rangle$ Negative

and the **difference** of **u** and **v** is

$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$ Add (–v). See figure 6.23.

$= \langle u_1 - v_1, u_2 - v_2 \rangle.$ Difference

Vector Operations

To represent **u** – **v** geometrically, you can use directed line segments with the *same* initial point. The difference **u** – **v** is the vector from the terminal point of **v** to the terminal point of **u**, which is equal to **u** + (–**v**) as shown in Figure 6.23.

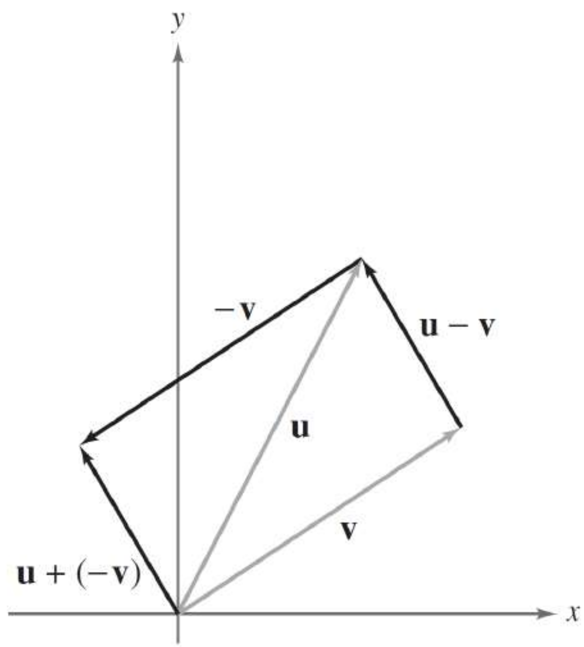

Figure 6.23

Vector Operations

The component definitions of vector addition and scalar multiplication are illustrated in Example 3.

In this example, notice that each of the vector operations can be interpreted geometrically.

Example 3 – *Vector Operations*

Let $\mathbf{v} = \langle -2, 5 \rangle$ and $\mathbf{w} = \langle 3, 4 \rangle$, of the following vectors.

a. 2**v** **b.** **w** – **v** **c.** **v** + 2**w**

Solution:

a. Because you have

$$\begin{aligned} \mathbf{v} &= \langle -2, 5 \rangle \\ 2\mathbf{v} &= 2\langle -2, 5 \rangle \\ &= \langle 2(-2), 2(5) \rangle \\ &= \langle -4, 10 \rangle. \end{aligned}$$

A sketch of 2**v** is shown in Figure 6.24.

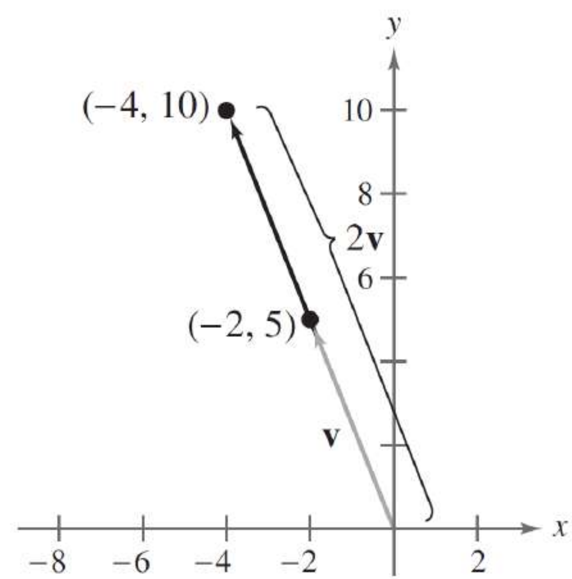

Figure 6.24

Example 3(b) – *Solution*

cont'd

The difference of **w** and **v** is

$$\begin{aligned} \mathbf{w} - \mathbf{v} &= \langle 3, 4 \rangle - \langle -2, 5 \rangle \\ &= \langle 3 - (-2), 4 - 5 \rangle \\ &= \langle 5, -1 \rangle. \end{aligned}$$

A sketch of **w** – **v** is shown in Figure 6.25. Note that the figure shows the vector difference **w** – **v** as the sum **w** + (– **v**).

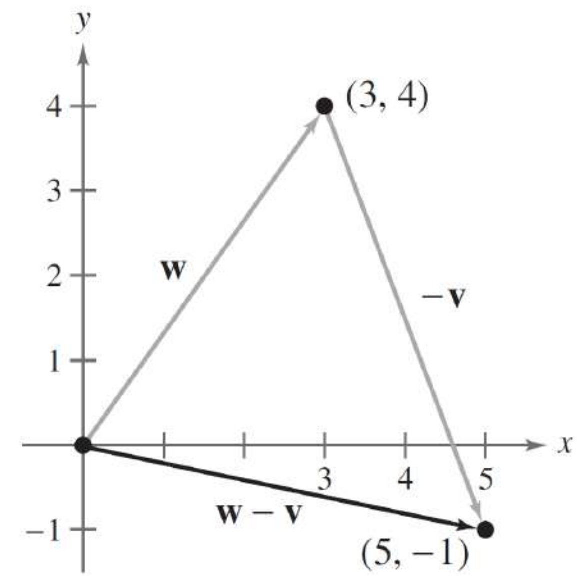

Figure 6.25

Example 3(c) – *Solution*

cont'd

The sum of **v** and 2**w** is

$$\mathbf{v} + 2\mathbf{w} = \langle -2, 5\rangle + 2\langle 3, 4\rangle$$

$$= \langle -2, 5\rangle + \langle 2(3), 2(4)\rangle$$

$$= \langle -2, 5\rangle + \langle 6, 8\rangle$$

$$= \langle -2 + 6, 5 + 8\rangle$$

$$= \langle 4, 13\rangle.$$

A sketch of **v** + 2**w** is shown in Figure 6.26.

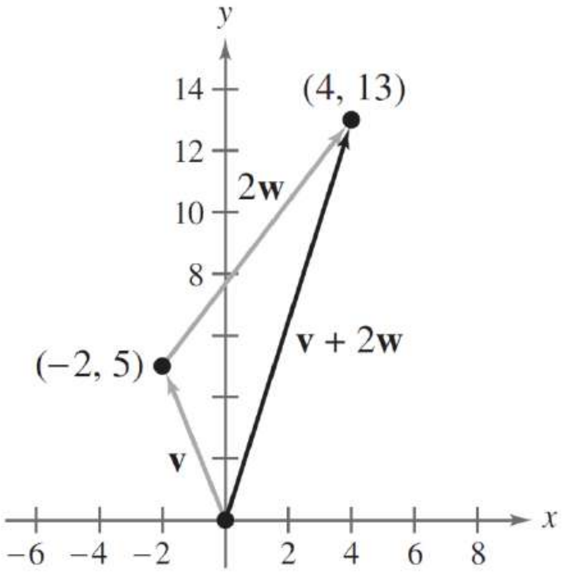

Figure 6.26

Vector Operations

Properties of Vector Addition and Scalar Multiplication

Let **u**, **v**, and **w** be vectors and let c and d be scalars. Then the following properties are true.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $c(d\mathbf{u}) = (cd)\mathbf{u}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $1(\mathbf{u}) = \mathbf{u}, 0(\mathbf{u}) = \mathbf{0}$
9. $\|c\mathbf{v}\| = |c| \, \|\mathbf{v}\|$

Unit Vectors

Unit Vectors

In many applications of vectors, it is useful to find a unit vector that has the same direction as a given nonzero vector **v**. To do this, you can divide **v** by its length to obtain

$$\mathbf{u} = \text{unit vector} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{1}{\|\mathbf{v}\|}\right)\mathbf{v}.$$

Unit vector in direction of **v**

Note that **u** is a scalar multiple of **v**. The vector **u** has a magnitude of 1 and the same direction as **v**. The vector **u** is called a **unit vector in the direction of v.**

Example 4 – *Finding a Unit Vector*

Find a unit vector in the direction of $\mathbf{v} = \langle -2, 5 \rangle$ that the result has a magnitude of 1.

Solution:

The unit vector in the direction of **v** is

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -2, 5 \rangle}{\sqrt{(-2)^2 + 5^2}}$$

$$= \frac{1}{\sqrt{4 + 25}} \langle -2, 5 \rangle$$

$$= \frac{1}{\sqrt{29}} \langle -2, 5 \rangle$$

Example 4 – *Solution*

cont'd

$$= \left\langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$$

$$= \left\langle \frac{-2\sqrt{29}}{29}, \frac{5\sqrt{29}}{29} \right\rangle.$$

This vector has a magnitude of 1 because

$$\sqrt{\left(\frac{-2\sqrt{29}}{29}\right)^2 + \left(\frac{5\sqrt{29}}{29}\right)^2} = \sqrt{\frac{116}{841} + \frac{725}{841}}$$

$$= \sqrt{\frac{841}{841}} = 1.$$

Unit Vectors

The unit vectors $\langle 1, 0 \rangle$d $\langle 0, 1 \rangle$ called the **standard unit vectors** and are denoted by

and $\mathbf{i} = \langle 1, 0 \rangle$ $\mathbf{j} = \langle 0, 1 \rangle$

as shown in Figure 6.27.(Note that the lowercase letter is written in boldface to distinguish it from the imaginary number $i = \sqrt{-1}$.

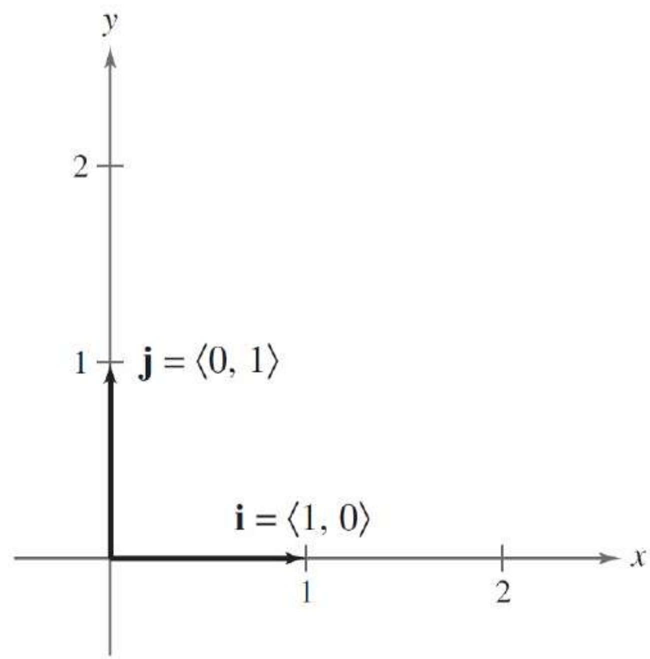

Figure 6.27

Unit Vectors

These vectors can be used to represent any vector $\mathbf{v} = \langle v_1, v_2 \rangle$ as follows.

$$\begin{aligned} \mathbf{v} &= \langle v_1, v_2 \rangle \\ &= v_1\langle 1, 0 \rangle + v_2\langle 0, 1 \rangle \\ &= v_1\mathbf{i} + v_2\mathbf{j} \end{aligned}$$

The scalars v_1 and v_2 are called the **horizontal and vertical components of v,** respectively. The vector sum $v_1\mathbf{i} + v_2\mathbf{j}$ a **linear combination** of the vectors **i** and **j**. Any vector in the plane can be written as a linear combination of the standard unit vectors **i** and **j**.

Example 5 – *Writing a Linear Combination of Unit Vectors*

Let **u** be the vector with initial point (2, –5) and terminal point (–1, 3). Write **u** as a linear combination of the standard unit vectors **i** and **j**.

Solution:

Begin by writing the component form of the vector **u**.

$$\mathbf{u} = \langle -1 - 2, 3 - (-5) \rangle$$

$$= \langle -3, 8 \rangle$$

$$= -3\mathbf{i} + 8\mathbf{j}$$

Example 5 – *Solution*

cont'd

This result is shown graphically in Figure 6.28.

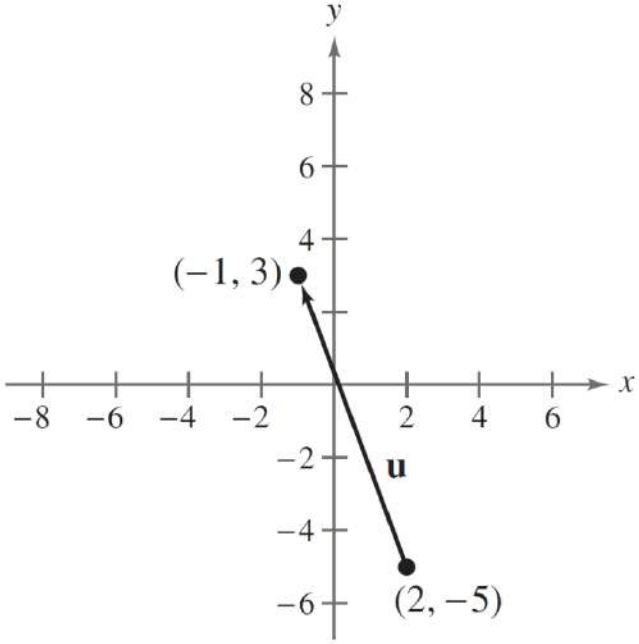

Figure 6.28

Direction Angles

Direction Angles

If **u** is a *unit vector* such that θ is the angle (measured counterclockwise) from the positive *x*-axis to **u**, then the terminal point of **u** lies on the unit circle and you have as

$$\mathbf{u} = \langle x, y \rangle = \langle \cos\theta, \sin\theta \rangle = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$$

shown in Figure 6.29. The angle θ is the **direction angle** of the vector **u**.

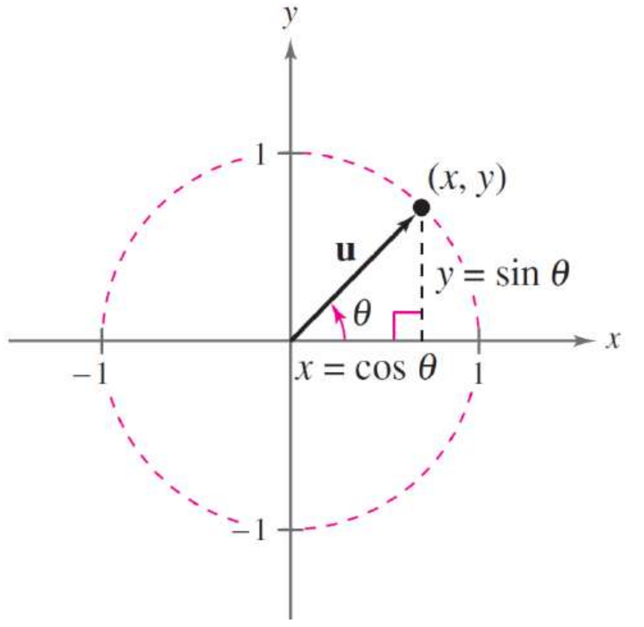

Figure 6.29

Direction Angles

Suppose that **u** is a unit vector with direction angle θ. If $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is any vector that makes an angle θ with the positive *x*-axis, then it has the same direction as **u** and you can write

$$\mathbf{v} = \|\mathbf{v}\|\langle\cos\theta, \sin\theta\rangle = \|\mathbf{v}\|(\cos\theta)\mathbf{i} + \|\mathbf{v}\|(\sin\theta)\mathbf{j}.$$

For instance, the vector **v** of length 3 that makes an angle of 30° with the positive *x*-axis is given by

$$\mathbf{v} = 3(\cos 30^\circ)\mathbf{i} + 3(\sin 30^\circ)\mathbf{j} = \frac{3\sqrt{3}}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}$$

where

$$\|\mathbf{v}\| = 3.$$

Direction Angles

Because $\mathbf{v} = a\mathbf{i} + b\mathbf{j} = \|\mathbf{v}\|(\cos\theta)\mathbf{i} + \|\mathbf{v}\|(\sin\theta)\mathbf{j}$, r **v** is determined from

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$ Quotient identity

$$= \frac{\|\mathbf{v}\|\sin\theta}{\|\mathbf{v}\|\cos\theta}$$ Multiply numerator and denominator by $\|\mathbf{v}\|$

$$= \frac{b}{a}.$$ Simplify.

Example 7 – *Finding Direction Angles of Vectors*

Find the direction angle of each vector.

a. u = 3**i** + 3**j**   **b. v** = 3**i** – 4**j**

Solution:

a. The direction angle is

$$\tan \theta = \frac{b}{a} = \frac{3}{3} = 1.$$

So, $\theta = 45°$, as shown in Figure 6.30.

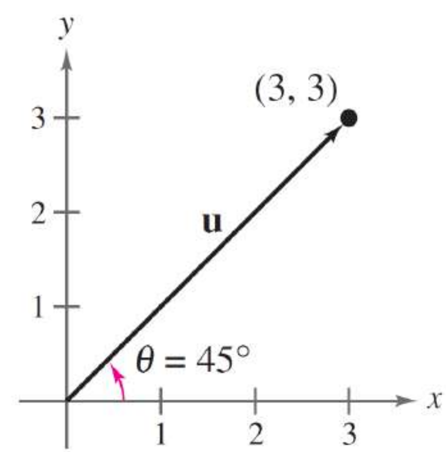

Figure 6.30

Example 7(b) – *Solution*

cont'd

The direction angle is

$$\tan\theta = \frac{b}{a} = \frac{-4}{3}.$$

Moreover, because **v** = 3**i** – 4**j** lies in Quadrant IV, θ lies in Quadrant IV and its reference angle is

$$\theta' = \left|\arctan\left(-\frac{4}{3}\right)\right| \approx |-53.13°|$$
$$= 53.13°.$$

Example 7 – *Solution*

cont'd

So, it follows that

$\theta \approx 360° - 53.13°$ $= 306.87°$

as shown in Figure 6.31.

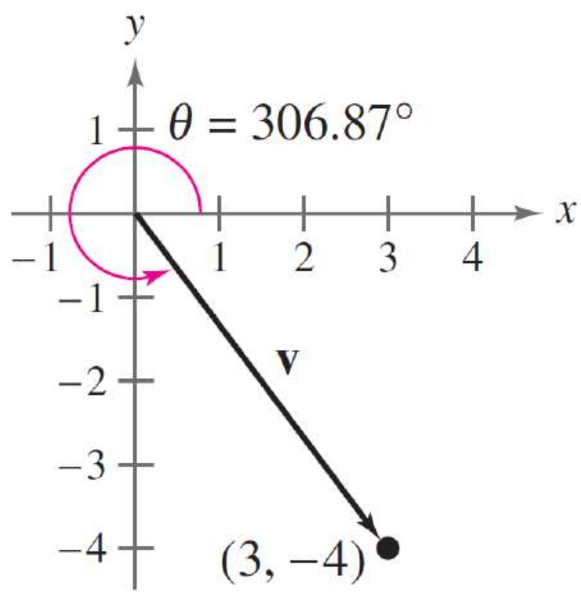

Figure 6.31

Applications

Example 9 – *Using Vectors to Determine Weight*

A force of 600 pounds is required to pull a boat and trailer up a ramp inclined at 15° from the horizontal. Find the combined weight of the boat and trailer.

Solution:

Based on Figure 6.33, you can make the following observations.

$\|\overrightarrow{BA}\|$ orce of gravity

= combined weight of boat and trailer

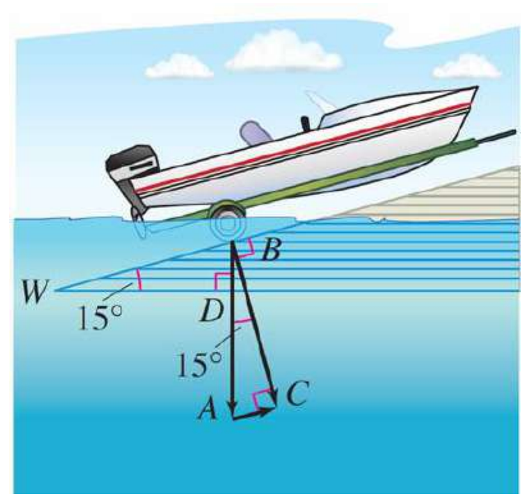

Figure 6.33

Example 9 – *Solution*

cont'd

$\|\overrightarrow{BC}\|$ = force against ramp

$\|\overrightarrow{AC}\|$ = force required to move boat up ramp = 600 pounds

By construction, triangles *BWD* and *ABC* are similar. So, angle *ABC* is 15°.

In triangle *ABC* you have

$$\sin 15° = \frac{\|\overrightarrow{AC}\|}{\|\overrightarrow{BA}\|}$$

$$\sin 15° = \frac{600}{\|\overrightarrow{BA}\|}$$

Example 9 – *Solution*

cont'd

$$\|\overrightarrow{BA}\| = \frac{600}{\sin 15^\circ}$$

$$\|\overrightarrow{BA}\| \approx 2318.$$

So, the combined weight is approximately 2318 pounds. (In Figure 6.33, note that $\overrightarrow{AC}$ is parallel to the ramp.)

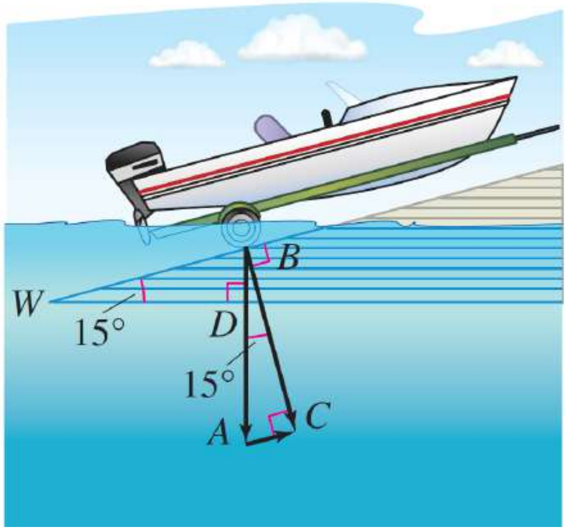

Figure 6.33