6

Additional Topics in Trigonometry

6.1 Law of Sines

What You Should Learn

- I had to leave that picture in there for a holiday greeting. ☺
- Use the Law of Sines to solve oblique triangles (AAS or ASA)
- Use the Law of Sines to solve oblique triangles (SSA)
- Find areas of oblique triangles and use the Law of Sines to model and solve real-life problems

Introduction

Introduction

In this section let us solve **oblique triangles**—triangles that have no right angles. As standard notation, the angles of a triangle are labeled

A, B and C

and their opposite sides are labeled

a, b, c

as shown in Figure 6.1.

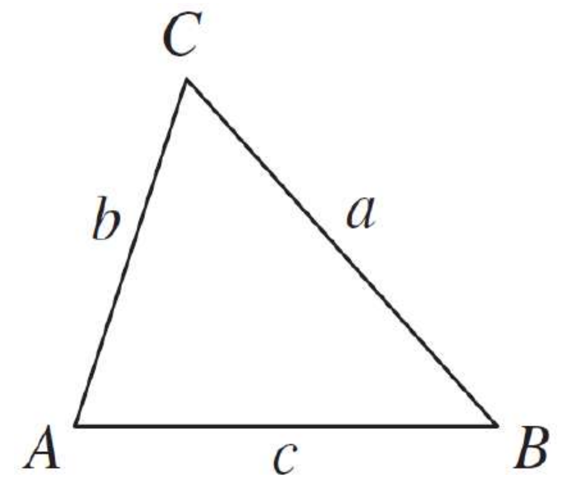


Figure 6.1

Introduction

To solve an oblique triangle, you need to know the measure of at least one side and the measures of any two other parts of the triangle—two sides, two angles, or one angle and one side.

This breaks down into the following four cases.

1. Two angles and any side (AAS or ASA)
2. Two sides and an angle opposite one of them (SSA or Angle Side Side)
3. Three sides (SSS)
4. Two sides and their included angle (SAS)

Introduction

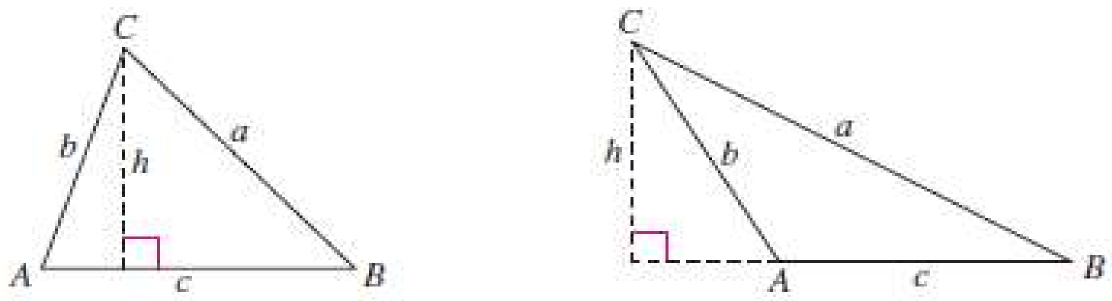
The first two cases can be solved using the **Law of Sines (AAS/ASA, SSA/Angle Side Side)**, whereas the last two cases can be solved using the Law of Cosines (SSS, SAS).

Law of Sines

If ABC is a triangle with sides a, b, and c, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Oblique Triangles



A is acute. *A is obtuse.*

Figure 6.2

Introduction

The Law of Sines can also be written in the reciprocal form. This is the form I usually use, but either is fine. Make sure you only use two fractions at a time, not all three.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Example 1 – *Given Two Angles and One Side—AAS*

For the triangle in Figure 6.3, $C = 102.3°$, $B = 28.7°$ and $b = 27.4$ feet. Find the remaining angle and sides.

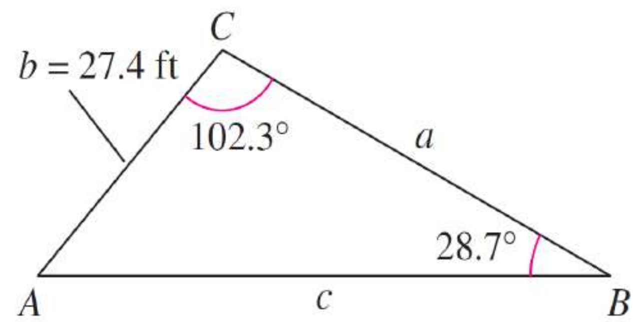


Figure 6.3

Solution:

The third angle of the triangle is

$$A = 180° - B - C$$

$$= 180° - 28.7° - 102.3°$$

$$= 49.0°$$

Example 1 – *Solution*

cont'd

By the Law of Sines, you have

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$= \frac{c}{\sin C}.$$

Using b = 27.4 produces

$$a = \frac{b}{\sin B}(\sin A) = \frac{27.4}{\sin 28.7^\circ}(\sin 49.0^\circ) \approx 43.06 \text{ feet}$$

and

$$c = \frac{b}{\sin B}(\sin C) = \frac{27.4}{\sin 28.7}(\sin 102.3^\circ) \approx 55.75 \text{ feet.}$$

The Ambiguous Case (SSA)

The Ambiguous Case (SSA)

In Example 1, you saw that two angles and one side determine a unique triangle.

However, if two sides and one opposite angle are given, then three possible situations can occur:

(1) no such triangle exists,

(2) one such triangle exists, or

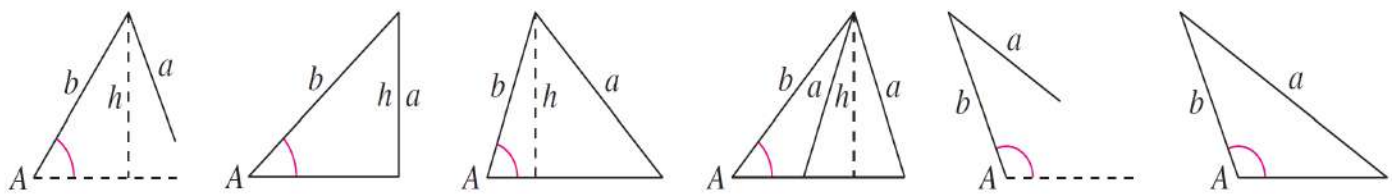
(3) two distinct triangles satisfy the conditions.

The Ambiguous Case (SSA)

The Ambiguous Case (SSA)

Consider a triangle in which you are given a, b, and A. (Notice that $h = b \sin A$.)

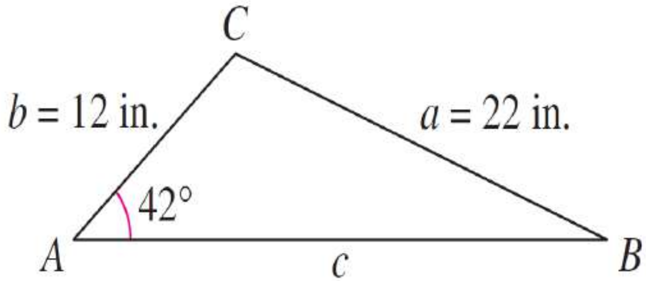
Sketch



	A is acute.	A is acute.	A is acute.	A is acute.	A is obtuse.	A is obtuse.
Necessary condition	$a < h$	$a = h$	$a \geq b$	$h < a < b$	$a \leq b$	$a > b$
Possible triangles	None	One	One	Two	None	One

Example 3 – *Single-Solution Case—SSA*

For the triangle in Figure 6.5, a = 22 inches, b = 12 inches, and A = 42°. Find the remaining side and angles.



One solution: a ≥ b

Figure 6.5

Solution:

By the Law of Sines, you have

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

Reciprocal form

Example 3 – *Solution*

cont'd

$$\sin B = b\left(\frac{\sin A}{a}\right)$$

Multiply each side by *b*

$$\sin B = 12\left(\frac{\sin 42^\circ}{22}\right)$$

Substitute for *A*, *a*, and *b*.

$$B \approx 21.41^\circ.$$

B is acute.

Now you can determine that

$$C \approx 180^\circ - 42^\circ - 21.41^\circ$$

$$= 116.59^\circ$$

Example 3 – *Solution*

cont'd

Then the remaining side is given by

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$ Law of Sines

$$c = \frac{a}{\sin A}(\sin C)$$ Multiply each side by sin *C*.

$$c = \frac{22}{\sin 42^\circ}(\sin 116.59^\circ)$$ Substitute for *a*, *A*, and *C*.

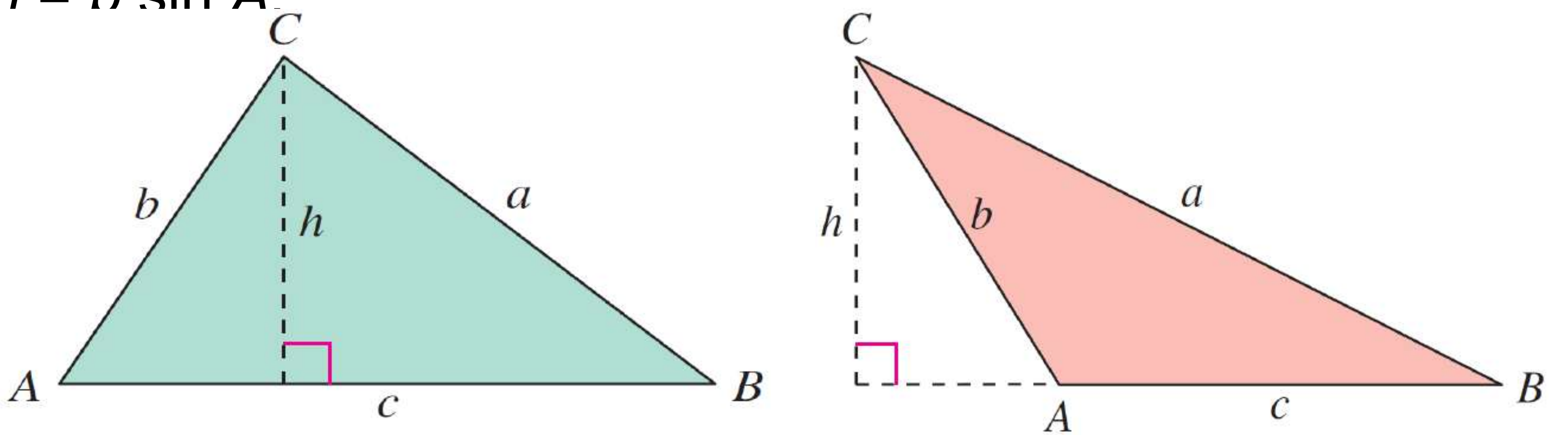
$$c \approx 29.40 \text{ inches.}$$ Simplify.

Area of an Oblique Triangle

Area of an Oblique Triangle

Read this slide and the next two, but do not copy them: The procedure used to prove the Law of Sines leads to a simple formula for the area of an oblique triangle. Referring to Figure 6.8, note that each triangle has a height of

$h = b \sin A$.



A is acute.

A is obtuse.

Figure 6.8

Area of an Oblique Triangle

To see this when A is obtuse, substitute the reference angle $180° - A$ for A. Now the height of the triangle is given by

$$h = b \sin (180° - A)$$

Using the difference formula for sine, the height is given by

$$h = b(\sin 180° \cos A - \cos 180° \sin A)$$

$\sin(u - v) = \sin u \cos v - \cos u \sin v$

$$= b[0 \cdot \cos A - (-1) \cdot \sin A]$$

$$= b \sin A.$$

Area of an Oblique Triangle

Consequently, the area of each triangle is given by

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height})$$

$$= \frac{1}{2}(c)(b \sin A)$$

$$= \frac{1}{2} bc \sin A.$$

By similar arguments, you can develop the formulas

$$\text{Area} = \frac{1}{2} ab \sin C$$

$$= \frac{1}{2} ac \sin B.$$

Area of an Oblique Triangle

Area of an Oblique Triangle

The area of any triangle is one-half the product of the lengths of two sides times the sine of their included angle. That is,

$$\text{Area} = \frac{1}{2}bc \sin A = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B.$$

Example 6 – *Finding the Area of an Oblique Triangle*

Find the area of a triangular lot having two sides of lengths 90 meters and 52 meters and an included angle of 102°.

Solution:

Consider a = 90 meters, b = 52 meters, and C = 102°, as shown in Figure 6.9.

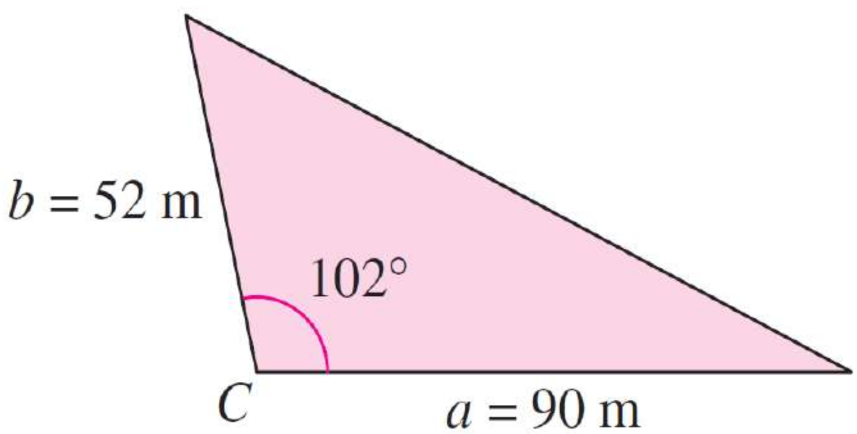


Figure 6.9

Example 6 – *Solution*

cont'd

Then the area of the triangle is

$$\text{Area} = a\frac{1}{2}\sin C$$

Formula for area

$$= \frac{1}{2}(90)(52)(\sin 102^\circ)$$

Substitute for *a*, *b*, and *C*.

$$\approx 2288.87 \text{ square meters.}$$

Simplify.