4.7 Inverse Trigonometric Functions

What You Should Learn

- Evaluate and graph inverse sine functions
- Evaluate and graph other inverse trigonometric functions
- Evaluate compositions of trigonometric functions

Inverse Sine Function

Inverse Sine Function

We have know that for a function to have an inverse function, it must be one-to-one—that is, it must pass the Horizontal Line Test.

In Figure 4.67 it is obvious that *y* = sin *x* does not pass the test because different values of *x* yield the same *y*-value.

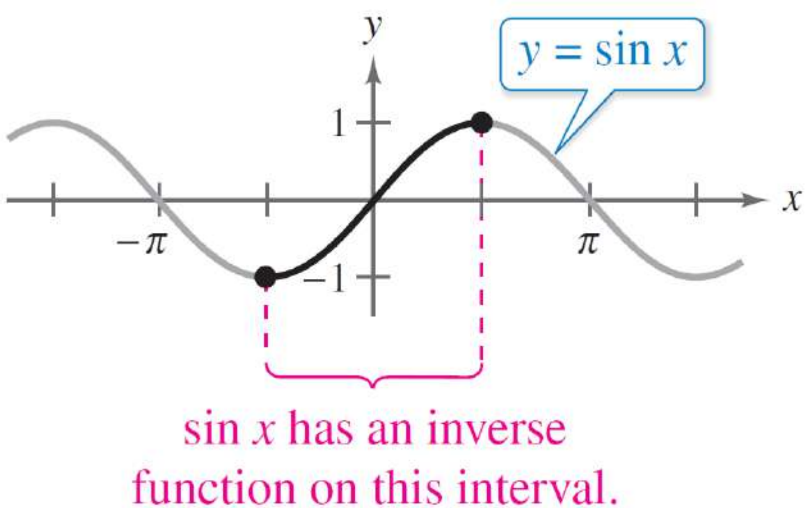

Figure 4.67

Inverse Sine Function

However, when you restrict the domain to the interval $-\pi/2 \le x \le \pi/2$ (corresponding to the black portion of the graph in Figure 4.67), the following properties hold.

1. On the interval $[-\pi/2, \pi/2]$, the function $y = \sin x$ is increasing.

2. On the interval $[-\pi/2, \pi/2]$, $y = \sin x$ takes on its full range
of values, $-1 \le \sin x \le 1$.

3. On the interval $[-\pi/2, \pi/2]$, $y = \sin x$ is one-to-one.

Inverse Sine Function

So, on the restricted domain $-\pi/2 \le x \le \pi/2$, $y = \sin x$ has a unique inverse function called the **inverse sine function.** It is denoted by

$y = \arcsin x$ or $y = \sin^{-1} x$.

The notation $\sin^{-1} x$ is consistent with the inverse function notation $f^{-1}(x)$. The arcsin x notation (read as "the arcsine of x") comes from the association of a central angle with its intercepted *arc length* on a unit circle.

Inverse Sine Function

So, arcsin *x* means the angle (or arc) whose sine is *x*. Both notations, arcsin *x* and $\sin^{-1} x$, are commonly used in mathematics, so remember that $\sin^{-1} x$ denotes the *inverse* sine function rather than $1/\sin x$. The values of arcsin *x* lie in the interval $-\pi/2 \le \arcsin x \le \pi/2$.

Definition of Inverse Sine Function

The **inverse sine function** is defined by

$$y = \arcsin x \qquad \text{if and only if} \qquad \sin y = x$$

where $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$. The domain of $y = \arcsin x$ is $[-1, 1]$ and the range is $[-\pi/2, \pi/2]$.

Example 1 – *Evaluating the Inverse Sine Function*

If possible, find the exact value.

a. $\arcsin\left(-\frac{1}{2}\right)$2 $\sin^{-1}\frac{\sqrt{3}}{2}$

Solution:

a. Because $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$s in $-\frac{\pi}{6}$, it $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
follows that

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}.$$

Angle whose sine is $-\frac{1}{2}$

Example 1 – *Solution*

cont'd

b. Because $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ lies in $\frac{\pi}{3}$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$'s that

$$\sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}.$$

Angle whose sine is $\sqrt{3}/2$

c. It is not possible to evaluate $y = \sin^{-1} x$ at $x = 2$ because there is no angle whose sine is 2. Remember that the domain of the inverse sine function is [–1, 1].

Example 2 – *Graphing the Arcsine Function*

Sketch a graph of $y = \arcsin x$ by hand.

Solution:

By definition, the equations

$$y = \arcsin x \quad \text{and} \quad \sin y = x$$

are equivalent for $-\pi/2 \le y \le \pi/2$. So, their graphs are the same. For the interval $[-\pi/2, \pi/2]$ you can assign values to y in the second equation to make a table of values.

y	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$x = \sin y$	-1	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	1

Example 2 – *Solution*

cont'd

Then plot the points and connect them with a smooth curve. The resulting graph of $y = \arcsin x$ is shown in Figure 4.68.

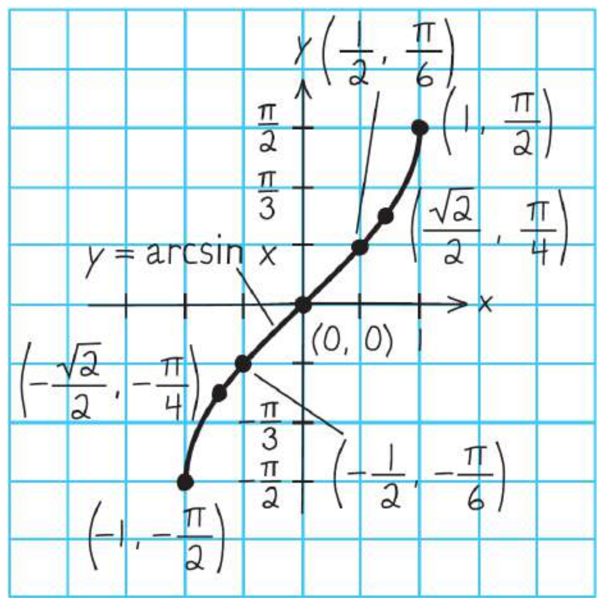

Figure 4.68

Example 2 – *Solution*

cont'd

Note that it is the reflection (in the line $y = x$) of the black portion of the graph in Figure 4.67.

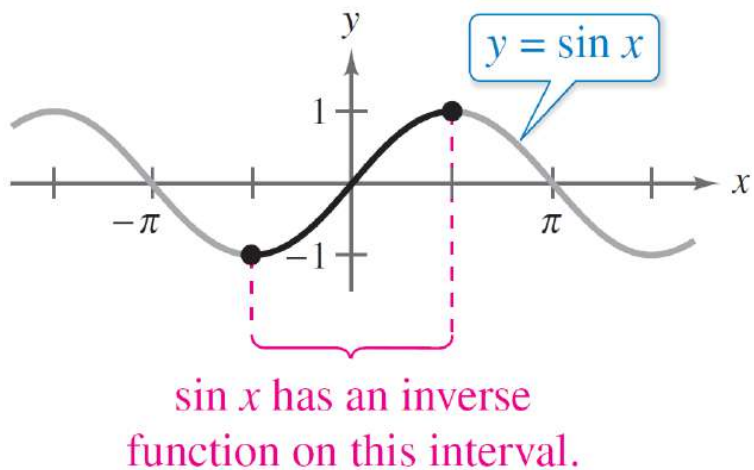

Figure 4.67

Remember that the domain of $y = \arcsin x$ is the closed interval [–1, 1] and the range is the closed interval $[-\pi/2, \pi/2]$.

Other Inverse Trigonometric Functions

Other Inverse Trigonometric Functions

The cosine function is decreasing and one-to-one on the interval $0 \leq x \leq \pi$, as shown in Figure 4.69.

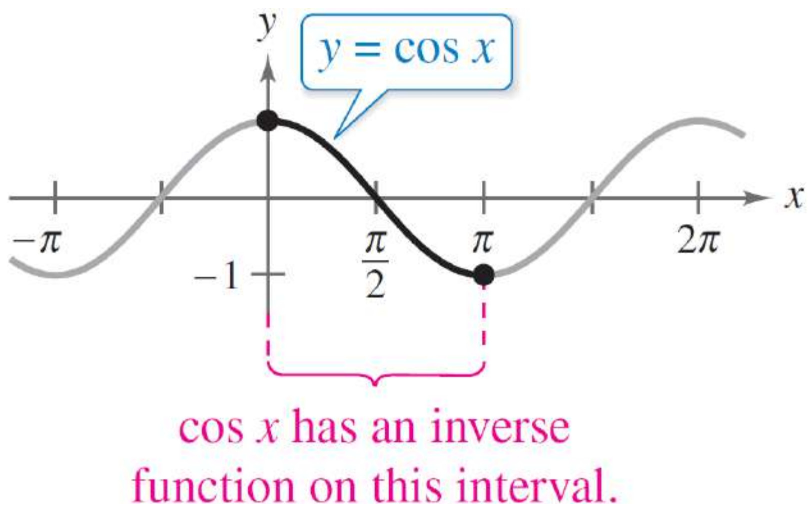

Figure 4.69

Other Inverse Trigonometric Functions

Consequently, on this interval the cosine function has an inverse function—the **inverse cosine function**—denoted by

$y = \arccos x$ or $y = \cos^{-1} x$.

Because $y = \arccos x$ and $x = \cos y$ are equivalent for $0 \le y \le \pi$, their graphs are the same, and can be confirmed by the following table of values.

y	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$x = \cos y$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Other Inverse Trigonometric Functions

Similarly, you can define an **inverse tangent function** by restricting the domain of $y = \tan x$ to the interval $(-\pi/2, \pi/2)$.

The inverse tangent function is denoted by

$$y = \arctan x \qquad \text{or } y = \tan^{-1} x.$$

Because $y = \arctan x$ and $x = \tan y$ are equivalent for $-\pi/2 < y < \pi/2$ their graphs are the same, and can be confirmed by the following table of values.

y	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
$x = \tan y$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1

Other Inverse Trigonometric Functions

The following list summarizes the definitions of the three most common inverse trigonometric functions.

Definition of the Inverse Trigonometric Functions

Function	*Domain*	*Range*
$y = \arcsin x$ if and only if $\sin y = x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \arccos x$ if and only if $\cos y = x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \arctan x$ if and only if $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

Example 3 – *Evaluating Inverse Trigonometric Functions*

Find the exact value.

a. $\arccos \dfrac{\sqrt{2}}{2}$ **b.** $\cos^{-1}(-1)$

c. arctan 0 **d.** $\tan^{-1}(-1)$

Solution:

a. Because $\cos(\pi/4) = \sqrt{2}/2$, and $\pi/4$ lies in $[0, \pi]$, it follows that

$$\arccos \frac{\sqrt{2}}{2} = \frac{\pi}{4}.$$

Angle whose cosine is $\dfrac{\sqrt{2}}{2}$

Example 3 – *Solution*

cont'd

b. Because $\cos \pi = -1$ and π lies in $[0, \pi]$ it follows that

$$\cos^{-1}(-1) = \pi.$$

Angle whose cosine is –1

c. Because $\tan 0 = 0$, and 0 lies in $(-\pi/2, \pi/2)$, it follows that

Angle whose tangent is 0s

$$\arctan 0 = 0.$$

d. Because $\tan(-\pi/4) = -1$ and $-\pi/4$ lies in $(-\pi/2, \pi/2)$, it follows that

$$\tan^{-1}(-1) = -\frac{\pi}{4}.$$

Angle whose tangent is –1

Compositions of Functions

Compositions of Functions

We have know that for all x in the domains of f and f^{-1}, inverse functions have the properties

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

Inverse Properties

If $-1 \leq x \leq 1$ and $-\pi/2 \leq y \leq \pi/2$, then

$$\sin(\arcsin x) = x \qquad \text{and} \qquad \arcsin(\sin y) = y.$$

If $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$, then

$$\cos(\arccos x) = x \qquad \text{and} \qquad \arccos(\cos y) = y.$$

If x is a real number and $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \qquad \text{and} \qquad \arctan(\tan y) = y.$$

Compositions of Functions

Keep in mind that these inverse properties do not apply for arbitrary values of *x* and *y*. For instance,

$$\arcsin\left(\sin\frac{3\pi}{2}\right) = \arcsin(-1) = -\frac{\pi}{2} \neq \frac{3\pi}{2}.$$

In other words, the property

$$\arcsin(\sin y) = y$$

is not valid for values of *y* outside the interval $[-\pi/2, \pi/2]$.

Example 6 – *Using Inverse Properties*

If possible, find the exact value.

a. tan[arctan(–5)]

b. $\arcsin\left(\sin\frac{5\pi}{3}\right)$

c. $(\cos^{-1}\pi)$

Solution:

a. Because –5 lies in the domain of the arctangent function, the inverse property applies, and you have

tan[arctan(–5)] = –5.

Example 6 – *Solution*

cont'd

b. In this case, $5\pi/3$ does not lie within the range of the arcsine function, $-\pi/2 \le y \le \pi/2$. However, $5\pi/3$ is coterminal with

$$\frac{5\pi}{3} - 2\pi = -\frac{\pi}{3}$$

which does lie in the range of the arcsine function, and you have

$$\arcsin\left(\sin\frac{5\pi}{3}\right) = \arcsin\left[\sin\left(-\frac{\pi}{3}\right)\right] = -\frac{\pi}{3}.$$

c. The expression $\cos(\cos^{-1}\pi)$ is not defined because $\cos^{-1}\pi$ is not defined. Remember that the domain of the inverse cosine function is $[-1, 1]$.