Trigonometric Functions

4.6 Graphs of Other Trigonometric Functions

What You Should Learn

- Sketch the graphs of tangent functions.
- Sketch the graphs of cotangent functions.
- Sketch the graphs of secant and cosecant functions.
- Sketch the graphs of damped trigonometric functions.

Graph of the Tangent Function

Graph of the Tangent Function

We have know that the tangent function is odd.

That is, $\tan(-x) = -\tan x$. Consequently, the graph of $y = \tan x$ is symmetric with respect to the origin.

You also know from the identity $\tan x = \sin x/\cos x$ that the tangent function is undefined when $\cos x = 0$.

Two such values are $x = \pm\pi/2 \approx \pm 1.5708$.

Graph of the Tangent Function

x	$-\frac{\pi}{2}$	-1.57	-1.5	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	1.5	1.57	$\frac{\pi}{2}$
$\tan x$	Undef.	-1255.8	-14.1	-1	0	1	14.1	1255.8	Undef.

tan x approaches $-\infty$ as x approaches $-\pi/2$ from the right.

tan x approaches ∞ as x approaches $\pi/2$ from the left.

As indicated in the table, tan x increases without bound as x approaches $\pi/2$ from the left, and it decreases without bound as x approaches from the right $-\pi/2$.

Graph of the Tangent Function

So, the graph of y = tan *x* has *vertical asymptotes* at and as shown in Figure 4.55. The basic characteristics of the parent tangent function are summarized below.

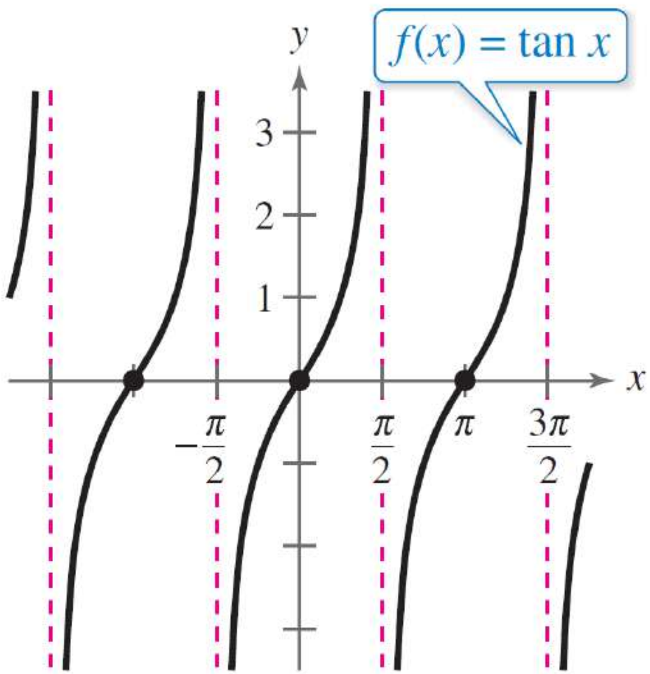


Figure 4.55

Domain: *all real numbers x,*

$$x \neq \frac{\pi}{2} + n\pi$$

Range: $(-\infty, \infty)$

Period: π

x-intercepts: $(n\pi, 0)$

y-intercept: $(0, 0)$

Vertical asymptotes: $x = \frac{\pi}{2} + n\pi$

Odd function

Origin symmetry

Graph of the Tangent Function

Moreover, because the period of the tangent function is π, vertical asymptotes also occur at $x = \pi/2 + n\pi$, where is an integer.

The domain of the tangent function is the set of all real numbers other than $x = \pi/2 + n\pi$, and the range is the set of all real numbers.

Example 1 – *Library of Parent Functions: f(x) = tan x*

Sketch the graph of $y = \tan \frac{x}{2}$ by hand.

Solution:

By solving the equations $x/2 = -\pi/2$ and $x/2 = \pi/2$ you can see that two consecutive asymptotes occur at $x = -\pi$ and $x = \pi$.

x	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π
$\tan \frac{x}{2}$	Undef.	-1	0	1	Undef.

Example 1 – *Solution*

cont'd

Between these two asymptotes, plot a few points, including the *x*-intercept, as shown in the table. Three cycles of the graph are shown in Figure 4.56. Use a graphing utility to confirm this graph.

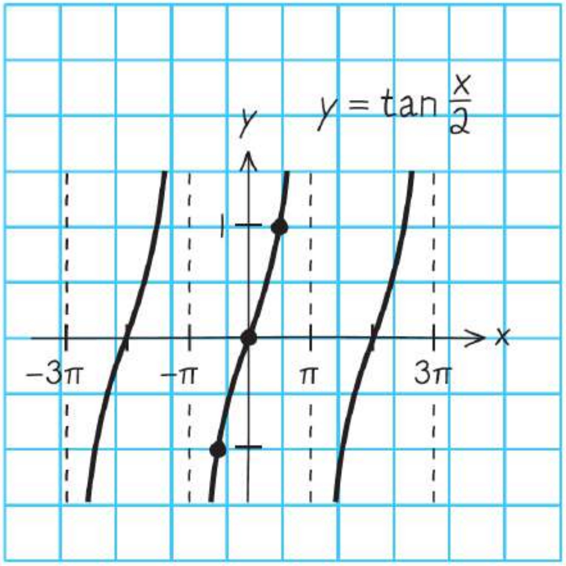


Figure 4.56

Graph of the Cotangent Function

Graph of the Cotangent Function

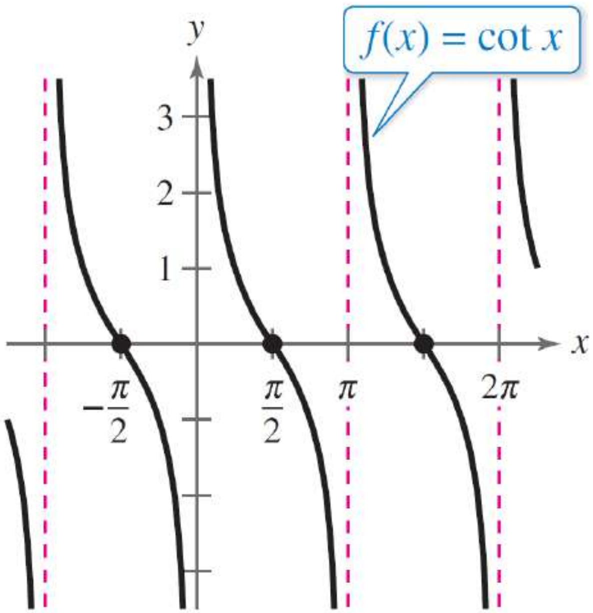
The graph of the parent cotangent function is similar to the graph of the parent tangent function.

It also has a period of π.

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$

Graph of the Cotangent Function

However, from the identity you can see that the cotangent function has vertical asymptotes when sin *x* is zero, which occurs at $x = n\pi$ where *n* is an integer. The basic characteristics of the parent cotangent function are summarized below



Domain: all real numbers x, $x \neq n\pi$

Range: $(-\infty, \infty)$

Period: π

x-intercepts: $\left(\frac{\pi}{2} + n\pi, 0\right)$

Vertical asymptotes: $x = n\pi$

Odd function

Origin symmetry

Example 3 – *Library of Parent Functions: f(x) = cot x*

Sketch the graph of $y = 2 \cot \frac{x}{3}$ by hand.

Solution:

To locate two consecutive vertical asymptotes of the graph, solve the equations $x/3 = 0$ and $x/3 = \pi$ to see that two consecutive asymptotes occur at $x = 0$ and $x = 3\pi$.

Then, between these two asymptotes, plot a few points, including the *x*-intercept, as shown in the table.

x	0	$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	$\frac{9\pi}{4}$	3π
$2 \cot \frac{x}{3}$	Undef.	2	0	-2	Undef.

Example 3 – *Solution*

cont'd

Three cycles of the graph are shown in Figure 4.59.

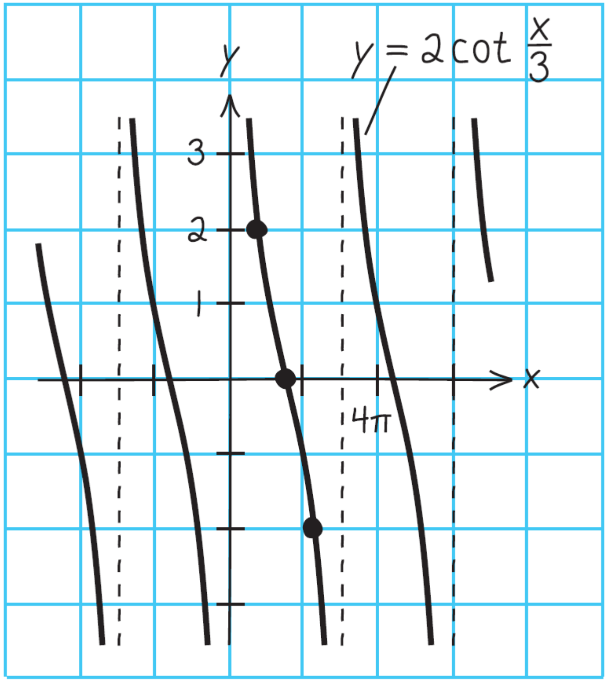


Figure 4.59

Example 3 – *Solution*

cont'd

Use a graphing utility to confirm this graph.

[Enter the function as y = 2/tan(x/3).]

Note that the period is 3π, the distance between consecutive asymptotes.

Graph of the Reciprocal Functions

Graph of the Reciprocal Functions

The graphs of the two remaining trigonometric functions can be obtained from the graphs of the sine and cosine functions using the reciprocal identities

$$\csc x = \frac{1}{\sin x} \quad \text{and} \quad \sec x = \frac{1}{\cos x}.$$

For instance, at a given value of *x*, the *y*-coordinate for sec *x* is the reciprocal of the *y*-coordinate for cos *x*. Of course, when *x* = 0, the reciprocal does not exist.

Graph of the Reciprocal Functions

Near such values of x, the behavior of the secant function is similar to that of the tangent function.

In other words, the graphs of

$$\tan x = \frac{\sin x}{\cos x} \quad \text{and} \quad \sec x = \frac{1}{\cos x}$$

have vertical asymptotes at $x = \pi/2 + n\pi$ where n is an integer (i.e., the values at which the cosine is zero).

Graph of the Reciprocal Functions

Similarly,

$$\cot x = \frac{\cos x}{\sin x} \qquad \text{and} \qquad \csc x = \frac{1}{\sin x}$$

have vertical asymptotes where sin x = 0—that is, at x = $n\pi$.

Graph of the Reciprocal Functions

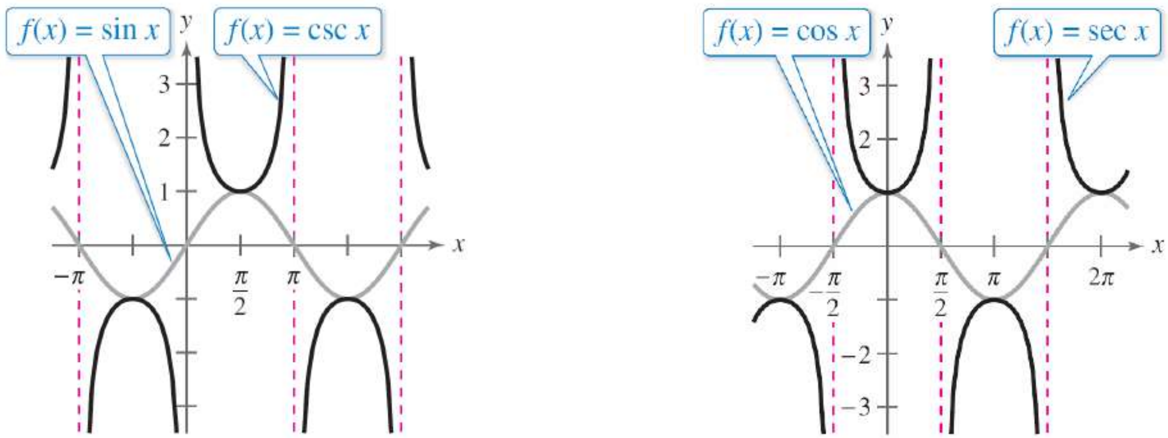
To sketch the graph of a secant or cosecant function, you should first make a sketch of its reciprocal function.

For instance, to sketch the graph of $y = \csc x$, first sketch the graph of $y = \sin x$.

Then take the reciprocals of the y-coordinates to obtain points on the graph of $y = \csc x$.

Graph of the Reciprocal Functions

The basic characteristics of the parent cosecant and secant functions are summarized below



Domain: all real numbers $x, x \neq n\pi$

Range: $(-\infty, -1] \cup [1, \infty)$

Period: 2π

No intercepts

Vertical asymptotes: $x = n\pi$

Odd function

Origin symmetry

Domain: all real numbers $x, x \neq \frac{\pi}{2} + n\pi$

Range: $(-\infty, -1] \cup [1, \infty)$

Period: 2π

y-intercept: $(0, 1)$

Vertical asymptotes: $x = \frac{\pi}{2} + n\pi$

Even function

y-axis symmetry

Figure 4.60

Example 4 – *Library of Parent Functions: f(x) = csc x*

Sketch the graph of $y = 2\csc\left(x + \frac{\pi}{4}\right)$ by hand.

Solution:

Begin by sketching the graph of $y = 2\sin\left(x + \frac{\pi}{4}\right)$.

For this function, the amplitude is 2 and the period is 2π.

Example 4 – *Solution*

cont'd

By solving the equations

$$x + \frac{\pi}{4} = 0 \qquad \text{and} \qquad x + \frac{\pi}{4} = 2\pi$$

you can see that one cycle of the sine function corresponds to the interval from

$$x = -\frac{\pi}{4} \qquad \text{to} \qquad x = \frac{7\pi}{4}.$$

Example 4 – *Solution*

cont'd

The graph of this sine function is represented by the gray curve in Figure 4.62.

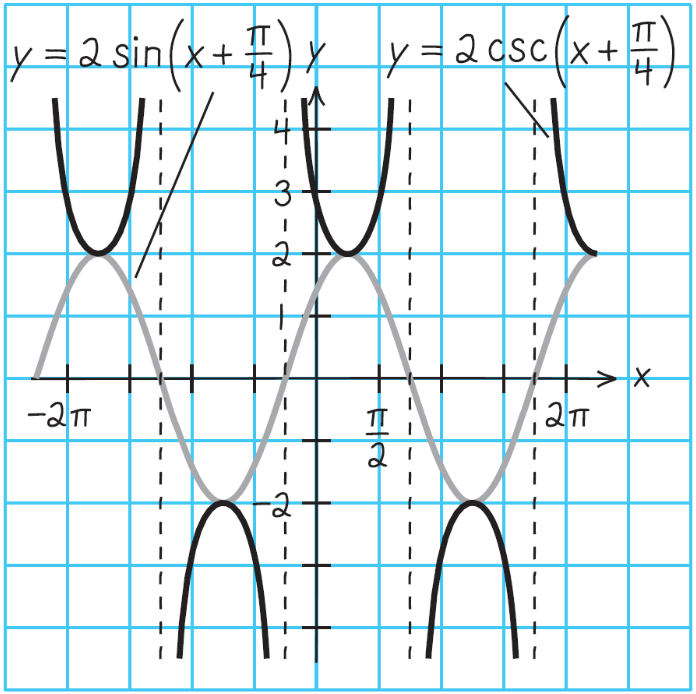


Figure 4.62

Example 4 – *Solution*

cont'd

Because the sine function is zero at the endpoints of this interval, the corresponding cosecant function

$$y = 2\csc\left(x + \frac{\pi}{4}\right) = 2\left(\frac{1}{\sin[x + (\pi/4)]}\right)$$

has vertical asymptotes at $x = -\frac{\pi}{4}, x = \frac{3\pi}{4}, x = \frac{7\pi}{4},$

and so on. The graph of the cosecant function is represented by the black curve in Figure 4.62.

Damped Trigonometric Graphs

Please read the next six slides so you are familiar with damped trig. graphs, but do not copy them down.

Damped Trigonometric Graphs

A *product* of two functions can be graphed using properties of the individual functions.

For instance, consider the function

$f(x) = x \sin x$

as the product of the functions $y = x$ and $y = \sin x$.

Using properties of absolute value and the fact that $|\sin x| \leq 1$, you have $0 \leq |x|\,|\sin x| \leq |x|$. Consequently,

$-|x| \leq x \sin x \leq |x|$

which means that the graph of $f(x) = x \sin x$ lies between

Damped Trigonometric Graphs

Furthermore, because

$f(x) = x \sin x = \pm x$ at $x = \frac{\pi}{2} + n\pi$

and

$f(x) = x \sin x = 0$ at $x = n\pi$

the graph of f touches the line $y = -x$ or the line at $y = x$ at $x = \pi/2 + n\pi$ and has x-intercepts at $x = n\pi$.

Damped Trigonometric Graphs

A sketch of *f* is shown in Figure 4.64.

In the function $f(x) = x \sin x$, the factor x is called the **damping factor.**

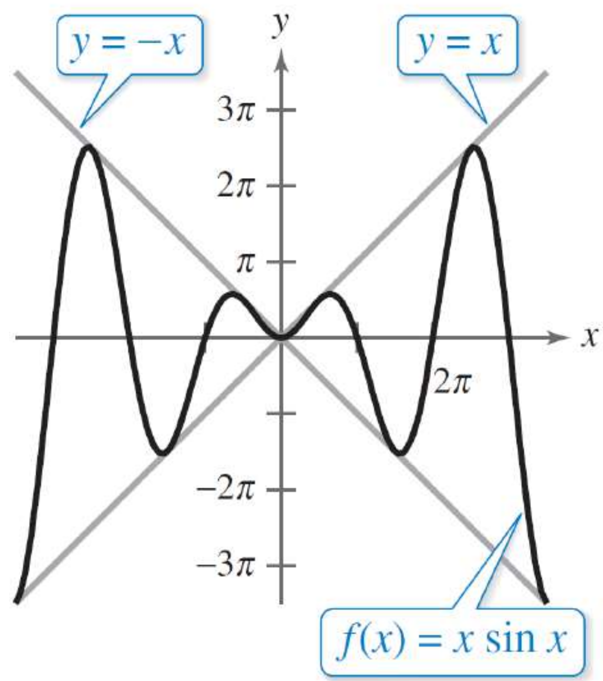


Figure 4.64

Example 4 – *Analyzing a Damped Sine Curve*

Analyze the graph of $f(x) = e^{-x} \sin 3x$.

Solution:

Consider $f(x)$ as the product of the two functions

$$y = e^{-x} \qquad \text{and} \qquad y = \sin 3x$$

each of which has the set of real numbers as its domain. For any real number x, you know that $e^{-x} \geq 0$ and $|\sin 3x| \leq 1$.

Example 4 – *Solution*

cont'd

So, $|e^{-x}|\ |\sin 3x| \leq e^{-x}$ which means that

$-e^{-x} \leq e^{-x} \sin 3x \leq e^{-x}$.

Furthermore, because

$f(x) = e^{-x} \sin 3x = \pm e^{-x}$ at $x = \dfrac{\pi}{6} + \dfrac{n\pi}{3}$

and

$f(x) = e^{-x} \sin 3x = 0$ at $x = \dfrac{n\pi}{3}$

the graph of f touches the curves $y = -e^{-x}$ and $y = e^{-x}$ at $x = \pi/6 + n\pi/3$ and has intercepts at $x = n\pi/3$.

Example 4 – *Solution*

cont'd

The graph is shown in Figure 4.65.

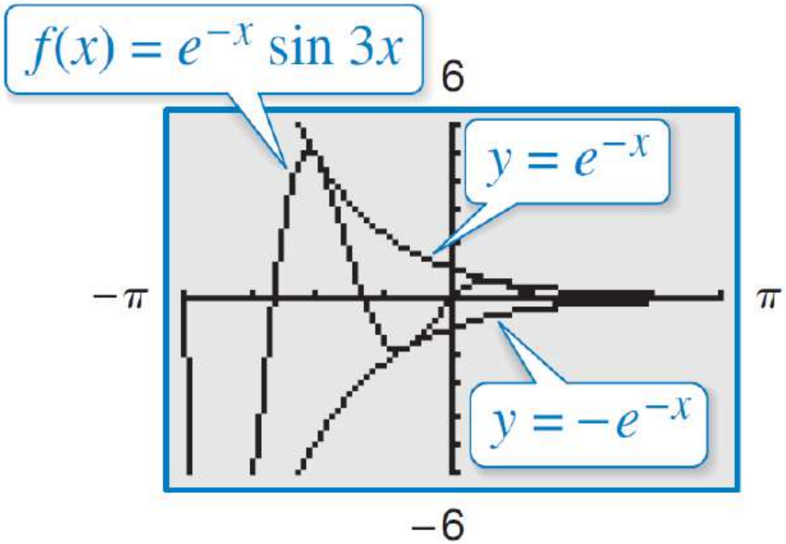


Figure 4.65