4.5

Graphs of Sine and Cosine Functions

What You Should Learn

- Sketch the graphs of basic sine and cosine functions
- Use amplitude and period to help sketch the graphs of sine and cosine functions
- Sketch translations of graphs of sine and cosine functions
- Use sine and cosine functions to model real-life data

Basic Sine and Cosine Curves

Basic Sine and Cosine Curves

The graph of the sine function is a **sine curve.** I drew this curve using the table on your unit circle.

In Figure 4.43, the black portion of the graph represents one period of the function and is called **one cycle** of the sine curve.

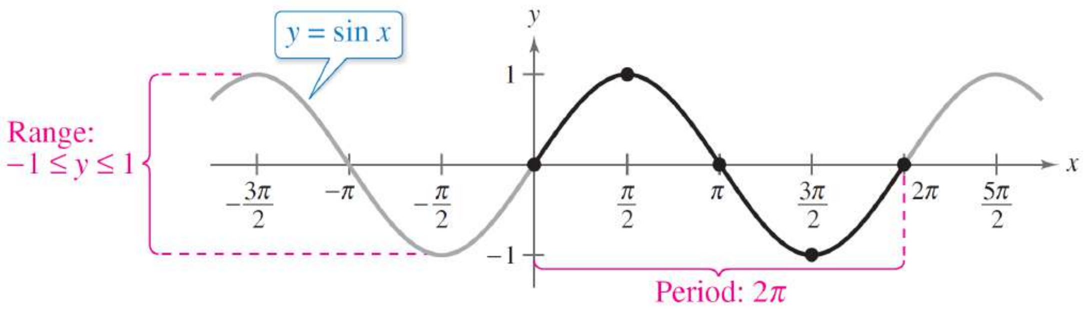

Figure 4.43

The gray portion of the graph indicates that the basic sine wave repeats indefinitely to the right and left.

Basic Sine and Cosine Curves

The graph of the cosine function is shown in Figure 4.44. This was also drawn using points from the table on your unit circle handout.

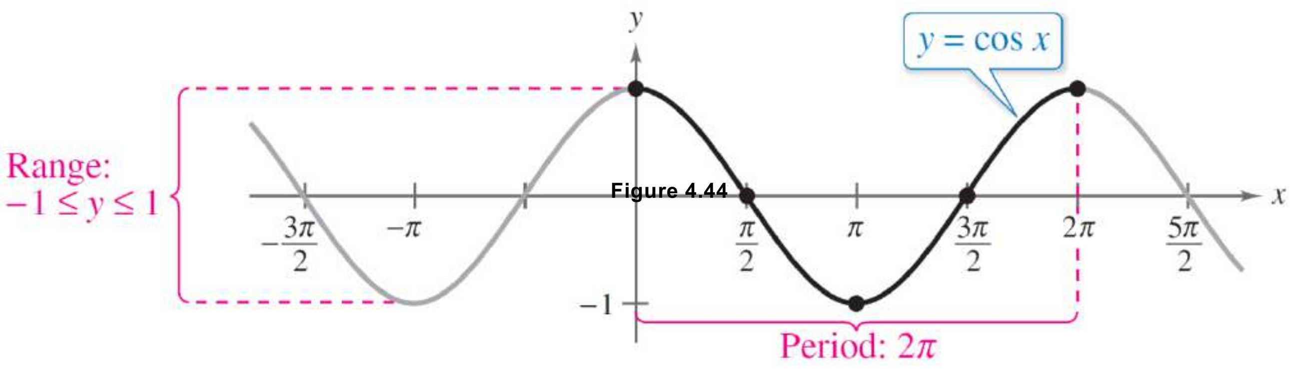

Figure 4.44

Basic Sine and Cosine Curves

The domain of the sine and cosine functions is the set of all real numbers. The range of each function is the interval

[–1, 1]

and each function has a period 2π which is the horizontal length of one cycle of each graph.

Basic Sine and Cosine Curves

Do you see how this information is consistent with the basic graphs shown in Figures 4.43 and 4.44?

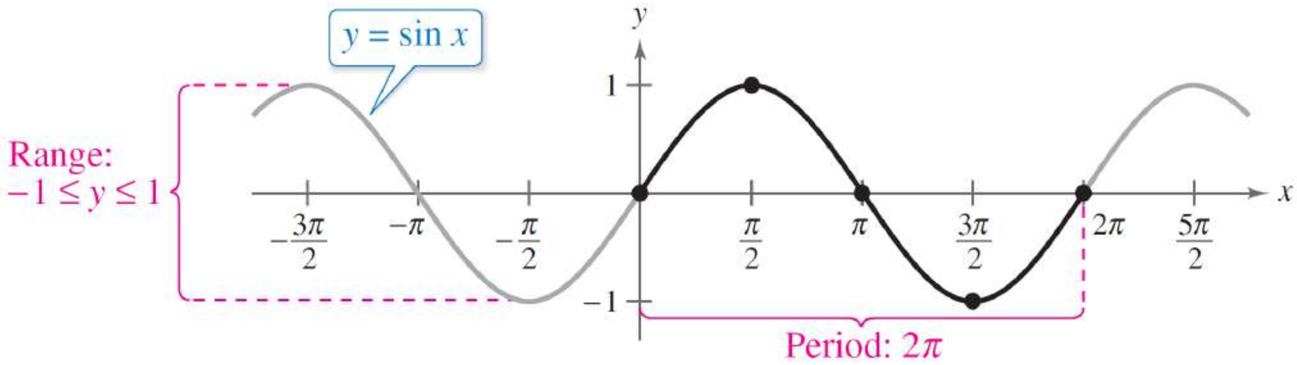

Figure 4.43

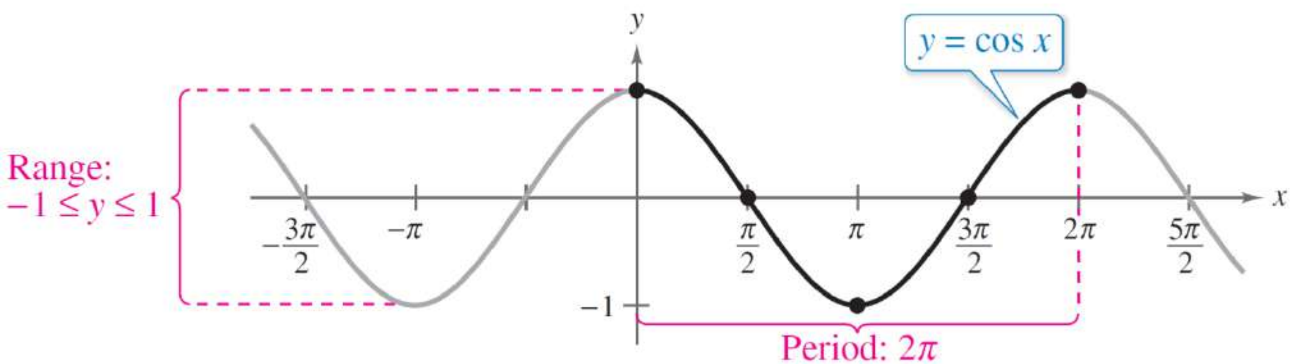

Figure 4.44

Basic Sine and Cosine Curves

To sketch the graphs of the basic sine and cosine functions by hand, it helps to note five *key points* in one period of each graph: the *intercepts*, the *maximum points*, and the *minimum points*. We will call this the Five Point Method.

The table below lists the five key points on the graphs of

$y = \sin x$ and $y = \cos x$.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	1	0	-1	0
$\cos x$	1	0	-1	0	1

Basic Sine and Cosine Curves

Note in Figures 4.43 and 4.44 that the sine curve is symmetric with respect to the *origin*, whereas the cosine curve is symmetric with respect to the *y-axis*.

These properties of symmetry follow from the fact that the sine function is odd whereas the cosine function is even.

Example 1 – *Library of Parent Functions: f(x) = sin x*

Sketch the graph of $g(x) = 2 \sin x$ by hand on the interval $[-\pi, 4\pi]$.

Solution:

Note that $g(x) = 2 \sin x$ = 2(sin x) indicates that the y-values of the key points will have twice the magnitude of those on the graph of $f(x) = \sin x$.

Divide the period 2π into four equal parts to get the key points

Intercept	*Maximum*	*Intercept*	*Minimum*		*Intercept*
$(0, 0)$,	$\left(\frac{\pi}{2}, 2\right)$,	$(\pi, 0)$,	$\left(\frac{3\pi}{2}, -2\right)$,	and	$(2\pi, 0)$.

Example 1 – *Solution*

cont'd

By connecting these key points with a smooth curve and extending the curve in both directions over the interval $[-\pi, 4\pi]$, you obtain the graph shown in Figure 4.45.

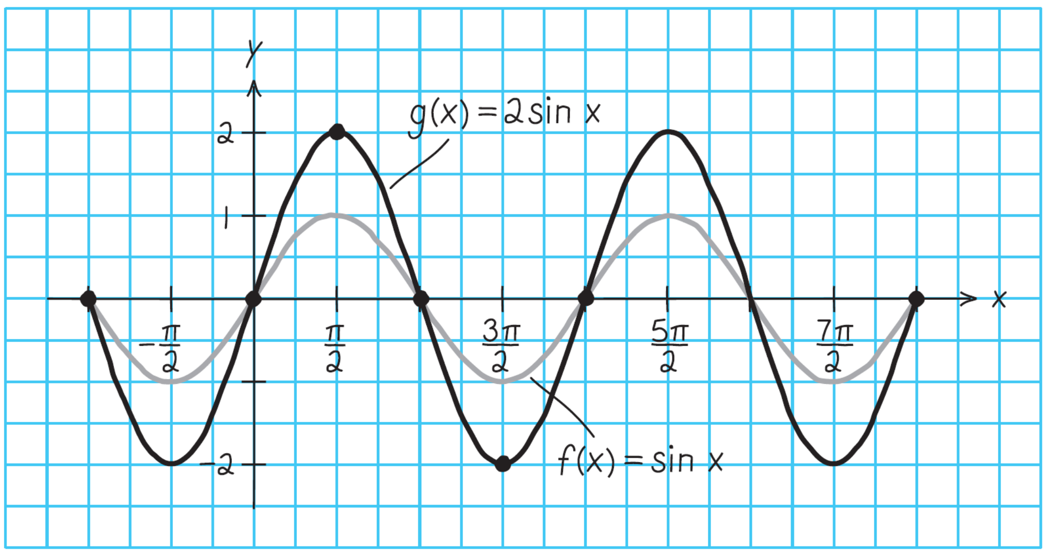

Figure 4.45

Amplitude and Period of Sine and Cosine Curves

Amplitude and Period of Sine and Cosine Curves

Let us discuss the graphic effect of each of the constants a, b, c, and d in equations of the forms

$$y = d + a\sin(bx - c) \quad \text{and} \quad y = d + a\cos(bx - c).$$

The constant factor a in $y = a \sin x$ acts as a *scaling factor* —a *vertical stretch* or *vertical shrink* of the basic sine curve "stretch factor" = amplitude.

When $|a| > 1$, the basic sine curve is stretched, and when $|a| < 1$, the basic sine curve is shrunk.

Amplitude and Period of Sine and Cosine Curves

The result is that the graph of $y = a \sin x$ ranges between $-a$ and a instead of between –1 and 1. The absolute value of a is the **amplitude** of the function $y = a \sin x$.

The range of the function $y = a \sin x$ for $a > 0$ is $-a \le y \le a$.

Definition of Amplitude of Sine and Cosine Curves

The **amplitude** of

$$y = a \sin x \text{ and } y = a \cos x$$

represents half the distance between the maximum and minimum values of the function and is given by

$$\text{Amplitude} = |a|.$$

Example 2 – *Scaling: Vertical Shrinking and Stretching*

On the same set of coordinate axes, sketch the graph of each function by hand.

a. $y = \frac{1}{2}\cos x$

Solution:

a. Because the amplitude of $y = \frac{1}{2}\cos x$ is $\frac{1}{2}$, the maximum value is $\frac{1}{2}$ and the minimum value is $-\frac{1}{2}$.

Divide one cycle, $0 \le x \le 2\pi$, into four equal parts to get the key points

Maximum	*Intercept*	*Minimum*	*Intercept*		*Maximum*
$\left(0, \frac{1}{2}\right)$,	$\left(\frac{\pi}{2}, 0\right)$,	$\left(\pi, -\frac{1}{2}\right)$,	$\left(\frac{3\pi}{2}, 0\right)$,	and	$\left(2\pi, \frac{1}{2}\right)$.

Example 2 – *Solution*

cont'd

b. A similar analysis shows that the amplitude of $y = 3 \cos x$ is 3, and the key points are

Maximum	*Intercept*	*Minimum*	*Intercept*		*Maximum*
$(0, 3),$	$\left(\frac{\pi}{2}, 0\right),$	$(\pi, -3),$	$\left(\frac{3\pi}{2}, 0\right),$	and	$(2\pi, 3).$

Example 2 – *Solution*

cont'd

The graphs of these two functions are shown in Figure 4.46.

Notice that the graph of

$$y = \frac{1}{2}\cos x$$

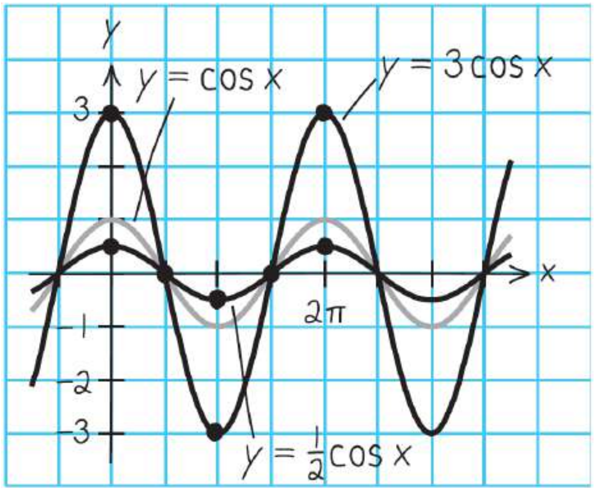

Figure 4.46

is a vertical compression of the graph of $y = \cos x$ and the graph of

$$y = 3\cos x$$

is a vertical stretch of the graph of $y = \cos x$.

Amplitude and Period of Sine and Cosine Curves

Next, consider the effect of the *positive* real number b on the graphs of $y = a \sin bx$ and $y = a \cos bx$.

Period of Sine and Cosine Functions

Let b be a positive real number. The **period** of

$$y = a \sin bx \qquad \text{and} \qquad y = a \cos bx$$

is given by

$$\text{Period} = \frac{2\pi}{b}.$$

Amplitude and Period of Sine and Cosine Curves

Note that when $0 < b < 1$, the period of $y = a \sin bx$ is greater than 2π and represents a *horizontal stretching* of the graph of $y = a \sin x$.

Similarly, when $b > 1$, the period of $y = a \sin bx$ is less than 2π and represents a *horizontal shrinking* of the graph of $y = a \sin x$.

Translations of Sine and Cosine Curves

Translations of Sine and Cosine Curves

The constant c in the general equations

$y = a \sin(bx - c)$ and $y = a \cos(bx - c)$

creates *horizontal translations* (shifts) of the basic sine and cosine curves.

To find the horizontal shift, first factor b out of the parentheses.

Translations of Sine and Cosine Curves

This implies that the period of $y = a \sin(bx - c)$ is $2\pi/b$, and the graph of $y = a \sin bx$ is shifted by an amount c/b. The number c/b is the **phase shift.**

Graphs of Sine and Cosine Functions

The graphs of $y = a \sin(bx - c)$ and $y = a \cos(bx - c)$ have the following characteristics. (Assume $b > 0$.)

$$\text{Amplitude} = |a| \qquad \text{Period} = \frac{2\pi}{b}$$

The left and right endpoints of a one-cycle interval can be determined by solving the equations

$$bx - c = 0 \qquad \text{and} \qquad bx - c = 2\pi.$$

Example 4 – *Horizontal Translation*

Analyze the graph of $y = \frac{1}{2}\sin\left(x - \frac{\pi}{3}\right)$.

Solution:

The amplitude is $\frac{1}{2}$, the horizontal shift is to the right, and the period is 2π.

Therefore, the interval which corresponds to one cycle of the graph is $\left[\frac{\pi}{3}, \frac{7\pi}{3}\right]$

Example 4 – *Solution*

cont'd

Dividing this interval into four equal parts produces the following key points.

Intercept	*Maximum*	*Intercept*	*Minimum*	*Intercept*
$\left(\frac{\pi}{3}, 0\right)$,	$\left(\frac{5\pi}{6}, \frac{1}{2}\right)$,	$\left(\frac{4\pi}{3}, 0\right)$,	$\left(\frac{11\pi}{6}, -\frac{1}{2}\right)$,	$\left(\frac{7\pi}{3}, 0\right)$

Mathematical Modeling

Mathematical Modeling

Read this slide and those that follow, but do not copy them down.

Sine and cosine functions can be used to model many real-life situations, including electric currents, musical tones, radio waves, tides, and weather patterns.

Example 8 – *Finding a Trigonometric Model*

Throughout the day, the depth of the water at the end of a dock varies with the tides. The table shows the depths (in feet) at various times during the morning.

Time	Depth, y
Midnight	3.4
2 A.M.	8.7
4 A.M.	11.3
6 A.M.	9.1
8 A.M.	3.8
10 A.M.	0.1
Noon	1.2

Example 8 – *Finding a Trigonometric Model* cont'd

a. Use a trigonometric function to model the data. Let t be the time, with $t = 0$ corresponding to midnight.

b. A boat needs at least 10 feet of water to moor at the dock. During what times in the evening can it safely dock?

Solution:

a. Begin by graphing the data, as shown in Figure 4.53. You can use either a sine or cosine model. Suppose you use a cosine model of the form

$$y = a\cos(bt - c) + d.$$

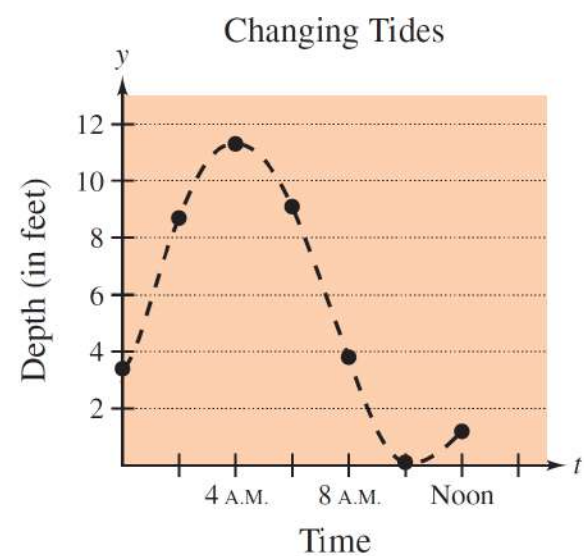

Figure 4.53c

Example 8 – *Solution*

cont'd

The difference between the maximum height and minimum height of the graph is twice the amplitude of the function.

So, the amplitude is

$$a = \frac{1}{2}[(\text{maximum depth}) - (\text{minimum depth})]$$

$$= \frac{1}{2}(11.3 - 0.1)$$

$$= 5.6.$$

Example 8 – *Solution*

cont'd

The cosine function completes one half of a cycle between the times at which the maximum and minimum depths occur. So, the period p is

$$p = 2[(\text{time of min. depth}) - (\text{time of max. depth})]$$

$$= 2(10 - 4)$$

$$= 12$$

which implies that $b = 2\pi/p \approx 0.524$.

Example 8 – *Solution*

cont'd

Because high tide occurs 4 hours after midnight, consider the left endpoint to be $c/b = 4$, so $c \approx 2.094$.

Moreover, because the average depth is

$$(1\tfrac{1}{2}.3 + 0.1) = 5.7$$

it follows that $d = 5.7$. So, you can model the depth with the function

$$y = 5.6 \cos(0.524t - 2.094) + 5.7.$$

Example 8 – *Solution*

cont'd

b. Using a graphing utility, graph the model with the line

$y = 10.$

Using the *intersect* feature, you can determine that the depth is at least 10 feet between 2:42 P.M. ($t \approx 14.7$) and 5:18 P.M. ($t \approx 17.3$), as shown in Figure 4.54.

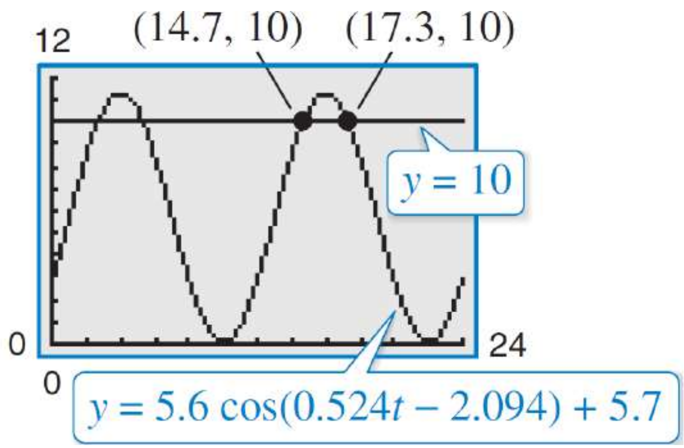

Figure 4.54