4.2

Trigonometric Functions: The Unit Circle

What You Should Learn

- Identify a unit circle and describe its relationship to real numbers.
- Evaluate trigonometric functions using the unit circle.
- Use domain and period to evaluate sine and cosine functions and use a calculator to evaluate trigonometric functions.

The Unit Circle

The Unit Circle

The two historical perspectives of trigonometry incorporate different methods of introducing the trigonometric functions.

Our first introduction to these functions is based on the unit circle.

Consider the **unit circle** given by

$$x^2 + y^2 = 1$$ Unit circle

It is called the unit circle because it has a radius of one unit.

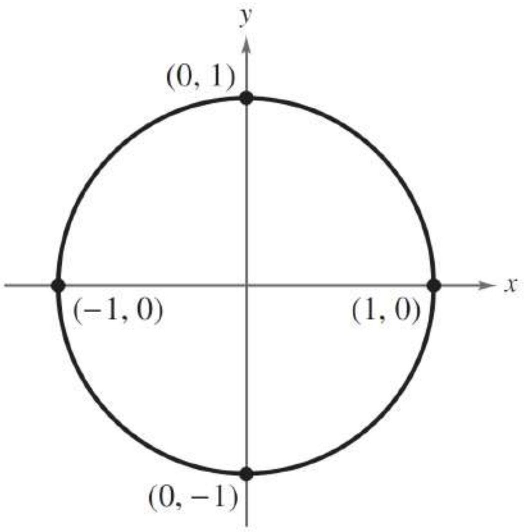


Figure 4.18

The Unit Circle

As you graph any angle on the unit circle, there is a point where its terminal side intersects the circle. The point is

$$(x, y)$$

If you graph an angle of 0 degrees or 0 radians, the terminal side intersection corresponds to the point (1, 0). Moreover, because the unit circle has a circumference of 2π, the angle 2π also corresponds to the point (1, 0).

The Trigonometric Functions

The Trigonometric Functions

You can use these coordinates to define the six trigonometric functions.

sine	**cosine**	**tangent**
cosecant	**secant**	**cotangent**

These six functions are normally abbreviated sin, cos, tan, csc, sec, and cot, respectively.

The Trigonometric Functions

Definitions of Trigonometric Functions

Let t be a real number and let (x, y) be the point on the unit circle corresponding to t.

$\sin t = y$ $\qquad$ $\cos t = x$ $\qquad$ $\tan t = \dfrac{y}{x}, \quad x \neq 0$

$\csc t = \dfrac{1}{y}, \quad y \neq 0$ $\qquad$ $\sec t = \dfrac{1}{x}, \quad x \neq 0$ $\qquad$ $\cot t = \dfrac{x}{y}, \quad y \neq 0$

The Trigonometric Functions

In the definitions of the trigonometric functions, note that the tangent and secant are not defined when $x = 0$.

For instance, because $t = \pi/2$ corresponds to $(x, y) = (0, 1)$, it follows that $\tan(\pi/2)$ and $\sec(\pi/2)$ are *undefined*.

Similarly, the cotangent and cosecant are not defined when $y = 0$.

For instance, because $t = 0$ corresponds to $(x, y) = (1, 0)$, $\cot 0$ and $\csc 0$ are *undefined.*

Example 1 – *Evaluating Trigonometric Functions*

Evaluate the six trigonometric functions at each real number.

a. $t = \frac{\pi}{6}$ **b.** $t = \frac{5\pi}{4}$ **c.** $t = \pi$

Solution:

For each *t*-value, begin by finding the corresponding point (*x*, *y*) on the unit circle. Then use the definitions of trigonometric functions.

Example1(a) – *Solution*

cont'd

t = π/6 corresponds to the point $(x, y) = \left(\sqrt{3}/2, 1/2\right)$.

$$\sin\frac{\pi}{6} = y = \frac{1}{2}$$

$$\csc\frac{\pi}{6} = \frac{1}{y} = \frac{1}{1/2} = 2$$

$$\cos\frac{\pi}{6} = x = \frac{\sqrt{3}}{2}$$

$$\sec\frac{\pi}{6} = \frac{1}{x} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan\frac{\pi}{6} = \frac{y}{x} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cot\frac{\pi}{6} = \frac{x}{y} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

Example1(b) – *Solution*

cont'd

$t = 5\pi/4$ corresponds to the point $(x, y) = \left(-\sqrt{2}/2, -\sqrt{2}/2\right)$.

$$\sin\frac{5\pi}{4} = y = -\frac{\sqrt{2}}{2} \qquad \csc\frac{5\pi}{4} = \frac{1}{y} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\cos\frac{5\pi}{4} = x = -\frac{\sqrt{2}}{2} \qquad \sec\frac{5\pi}{4} = \frac{1}{x} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\tan\frac{5\pi}{4} = \frac{y}{x} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1 \qquad \cot\frac{5\pi}{4} = \frac{x}{y} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$$

Example1(c) – *Solution*

cont'd

$t = \pi$ corresponds to the point $(x, y) = (-1, 0)$.

$$\sin \pi = y = 0$$

$$\csc \pi = \frac{1}{y} \text{ is undefined.}$$

$$\cos \pi = x = -1$$

$$\sec \pi = \frac{1}{x} = \frac{1}{-1} = -1$$

$$\tan \pi = \frac{y}{x} = \frac{0}{-1} = 0$$

$$\cot \pi = \frac{x}{y} \text{ is undefined.}$$

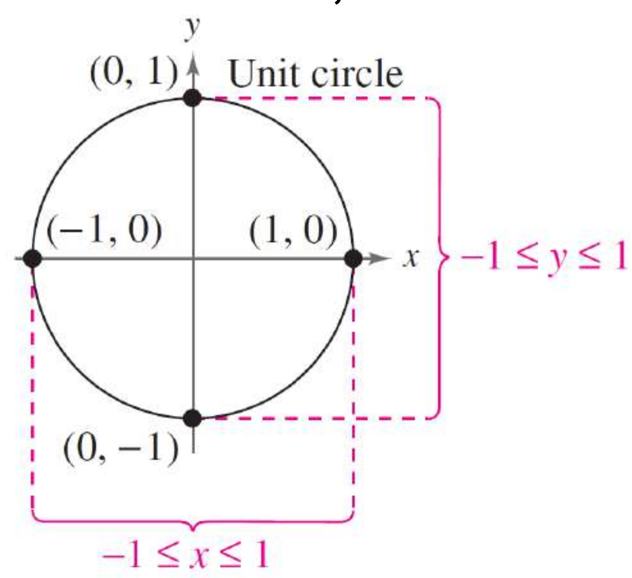
Domain and Period of Sine and Cosine

Domain and Period of Sine and Cosine

Please read this slide and the next three, but do not copy them down.

The *domain* of the sine and cosine functions is the set of all real numbers.

To determine the *range* of these two functions, consider the unit circle shown to the right.



Domain and Period of Sine and Cosine

Adding 2π to each value of in the interval $[0, 2\pi]$ completes a second revolution around the unit circle, as shown in Figure 4.23.

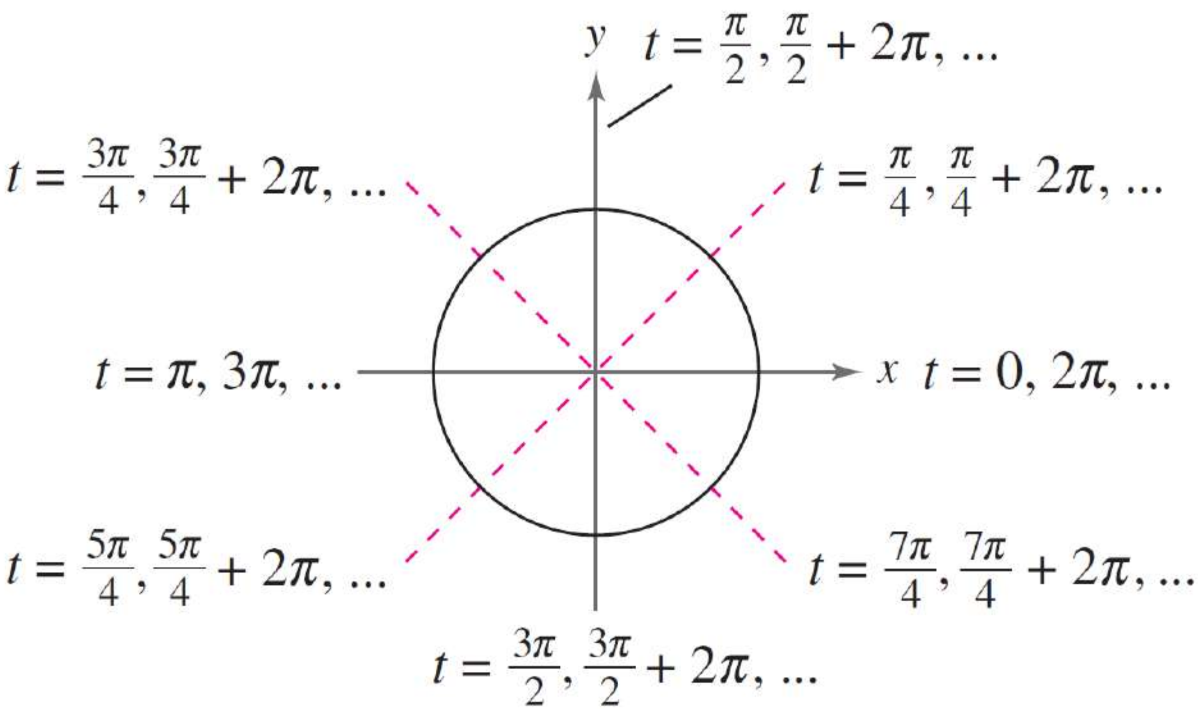


Figure 4.23

Domain and Period of Sine and Cosine

The values of $\sin(t + 2\pi)$ and $\cos(t + 2\pi)$ correspond to those of $\sin t$ and $\cos t$.

Similar results can be obtained for repeated revolutions (positive or negative) around the unit circle. This leads to the general result

$$\sin(t + 2\pi n) = \sin t \qquad \text{and} \qquad \cos(t + 2\pi n) = \cos t$$

for any integer n and real number t. Functions that behave in such a repetitive (or cyclic) manner are called **periodic.**

Domain and Period of Sine and Cosine

Definition of Periodic Function

A function f is **periodic** when there exists a positive real number c such that

$$f(t + c) = f(t)$$

for all t in the domain of f. The least number c for which f is periodic is called the **period** of f.

It follows from the definition of periodic function that the sine and cosine functions are periodic and have a period of 2π. The other four trigonometric functions are also periodic.

Domain and Period of Sine and Cosine

A function f is *even* when

$$f(-t) = f(t)$$

and is *odd* when

$$f(-t) = -f(t)$$

Of the six trigonometric functions, two are even and four are odd.

Example 2 – *Using the Period to Evaluate Sine and Cosine*

Because $\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$, you have

$$\sin \frac{13\pi}{6} = \sin\left(2\pi + \frac{\pi}{6}\right)$$

$$= \sin \frac{\pi}{6}$$

$$= \frac{1}{2}.$$