3 Exponential and Logarithmic Functions

3.2

Logarithmic Function and Their Graphs

What You Should Learn

- Recognize and evaluate logarithmic functions with base *a*.
- Graph logarithmic functions with base *a*.
- Recognize, evaluate, and graph natural logarithmic functions.
- Use logarithmic functions to model and solve real-life problems.

Logarithmic Functions

Logarithmic Function

When a function is one-to-one—that is, when the function has the property that no horizontal line intersects its graph more than once—the function must have an inverse function. Every function of the form

$f(x) = a^x$, $a > 0$, $a \neq 1$

passes the Horizontal Line Test and therefore must have an inverse function. This inverse function is called the **logarithmic function with base *a*.**

Logarithmic Function

Definition of Logarithmic Function

For $x > 0$, $a > 0$, and $a \neq 1$,

$$y = \log_a x \quad \text{if and only if} \quad x = a^y.$$

The function given by

$$f(x) = \log_a x$$

Read as "log base *a* of *x*."

is called the **logarithmic function with base *a*.**

Logarithmic Function

Every logarithmic equation can be written in an equivalent exponential form and every exponential equation can be written in logarithmic form.

The equations

$y = \log_a x$ and $x = a^y$

are equivalent.

Example 1 – *Evaluating Logarithms*

Use the definition of logarithmic function to evaluate each logarithm at the indicated value of *x*.

Function Value

a. $f(x) = \log_2 xx = 2$

b. $f(x) = \log_3 xx = 1$

c. $f(x) = \log_4 xx = 2$

d. $f(x) = \log_{10} xx =$ $\frac{1}{100}$

Example 1 – *Solution*

a. $f(32) = \log_2 32 = 5$ because $2^5 = 32$.

b. $f(1) = \log_3 1 = 0$ because $3^0 = 1$.

c. $f(2) = \log_4 2 = \frac{1}{2}$ because $3^{4/2} = \sqrt{4}$ 2.

d. $f(\frac{1}{100}) = \log_{10} \frac{1}{100} = -2$ because $10^{-2} = \frac{1}{10^2} \quad \frac{1}{100}$

Logarithmic Function

The logarithmic function with base 10 is called the **common logarithmic function.**

The following properties follow directly from the definition of the logarithmic function with base *a*.

Properties of Logarithms

1. $\log_a 1 = 0$ because $a^0 = 1$.
2. $\log_a a = 1$ because $a^1 = a$.
3. $\log_a a^x = x$ and $a^{\log_a x} = x$. — Inverse Properties
4. If $\log_a x = \log_a y$, then $x = y$. — One-to-One Property

Example 3 – *Using Properties of Logarithm*

a. Solve for x: $f(x) = \log_2 x = \log_2 3$

b. Solve for x: $f(x) = \log_4 4 = x$

c. Simplify : $\log_5 5^x$

d. Simplify : $7^{\log_7 14}$

Solution:

a. Using the One-to-One Property (Property 4), you can conclude that $x = 3$.

Example 3 – *Solution*

cont'd

b. Using Property 2, you can conclude that $x = 1$.

c. Using the Inverse Property (Property 3), it follows that $\log_5 5^x = x$.

d. Using the Inverse Property (Property 3), it follows that $7^{\log_7 14} = 14$.

Graphs of Logarithmic Functions

Graphs of Logarithmic Function

To sketch the graph of

$y = \log_a x$

you can use the fact that the graphs of inverse functions are reflections of each other in the line $y = x$.

Example 4 – *Graph of Exponential and Logarithmic Function*

In the same coordinate plane, sketch the graph of each function by hand.

a. $f(x) = 2^x$

b. $g(x) = \log_2 x$

Solution:

For $f(x) = 2^x$,construct a table of values. By plotting these points and Connecting them with a smooth curve, you obtain the graph of shown in Figure 3.16.

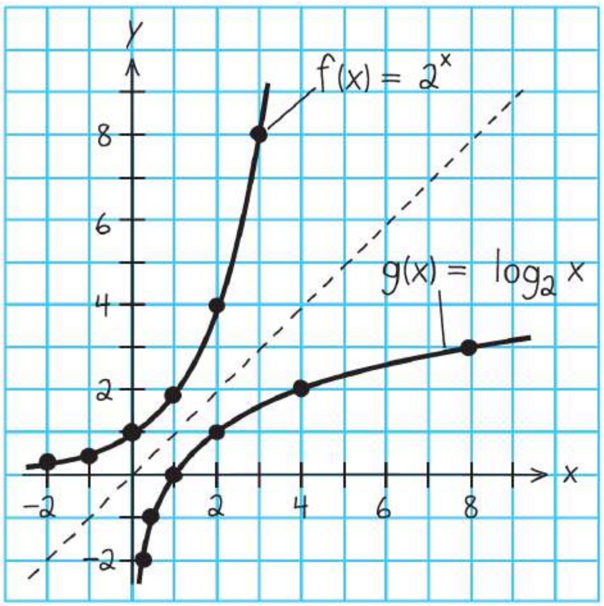


Figure 3.16

Example 4 – *Solution*

cont'd

x	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Because $g(x) = \log_2 x$ is the inverse function of $f(x) = 2^x$, the graph of is obtained by plotting the points $(f(x), x)$ and connecting them with a smooth curve.

The graph of g is a reflection of the graph of f in the line $y = x$, as shown in Figure 3.16.

Graphs of Logarithmic Function

The *parent logarithmic function*

$f(x) = \log_a x,\ a > 0,\ a \neq 1$

is the inverse function of the exponential function. Its domain is the set of positive real numbers and its range is the set of all real numbers. This is the opposite of the exponential function.

Moreover, the logarithmic function has the *y*-axis as a vertical asymptote, whereas the exponential function has the *x*-axis as a horizontal asymptote.

Graphs of Logarithmic Function

Many real-life phenomena with slow rates of growth can be modeled by logarithmic functions. The basic characteristics of the logarithmic function are summarized below

Graph of $f(x) = \log_a x$, $a > 1$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

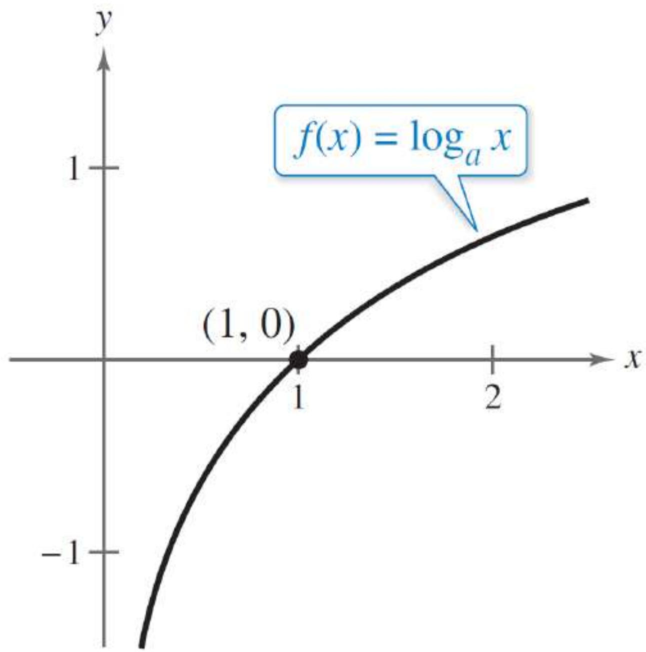
Intercept: (1,0)

Increasing on: $(0, \infty)$

y-axis is a vertical asymptote ($\log_a x \to -\infty$, as $x \to 0^+$)

Graphs of Logarithmic Function

Continuous Reflection of graph of $f(x) = a^x$ in the line $y = x$



Example 6 – *Library of Parent function* $f(x) = \log_a x$

Each of the following functions is a transformation of the graph of

$f(x) = \log_{10}x$.

a. Because $g(x) = \log_{10}(x - 1) = f(x - 1)$ the graph of g can be obtained by shifting the graph of one unit to the *right*, as shown in Figure 3.18.

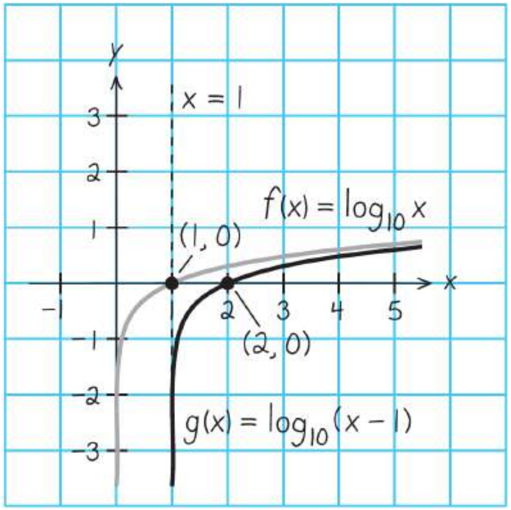


Figure 3.18

Example 6 – *Library of Parent function* $f(x) = \log_a x$ cont'd

b. Because $h(x) = 2 + \log_{10}x = 2 + f(x)$ the graph of *h* can be obtained by shifting the graph of f two units *upward*, as shown in Figure 3.18.

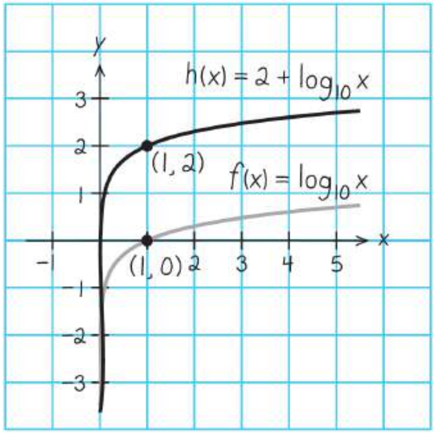


Figure 3.19

Notice that the transformation in Figure 3.19 keeps the *y*-axis as a vertical asymptote, but the transformation in Figure 3.18 yields the new vertical asymptote $x = 1$.

The Natural Logarithmic Functions

The Natural Logarithmic Function

The function $f(x) = e^x$ is one-to-one and so has an inverse function. This inverse function is called the **natural logarithmic function** and is denoted by the special symbol lnx read as "the natural log of *x*" or "el en of *x*."

The Natural Logarithmic Function

For $x > 0$,

$$y = \ln x \quad \text{if and only if} \quad x = e^y.$$

The function given by

$$f(x) = \log_e x = \ln x$$

is called the **natural logarithmic function.**

The Natural Logarithmic Function

The equations $y = \ln x$ and $x = e^y$ are equivalent. Note that the natural logarithm $\ln x$ is written without a base.

The base is understood to be e.

Example 7 – *Evaluating Natural Logarithmic function*

Use a calculator to evaluate the function

$f(x) = \ln x$

at each indicated value of x.

a. $x = 2$

b. $x = 0.3$

c. $x = -1$

Example 7 – *Solution*

Function Value	*Graphing Calculator Keystrokes*	*Display*
a. $f(2) = \ln 2$	LN 2 ENTER	0.6931472
b. $f(0.3) = \ln 0.3$	LN .3 ENTER	−1.2039728
c. $f(-1) = \ln(-1)$	LN (−) 1 ENTER	ERROR

The Natural Logarithmic Function

The *following* properties of logarithms are valid for natural logarithms.

Properties of Natural Logarithms

1. $\ln 1 = 0$ because $e^0 = 1$.
2. $\ln e = 1$ because $e^1 = e$.
3. $\ln e^x = x$ and $e^{\ln x} = x$. — Inverse Properties
4. If $\ln x = \ln y$, then $x = y$. — One-to-One Property

Example 8 – *Using Properties of Natural Algorithm*

Use the properties of natural logarithms to rewrite each expression.

a. $\ln \frac{1}{e}$ **b.** $e^{\ln 5}$ **c.** $4 \ln 1$ **d.** $2 \ln e$

Solution:

a. $\ln \frac{1}{e} = \ln e^{-1} = -1$ Inverse Property

b. $e^{\ln 5} = 5$ Inverse Property

c. $4 \ln 1 = 4(0) = 0$ Property 1

d. $2 \ln e = 2(1) = 2$ Property 2

Example 9 – *Finding the Domains of Logarithmic Functions*

Find the domain of each function.

a. $f(x) = \ln(x - 2)$ **b.** $g(x) = \ln(2 - x)$ **c.** $h(x) = \ln x^2$

Solution:

a. Because $\ln(x - 2)$ is defined only when

$x - 2 > 0$

it follows that the domain of f is $(2, \infty)$.

Example 9 – *Solution*

cont'd

b. Because $\ln(2 - x)$ is defined only when

$2 - x > 0$

it follows that the domain of is $(-\infty, 2)$.

c. Because $\ln x^2$ defined only when

$x^2 > 0$

it follows that the domain of is all real numbers except $x = 0$

Application

Application

Logarithmic functions are used to model many situations in real life, as shown in the next example.

Example 10 – *Psychology*

Students participating in a psychology experiment attended several lectures on a subject and were given an exam. Every month for a year after the exam, the students were retested to see how much of the material they remembered. The average scores for the group are given by the *human memory model*

$f(t) = 75 - 6 \ln(t + 1), \quad 0 \leq t \leq 12$

where t is the time in months. The graph of f is shown in Figure 3.21.

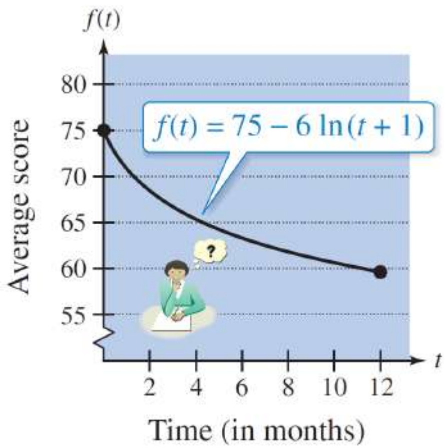


Figure 3.21

Example 10 – *Psychology*

cont'd

a. What was the average score on the original exam ($t = 0$)

b. What was the average score at the end of $t = 2$ months?

c. What was the average score at the end of $t = 6$ months?

Solution:

a. The original average score was

$$f(0) = 75 - 6 \ln(0 + 1)$$

$$= 75 - 6 \ln 1$$

$$= 75 - 6(0)$$

$$= 75.$$

Example 10 – *Solution*

cont'd

b. After 2 months, the average score was

$$f(2) = 75 - 6\ln(2 + 1)$$

$$= 75 - 6\ln 3$$

$$\approx 75 - (61.0986)$$

$$\approx 68.41.$$

Example 10 – *Solution*

cont'd

c. After 6 months, the average score was

$$f(6) = 75 - 6\ln(6 + 1)$$

$$= 75 - 6\ln 7$$

$$\approx 75 - (1.9459)$$

$$\approx 63.32.$$