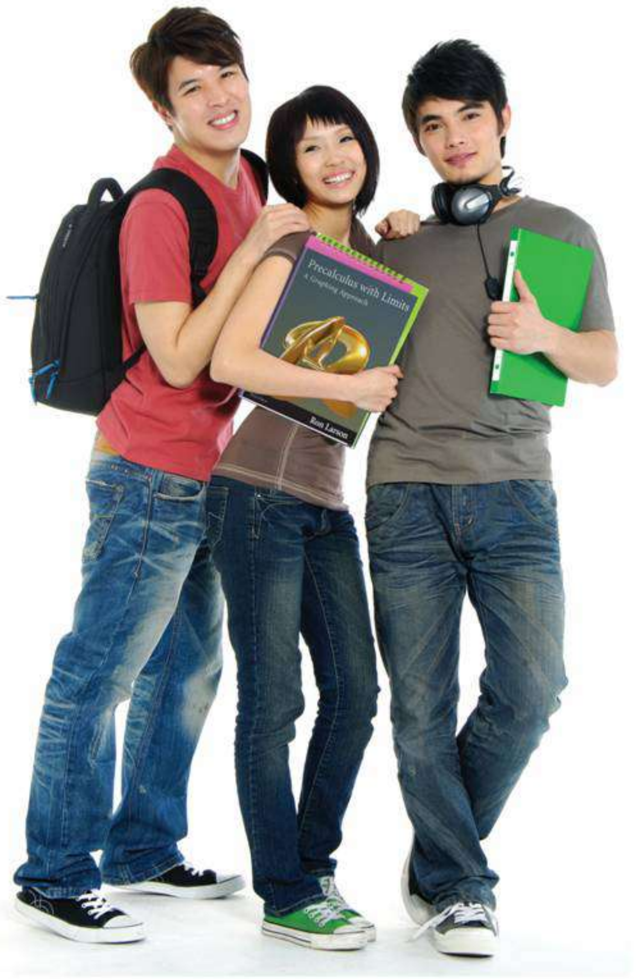
2

Polynomial and Rational Functions



2.2

Polynomial Functions of Higher Degree

What You Should Learn

- Use transformations to sketch graphs of polynomial functions.
- Use the Leading Coefficient Test to determine the end behavior of graphs of polynomial functions.
- Find and use zeros of polynomial functions as sketching aids.
- Use the Intermediate Value Theorem to help locate zeros of polynomial functions.

Graphs of Polynomial Functions

Graphs of Polynomial Functions

At this point, you should be able to sketch accurate graphs of polynomial functions of degrees 0, 1, and 2.

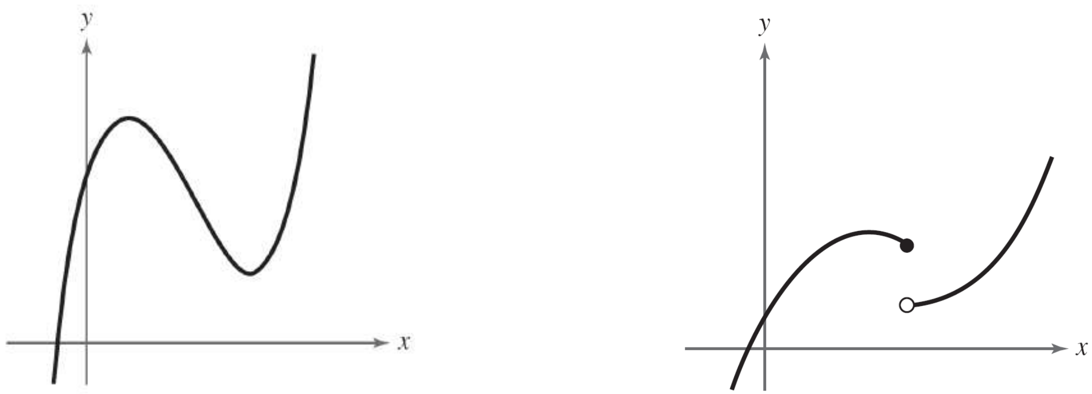
Function	*Graph*
$f(x) = a$	Horizontal line
$f(x) = ax + b$	Line of slope a
$f(x) = ax^2 + b + c$	Parabola

The graphs of polynomial functions of degree greater than 2 are more difficult to sketch by hand.

However, in this section you will learn how to recognize some of the basic features of the graphs of polynomial functions.

Graphs of Polynomial Functions

Using these features along with point plotting, intercepts, and symmetry, you should be able to make reasonably accurate sketches *by hand*. The graph of a polynomial function is **continuous.** Essentially, this means that the graph of a polynomial function has no breaks, holes, or gaps, as shown in Figure 2.7.



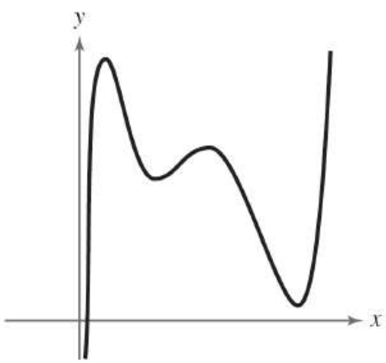
(a) Polynomial functions have continuous graphs.

(b) Functions with graphs that are not continuous are not polynomial functions.

Figure 2.7

Graphs of Polynomial Functions

Informally, you can say that a function is continuous when its graph can be drawn with a pencil without lifting the pencil from the paper. Another feature of the graph of a polynomial function is that it has only smooth, rounded turns, as shown in Figure 2.8(a).

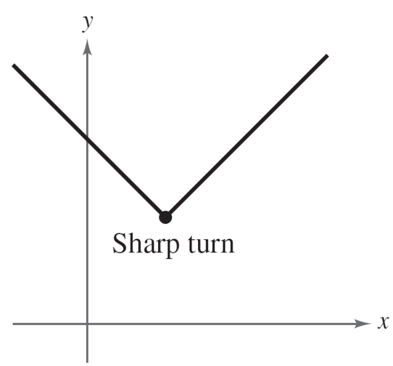


(a) Polynomial functions have graphs with smooth, rounded turns.

Figure 2.8

Graphs of Polynomial Functions

It cannot have a sharp turn such as the one shown in Figure 2.8(b).



(b) Functions with graphs that have sharp turns are not polynomial functions.

Figure 2.8

The graphs of polynomial functions of degree 1 are lines, and those of functions of degree 2 are parabolas.

Graphs of Polynomial Functions

The graphs of all polynomial functions are smooth and continuous. A polynomial function of degree n has the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_2 x^2 + a_1 x + a_0$$

where n is a positive integer and $a_n \neq 0$.

The polynomial functions that have the simplest graphs are monomials of the form $f(x) = x^n$, where n is an integer greater than zero.

The greater the value of n, the flatter the graph near the origin.

Graphs of Polynomial Functions

When n is even, the graph is similar to the graph of $f(x) = x^2$ and touches the x-axis at the x-intercept.

When n is odd, the graph is similar to the graph of $f(x) = x^3$ and crosses the x-axis at the x-intercept.

Polynomial functions of the form $f(x) = x^n$ are often referred to as **power functions**.

Example 1 – *Library of Parent Functions: $f(x) = x^3$*

Sketch the graphs of

(a) $g(x) = -x^3$

(b) $h(x) = x^3 + 1$

(c) $k(x) = (x - 1)^3$.

Example 1(a) – *Solution*

With respect to the graph of $f(x) = x^3$, the graph of g is obtained by a *reflection* in the x-axis, as shown in Figure 2.9.

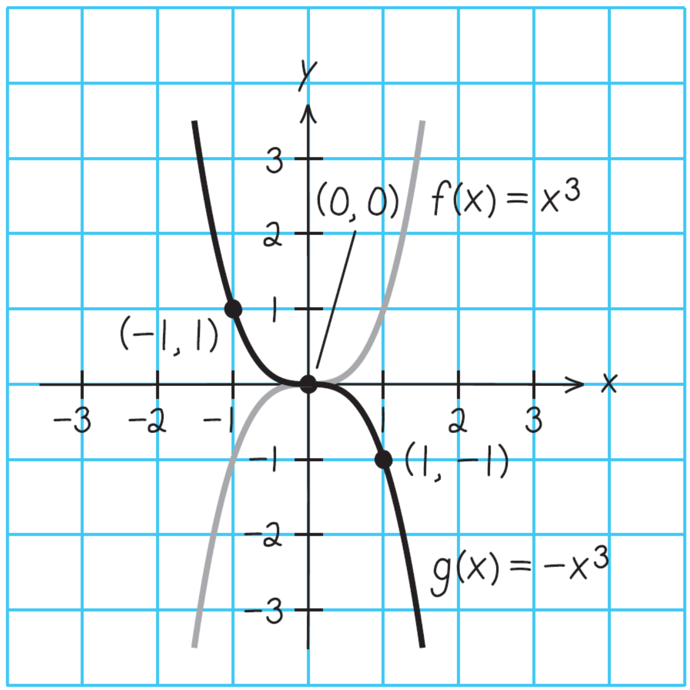


Figure 2.9

Example 1(b) – *Solution*

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With respect to the graph of $f(x) = x^3$, the graph of h is obtained by a *vertical shift* one unit *upward*, as shown in Figure 2.10.

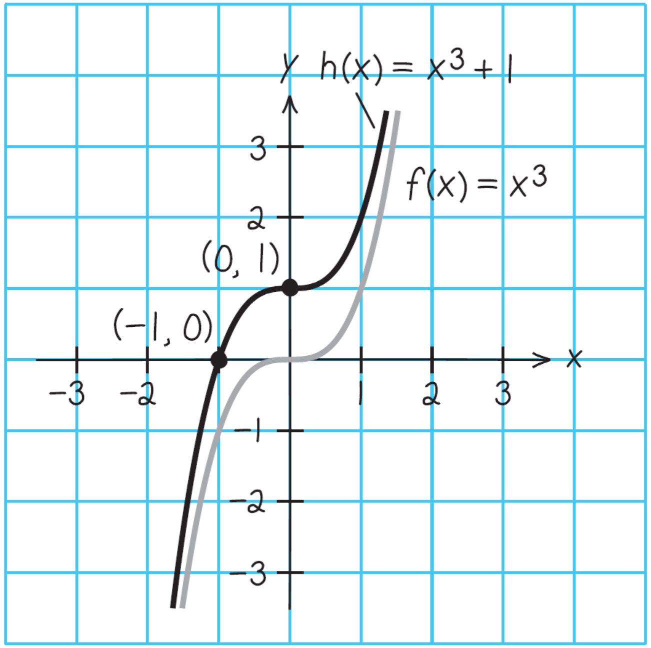

Figure 2.10

Example 1(c) – *Solution*

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With respect to the graph of $f(x) = x^3$, the graph of k is obtained by a horizontal shift one unit *to the right*, as shown in Figure 2.11.

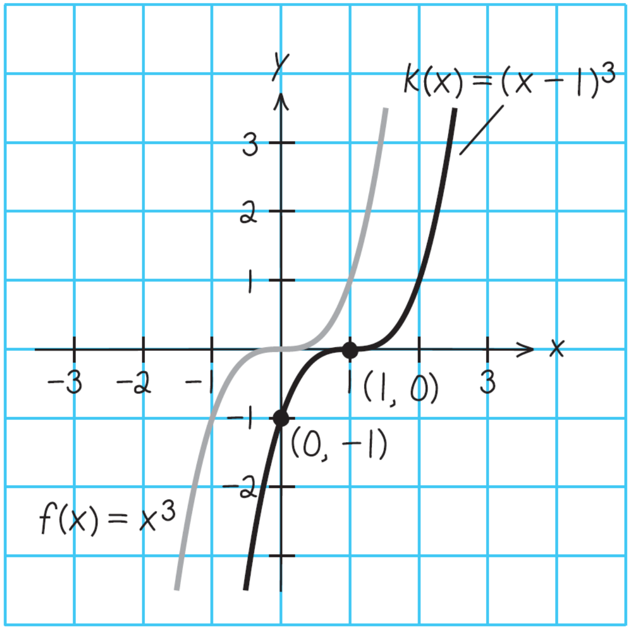


Figure 2.11

The Leading Coefficient Test

The Leading Coefficient Test

In Example 1, note that all three graphs eventually rise or fall without bound (this means it goes to positive infinity on the right) as *x* moves to the right.

Whether the graph of a polynomial eventually rises or falls can be determined by the polynomial function's degree (even or odd) and by its leading coefficient, as indicated in the **Leading Coefficient Test** .

The Leading Coefficient Test

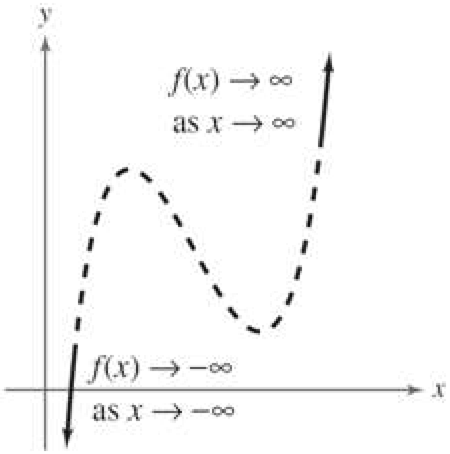
Leading Coefficient Test

As x moves without bound to the left or to the right, the graph of the polynomial function

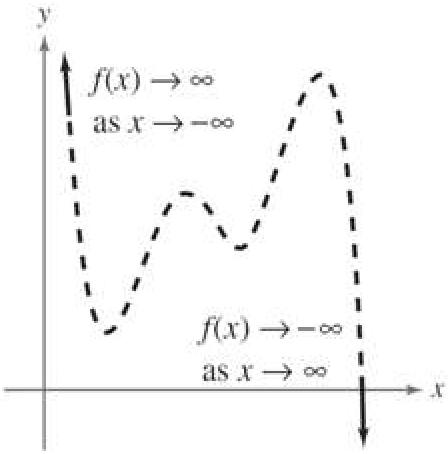
$$f(x) = a_n x^n + \cdot \cdot \cdot + a_1 x + a_0, \quad a_n \neq 0$$

eventually rises or falls in the following manner.

1. When n is odd:



If the leading coefficient is positive ($a_n > 0$), then the graph falls to the left and rises to the right.

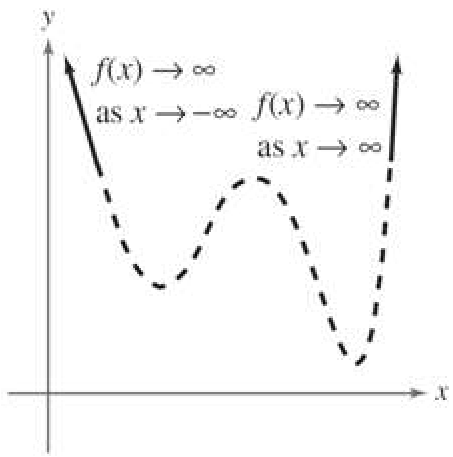


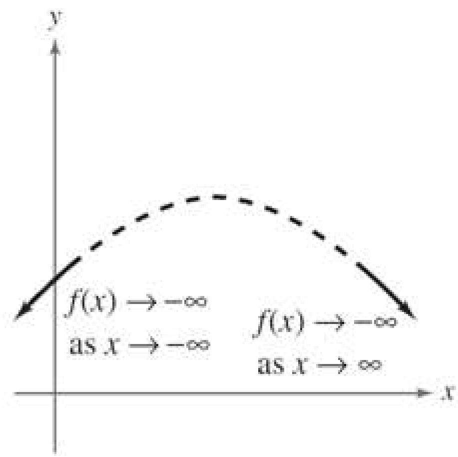
If the leading coefficient is negative ($a_n < 0$), then the graph rises to the left and falls to the right.

The Leading Coefficient Test

Leading Coefficient Test

2. When n is even:





If the leading coefficient is positive ($a_n > 0$), then the graph rises to the left and right.

If the leading coefficient is negative ($a_n < 0$), then the graph falls to the left and right.

Note that the dashed portions of the graphs indicate that the test determines only the right-hand and left-hand behavior of the graph.

Example 2 – *Applying the Leading Coefficient Test*

Use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of

$$f(x) = -x^3 + 4x.$$

Example 2 – *Solution*

Because the degree is odd and the leading coefficient is negative, the graph rises to the left and falls to the right, as shown in Figure 2.12.

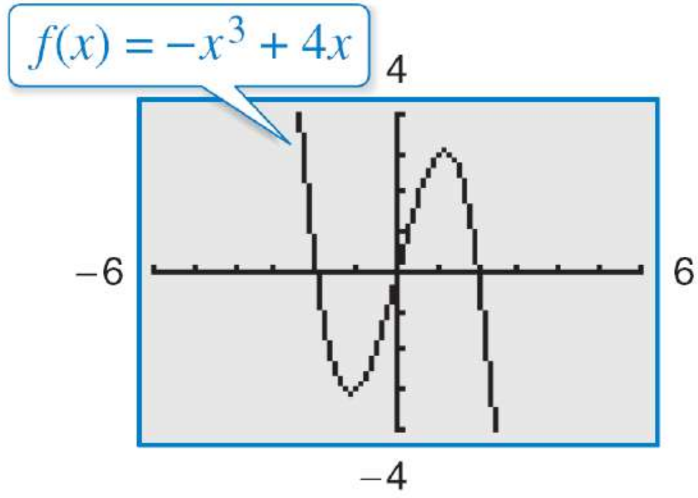


Figure 2.12

Zeros of Polynomial Functions

Zeros of Polynomial Functions

It can be shown that for a polynomial function *f* of degree *n*, the following statements are true.

1. The function *f* has at most *n* real zeros.

2. The graph of *f* has at most $n - 1$ relative **extrema** (relative minima or maxima).

Recall that a zero of a function *f* is a number *x* for which $f(x) = 0$.

We also call zeros x-intercepts or roots.

Zeros of Polynomial Functions

Real Zeros of Polynomial Functions

If f is a polynomial function and a is a real number, then the following statements are equivalent.

1. $x = a$ is a *zero* of the function f.

2. $x = a$ is a *solution* of the polynomial equation $f(x) = 0$.

3. $(x - a)$ is a *factor* of the polynomial $f(x)$.

4. $(a, 0)$ is an x-intercept of the graph of f.

Example 4 – *Finding Zeros of a Polynomial Function*

Find all real zeros of $f(x) = x^3 - x^2 - 2x$.

Solution:

$$f(x) = x^3 - x^2 - 2x.$$

Write original function.

$$0 = x^3 - x^2 - 2x$$

Substitute 0 for $f(x)$.

$$0 = x(x^2 - x - 2)$$

Remove common monomial factor.

$$0 = x(x - 2)(x + 1)$$

Factor completely and do the zero product property.

Example 4 – *Solution*

cont'd

So, the real zeros are

$$x = 0, \quad x = 2, \quad \text{and} \quad x = -1$$

and the corresponding x-intercepts are

$$(0, 0), (2, 0), \text{ and } (-1, 0).$$

Check

$$(0)^3 - (0)^2 - 2(0) = 0$$ $x = 0$ is a zero. ✓

$$(2)^3 - (2)^2 - 2(2) = 0$$ $x = 2$ is a zero. ✓

$$(-1)^3 - (-1)^2 - 2(-1) = 0$$ $x = -1$ is a zero. ✓

Zeros of Polynomial Functions

Repeated Zeros

For a polynomial function, a factor of $(x - a)^k$, $k > 1$, yields a **repeated zero** $x = a$ of **multiplicity** k.

1. If k is odd, then the graph *crosses* the x-axis at $x = a$.
2. If k is even, then the graph *touches* the x-axis (but does not cross the x-axis) at $x = a$.

Example 8 – *Sketching the Graph of a Polynomial Function*

Sketch the graph of

$$f(x) = 3x^4 - 4x^3$$

by hand.

Example 8 – *Solution*

1. *Apply the Leading Coefficient Test*. Because the leading coefficient is positive and the degree is even, you know that the graph eventually rises to the left and to the right (see Figure 2.18).

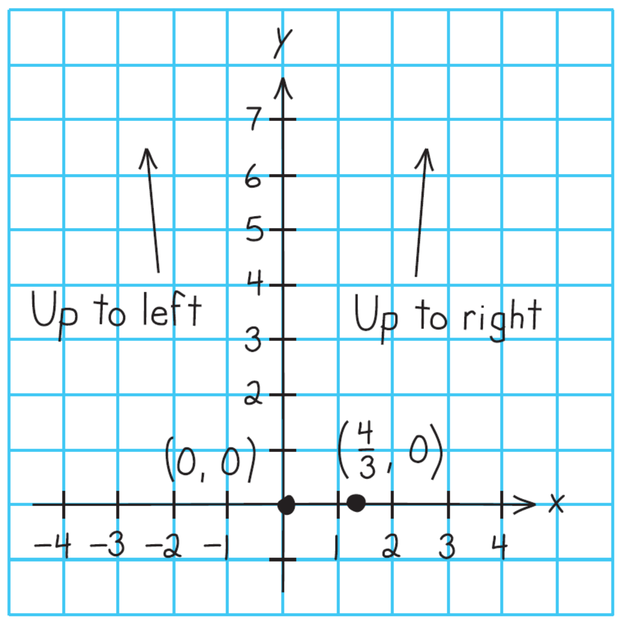


Figure 2.18

Example 8 – *Solution*

cont'd

2. *Find the Real Zeros of the Polynomial* by factoring and setting each factor equal to zero separately:

$$f(x) = 3x^4 - 4x^3 = x^3(3x - 4)$$

$0 = x^3 0 = 3x - 4$

you can see that the real zeros of f are $x = 0$ (of odd multiplicity 3) and $x = \frac{4}{3}$ (of odd multiplicity 1).

So, the x-intercepts occur at (0, 0) and (, ($\frac{4}{3}$. Add these points to your graph, as shown in Figure 2.18.

Example 8 – *Solution*

cont'd

3. *Plot a Few Additional Points*. To sketch the graph by hand, find a few additional points, as shown in the table.

Be sure to choose points between the zeros and to the left and right of the zeros.

x	-1	0.5	1	1.5
$f(x)$	7	-0.31	-1	1.69

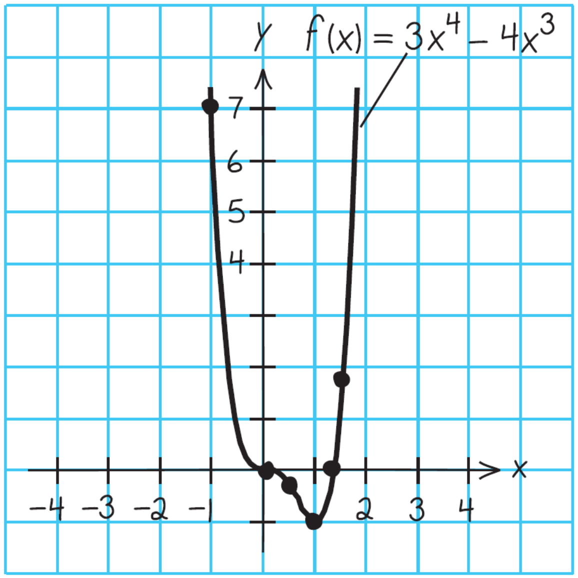


Figure 2.19

The Intermediate Value Theorem

The Intermediate Value Theorem

The **Intermediate Value Theorem** concerns the existence of real zeros of polynomial functions. The theorem states that if

$$(a, f(a)) \quad \text{and} \quad (b, f(b))$$

are two points on the graph of a polynomial function such that $f(a) \neq f(b)$, then for any number d between $f(a)$ and $f(b)$ there must be a number c between a and b such that $f(c) = d$.

The Intermediate Value Theorem

(See Figure 2.22.)

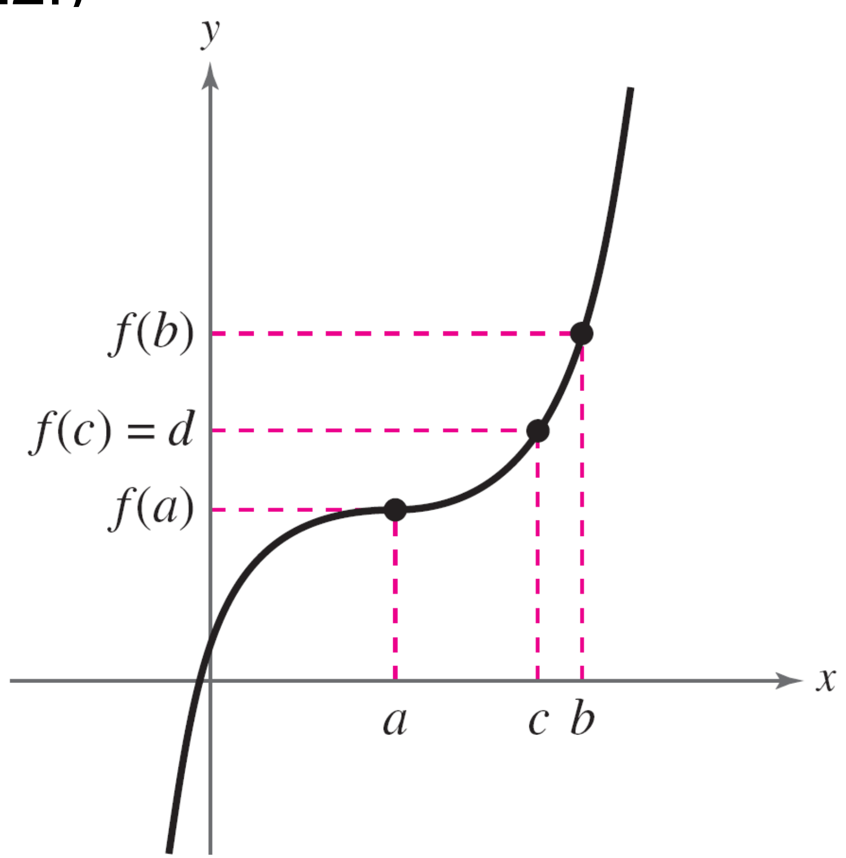


Figure 2.22

The Intermediate Value Theorem

Intermediate Value Theorem

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.

Example 10 – *Approximating the Zeros of a Function*

Find three intervals of length 1 in which the polynomial

$$f(x) = 12x^3 - 32x^2 + 3x + 5$$

is guaranteed to have a zero.

Solution:

From the table in Figure 2.24, you can see that $f(-1)$ and $f(0)$ differ in sign.

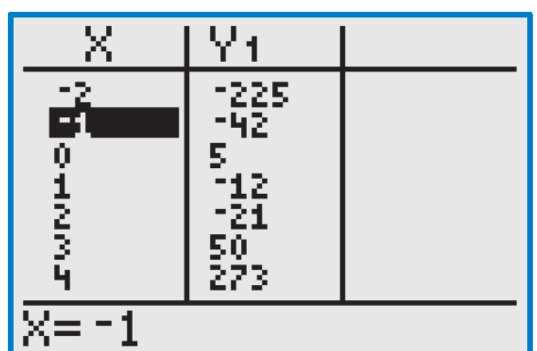


Figure 2.24

Example 10 – *Solution*

cont'd

So, you can conclude from the Intermediate Value Theorem that the function has a zero between –1 and 0. Similarly, $f(0)$ and $f(1)$ differ in sign, so the function has a zero between 0 and 1.

Likewise, $f(2)$ and $f(3)$ differ in sign, so the function has a zero between 2 and 3. So, you can conclude that the function has zeros in the intervals (–1, 0), (0, 1), and (2, 3).