2.3

Real Zeros of Polynomial Functions

What You Should Learn

- Use long division to divide polynomials by other polynomials.
- Use synthetic division to divide polynomials by binomials of the form $(x - k)$.
- Use the Remainder and Factor Theorems.

What You Should Learn

- Use the Rational Zero Test to determine possible rational zeros of polynomial functions.

- Use Descartes's Rule of Signs and the Upper and Lower Bound Rules to find zeros of polynomials.

Long Division of Polynomials

Example 1 – *Long Division of Polynomials*

Divide $6x^3 - 19x^2 + 16x - 4$ by $x - 2$, and use the result to factor the polynomial completely.

Solution:

$$
\begin{array}{rrrrrrrrl}
 & & & \multicolumn{5}{c}{\text{Partial quotients}} & \\
 & & & \downarrow & & \downarrow & & \downarrow & \\
 & & & 6x^2 & - & 7x & + & 2 & \\
x - 2 \big) & \overline{6x^3} & \overline{-} & \overline{19x^2} & \overline{+} & \overline{16x} & \overline{-} & \overline{4} & \\
 & \underline{6x^3} & \underline{-} & \underline{12x^2} & & & & & \text{Multiply: } 6x^2(x - 2). \\
 & & & -7x^2 & + & 16x & & & \text{Subtract.} \\
 & & & \underline{-7x^2} & \underline{+} & \underline{14x} & & & \text{Multiply: } -7x(x - 2). \\
 & & & & & 2x & - & 4 & \text{Subtract.} \\
 & & & & & \underline{2x} & \underline{-} & \underline{4} & \text{Multiply: } 2(x - 2). \\
 & & & & & & & 0 & \text{Subtract.}
\end{array}
$$

Example 1 – *Solution*

cont'd

You can see that

$$6x^3 - 19x^2 + 16x - 4 = (x - 2)(6x^2 - 7x + 2)$$

$$= (x - 2)(2x - 1)(3x - 2).$$

$\frac{1}{2}$ $\frac{2}{3}$

Long Division of Polynomials

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$\underset{\substack{\uparrow\\ \text{Dividend}}}{f(x)} = \underset{\substack{\uparrow\\ \text{Divisor}}}{d(x)}\underset{\substack{\uparrow\\ \text{Quotient}}}{q(x)} + \underset{\substack{\uparrow\\ \text{Remainder}}}{r(x)}$$

where $r(x) = 0$ *or* the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, then $d(x)$ *divides evenly* into $f(x)$.

Synthetic Division

Synthetic Division

There is a nice shortcut for long division of polynomials when dividing by divisors of the form

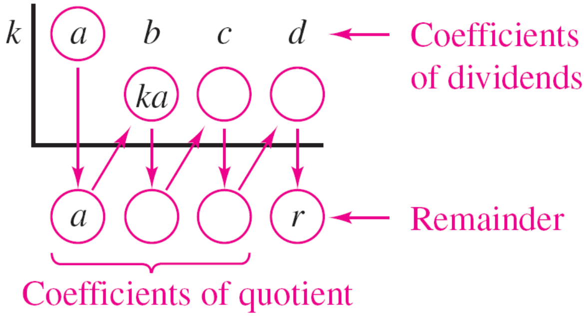
$$x - k.$$

The shortcut is called **synthetic division.** The pattern for synthetic division of a cubic polynomial is summarized as follows. (The pattern for higher-degree polynomials is similar.)

Synthetic Division

Synthetic Division (of a Cubic Polynomial)

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following pattern.



Vertical pattern: Add terms.
Diagonal pattern: Multiply by k.

This algorithm for synthetic division works *only* for divisors of the form $x - k$.

Remember that

$$x + k = x - (-k).$$

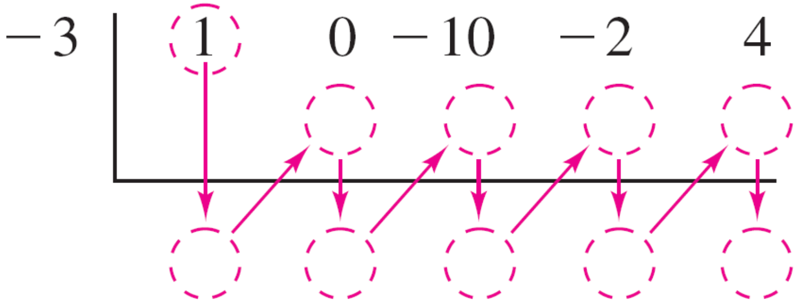
Example 4 – *Using Synthetic Division*

Use synthetic division to divide

$$x^4 - 10x^2 - 2x + 4 \text{ by } x + 3.$$

Solution:

You should set up the array as follows. Note that a zero is included for each missing term in the dividend.



Example 4 – *Solution*

cont’d

Then, use the synthetic division pattern by adding terms in columns and multiplying the results by –3.

Divisor: $x + 3$

Dividend: $x^4 - 10x^2 - 2x + 4$

$$\begin{array}{r|rrrrr} -3 & 1 & 0 & -10 & -2 & 4 \\ & & -3 & 9 & 3 & -3 \\ \hline & 1 & -3 & -1 & 1 & \boxed{1} \end{array}$$

Remainder: 1

Quotient: $x^3 - 3x^2 - x + 1$

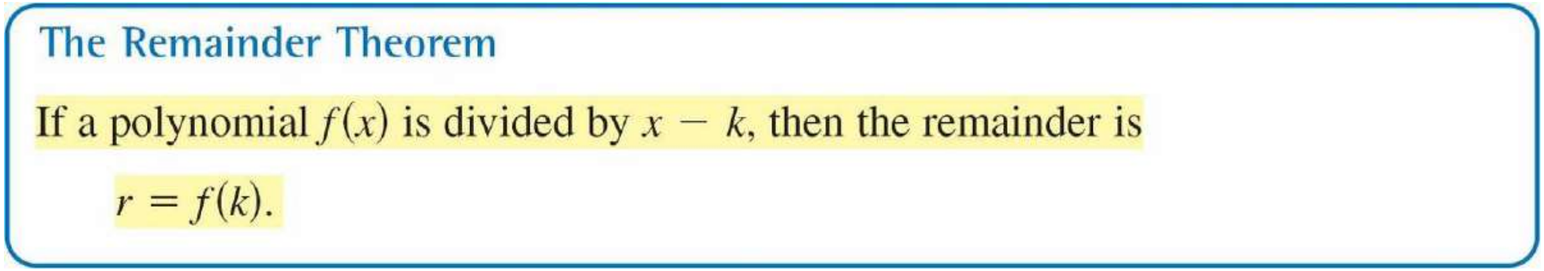
So, you have

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - x + 1 + \frac{1}{x + 3}.$$

The Remainder and Factor Theorems

The Remainder and Factor Theorems

The remainder obtained in the synthetic division process has an important interpretation, as described in the **Remainder Theorem.**



The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is

$$r = f(k).$$

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. That is, to evaluate a polynomial function $f(x)$ when $x = k$,
Divide $f(x)$ by $x - k$ the remainder will be $f(k)$.

Example 5 – *Using the Remainder Theorem*

Use the Remainder Theorem to evaluate the following function at $x = -2$.

$f(x) = 3x^3 + 8x^2 + 5x - 7$

Solution:

Using synthetic division, you obtain the following.

$$\begin{array}{r|rrrr} -2 & 3 & 8 & 5 & -7 \\ & & -6 & -4 & -2 \\ \hline & 3 & 2 & 1 & -9 \end{array}$$

Because the remainder is $r = -9$, you can conclude that

$f(-2) = -9$.

$r = f(k)$

Example 5 – *Solution*

cont'd

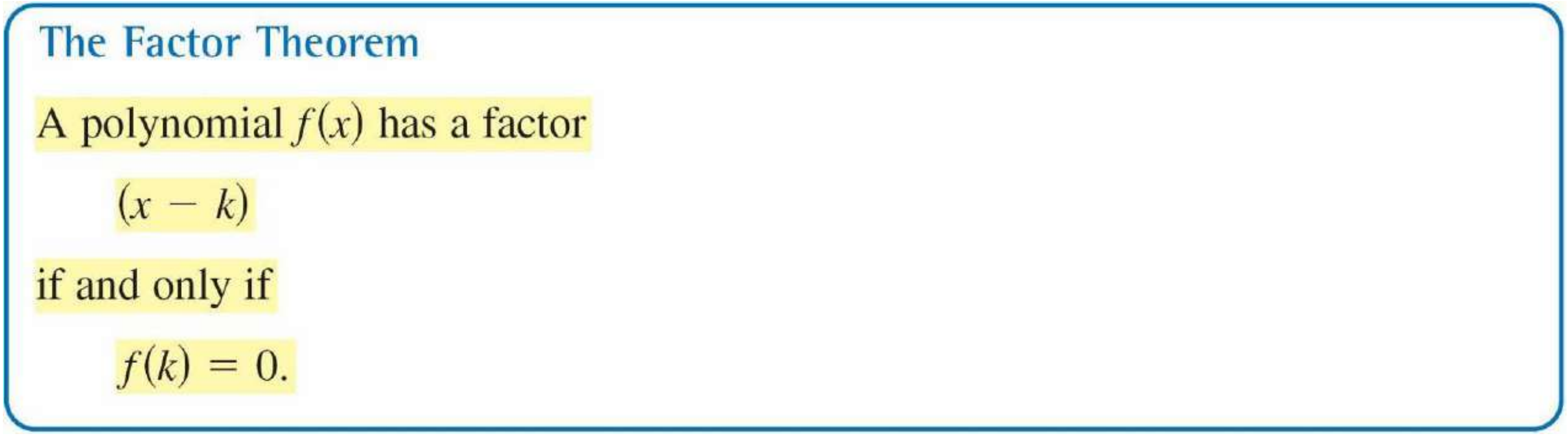
This means that (–2, –9) is a point on the graph of *f*. You can check this by substituting $x = -2$ in the original function.

Check

$$\begin{aligned} f(-2) &= 3(-2)^3 + 8(-2)^2 + 5(-2) - 7 \\ &= 3(-8) + 8(4) - 10 - 7 \\ &= -24 + 32 - 10 - 7 \\ &= -9 \end{aligned}$$

The Remainder and Factor Theorems

Another important theorem is the **Factor Theorem.** This theorem states that you can test whether a polynomial has (x – k) as a factor by evaluating the polynomial at $x = k$. If the result is 0, then (x – k) is a factor.



The Factor Theorem

A polynomial $f(x)$ has a factor

$$(x - k)$$

if and only if

$$f(k) = 0.$$

Example 6 – *Factoring a Polynomial: Repeated Division*

Show that $(x - 2)$ and $(x + 3)$ are factors of

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18.$$

Then find the remaining factors of $f(x)$.

Solution:

Using synthetic division with the factor $(x - 2)$, you obtain the following.

$$\begin{array}{r|rrrrr} 2 & 2 & 7 & -4 & -27 & -18 \\ & & 4 & 22 & 36 & 18 \\ \hline & 2 & 11 & 18 & 9 & 0 \end{array} \longrightarrow$$

0 remainder;
$(x - 2)$ is
a factor.

Example 6 – *Solution*

cont'd

Take the result of this division and perform synthetic division again using the factor (x + 3).

$$\begin{array}{r|rrrr} -3 & 2 & 11 & 18 & 9 \\ & & -6 & -15 & -9 \\ \hline & 2 & 5 & 3 & 0 \end{array}$$

$$\underbrace{2 \quad 5 \quad 3}_{2x^2 + 5x + 3}$$

→ 0 remainder; (x + 3) is a factor.

Because the resulting quadratic factors as

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

the complete factorization of $f(x)$ is

$$f(x) = (x - 2)(x + 3)(2x + 3)(x + 1).$$

The Remainder and Factor Theorems

Using the Remainder in Synthetic Division

In summary, the remainder r, obtained in the synthetic division of $f(x)$ by $x - k$, provides the following information.

1. The remainder r gives the value of f at $x = k$. That is, $r = f(k)$.

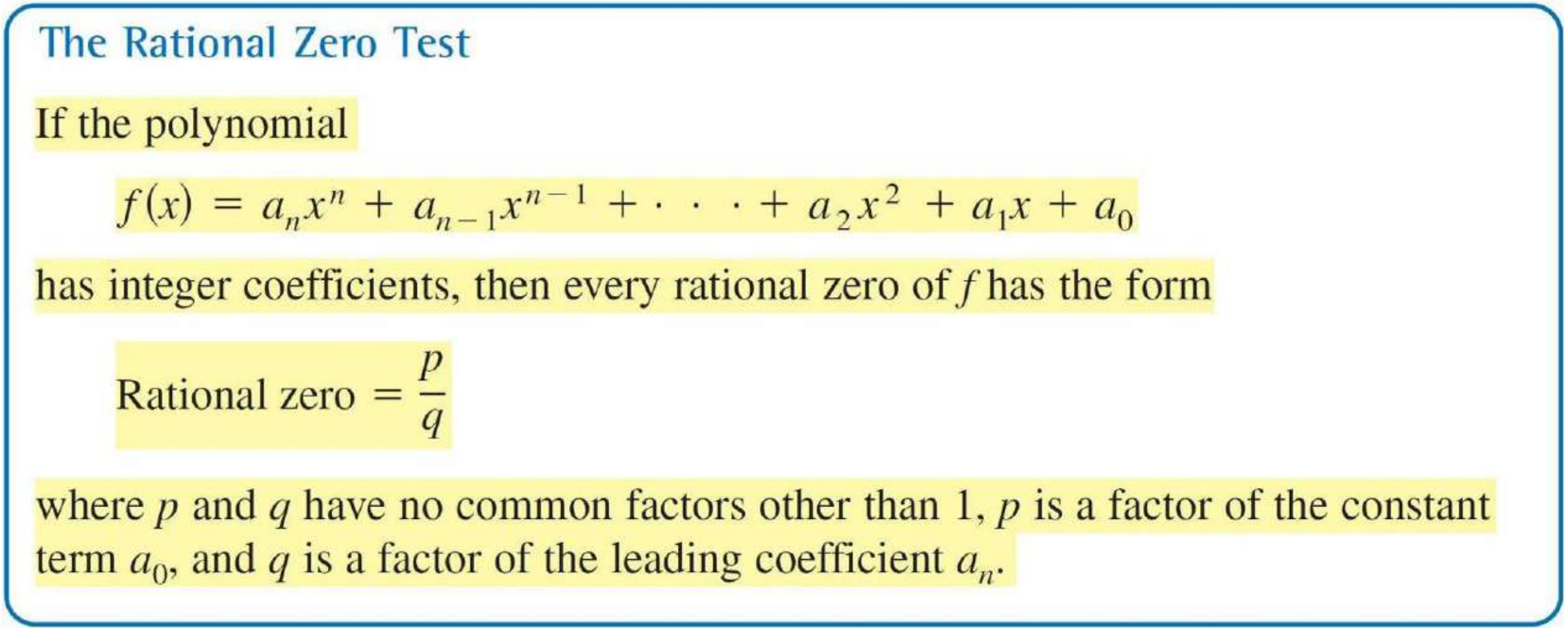
2. If $r = 0$, then $(x - k)$ is a factor of $f(x)$.

3. If $r = 0$, then $(k, 0)$ is an x-intercept of the graph of f.

The Rational Zero Test

The Rational Zero Test

The **Rational Zero Test** relates the possible rational zeros of a polynomial (having integer coefficients) to the leading coefficient and to the constant term of the polynomial.



The Rational Zero Test

If the polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$$

has integer coefficients, then every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q}$$

where p and q have no common factors other than 1, p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

The Rational Zero Test

To use the Rational Zero Test, first list all rational numbers whose numerators are factors of the constant term and whose denominators are factors of the leading coefficient.

$$\text{Possible rational zeros} = \frac{\text{factors of constant term}}{\text{factors of leading coefficient}}$$

Example 7 – *Rational Zero Test with Leading Coefficient of 1*

Find the rational zeros of $f(x) = x^3 + x + 1$.

Solution:

Because the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term.

Possible rational zeros: ±1

By testing these possible zeros, you can see that neither works.

$$f(1) = (1)^3 + 1 + 1$$

$$= 3$$

$$f(-1) = (-1)^3 + (-1) + 1$$

$$= -1$$

Example 7 – *Solution*

cont'd

So, you can conclude that the polynomial has *no* rational zeros. Note from the graph of *f* in Figure 2.27 that *f* does have one real zero between –1 and 0. However, by the Rational Zero Test, you know that this real zero is *not* a rational number.

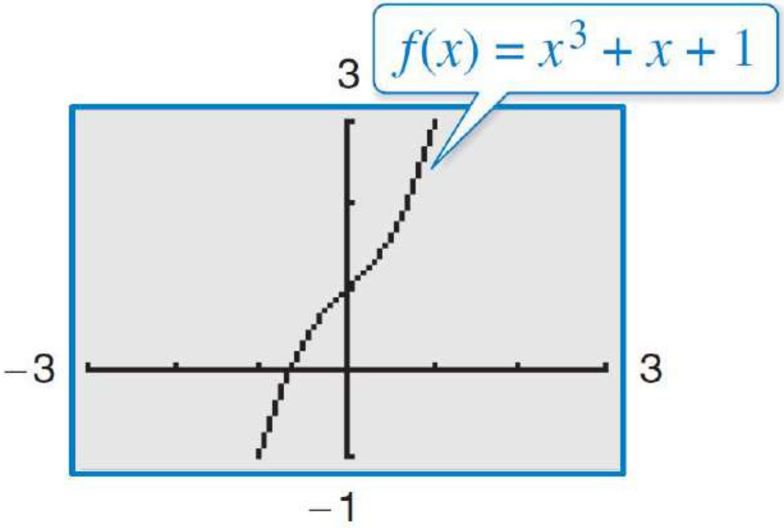


Figure 2.27

Example 8 – *Using the Rational Zero Test*

Find the rational zeros of

$$f(x) = 2x^3 + 3x^2 - 8x + 3.$$

Solution:

The leading coefficient is 2 and the constant term is 3.

Possible rational zeros:

$$\frac{\text{Factors of } 3}{\text{Factors of } 2} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}$$

By synthetic division, you can determine that $x = 1$ is a rational zero.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

Example 8 – *Solution*

cont'd

So, $f(x)$ factors as

$$f(x) = (x - 1)(2x^2 + 5x - 3)$$

$$= (x - 1)(2x - 1)(x + 3)$$

and you can conclude that the rational zeros of f are $x = 1$, $x = \frac{1}{2}$, and $x = -3$, as shown in Figure 2.28.

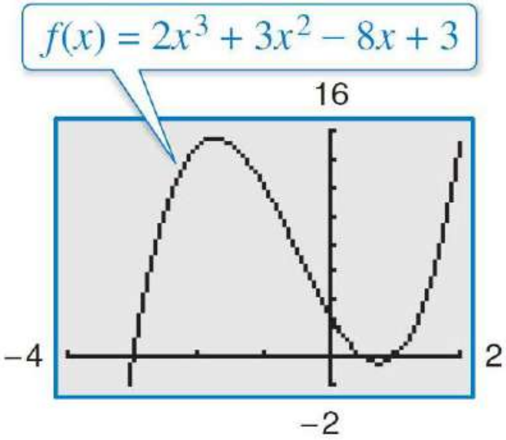


Figure 2.28

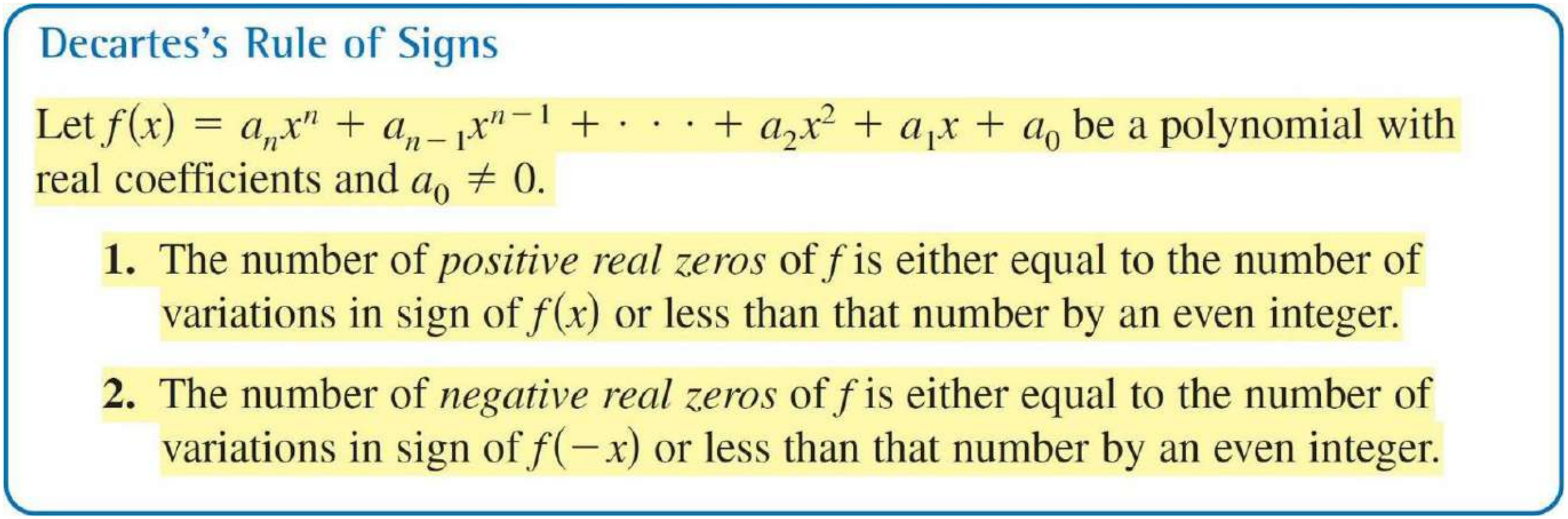
Other Tests for Zeros of Polynomials

Other Tests for Zeros of Polynomials

You know that an *n*th-degree polynomial function can have *at most n* real zeros. Of course, many *n*th-degree polynomials do not have that many real zeros.

For instance, $f(x) = x^2 + 1$ has no real zeros, and $f(x) = x^3 + 1$ has only one real zero. The following theorem, called **Descartes's Rule of Signs,** sheds more light on the number of real zeros of a polynomial.

Other Tests for Zeros of Polynomials



Decartes's Rule of Signs

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$.

1. The number of *positive real zeros* of f is either equal to the number of variations in sign of $f(x)$ or less than that number by an even integer.
2. The number of *negative real zeros* of f is either equal to the number of variations in sign of $f(-x)$ or less than that number by an even integer.

Other Tests for Zeros of Polynomials

A **variation in sign** means that two consecutive (nonzero) coefficients have opposite signs.

When using Descartes's Rule of Signs, a zero of multiplicity *k* should be counted as zeros. For instance, the polynomial $x^3 - 3x + 2$ has two variations in sign, and so has either two positive or no positive real zeros. Because

$$x^3 - 3x + 2 = (x - 1)(x - 1)(x + 2)$$

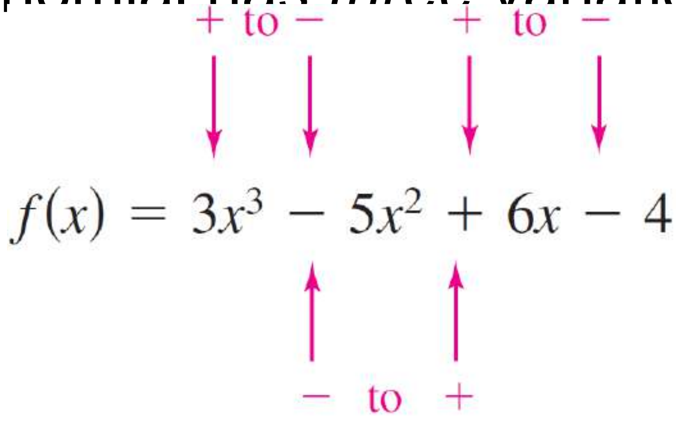
you can see that the two positive real zeros are $x = 1$ of multiplicity 2.

Example 9 – *Using Descartes's Rule of Signs*

Describe the possible real zeros of $f(x) = 3x^3 - 5x^2 + 6x - 4$.

Solution:

The original polynomial has *three* variations in sign.



$$f(x) = 3x^3 - 5x^2 + 6x - 4$$

Example 9 – *Solution*

cont'd

The polynomial

$$f(-x) = 3(-x)^3 - 5(-x)^2 + 6(-x) - 4$$
$$= -3x^3 - 5x^2 - 6x - 4$$

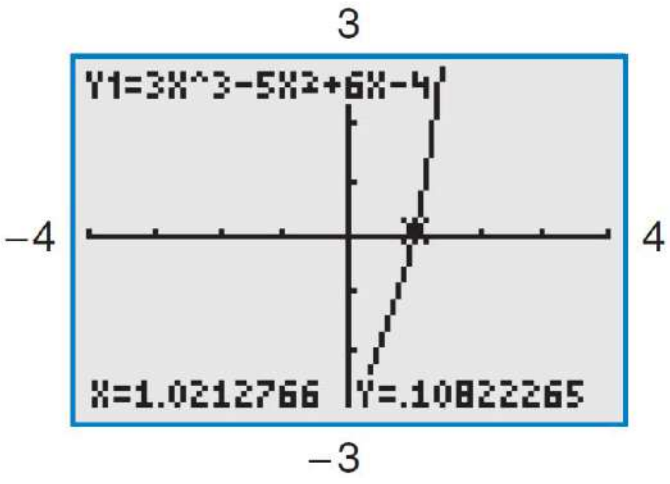
has no variations in sign.

So, from Descartes's Rule of Signs, the polynomial $f(x) = 3x^3 - 5x^2 + 6x - 4$ has either three positive real zeros or one positive real zero, and has no negative real zeros.

Example 9 – *Solution*

cont’d

By using the *trace* feature of a graphing utility, you can see that the function has only one real zero (it is a positive number near $x = 1$), as shown in Figure 2.29.



```
Y1=3X^3-5X²+6X-4
X=1.0212766    Y=.10822265
```

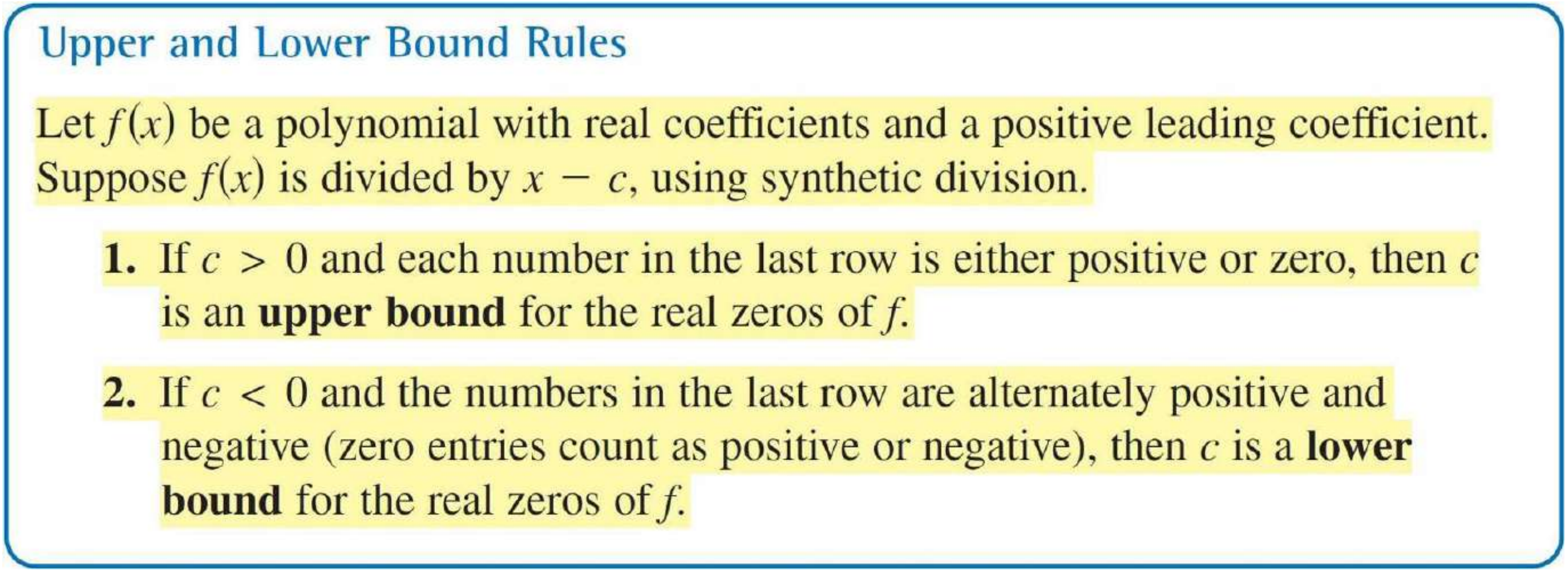
Figure 2.29

Other Tests for Zeros of Polynomials

Another test for zeros of a polynomial function is related to the sign pattern in the last row of the synthetic division array.

This test can give you an upper or lower bound of the real zeros of *f*, which can help you eliminate possible real zeros.

Other Tests for Zeros of Polynomials



Upper and Lower Bound Rules

Let $f(x)$ be a polynomial with real coefficients and a positive leading coefficient. Suppose $f(x)$ is divided by $x - c$, using synthetic division.

1. If $c > 0$ and each number in the last row is either positive or zero, then c is an **upper bound** for the real zeros of f.
2. If $c < 0$ and the numbers in the last row are alternately positive and negative (zero entries count as positive or negative), then c is a **lower bound** for the real zeros of f.