2.1 Quadratic Functions

What You Should Learn

- Analyze graphs of quadratic functions.
- Write quadratic functions in standard form and use the results to sketch graphs of functions.
- Find minimum and maximum values of quadratic functions in real-life applications.

The Graph of a Quadratic Function

The Graph of a Quadratic Function

Definition of Polynomial Function

Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function given by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a **polynomial function of x of degree n.**

The Graph of a Quadratic Function

Polynomial functions are classified by degree. For instance, the polynomial function

$f(x) = a, \qquad a \neq 0$ *Constant function*

has degree 0 and is called a **constant function.**

We have learned that the graph of this type of function is a horizontal line. The polynomial function

$f(x) = mx + b, \qquad m \neq 0$ *Linear function*

has degree 1 and is called a **linear function.**

The Graph of a Quadratic Function

We have learned that the graph of $f(x) = mx + b$ is a line whose slope is m and whose y-intercept is $(0, b)$. In this section, you will study second-degree polynomial functions, which are called **quadratic functions.**

Definition of Quadratic Function

Let a, b, and c be real numbers with $a \neq 0$. The function given by

$$f(x) = ax^2 + bx + c \qquad \text{Quadratic function}$$

is called a **quadratic function.**

The Graph of a Quadratic Function

The graph of a quadratic function is a special type of U-shaped curve called a **parabola.**

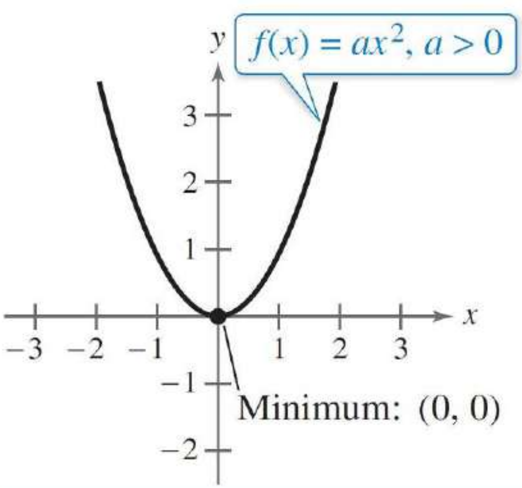
Parabolas occur in many real-life applications, especially those involving reflective properties, such as satellite dishes or flashlight reflectors.

All parabolas are symmetric with respect to a line called the **axis of symmetry,** or simply the **axis** of the parabola. The point where the axis intersects the parabola is called the **vertex** of the parabola.

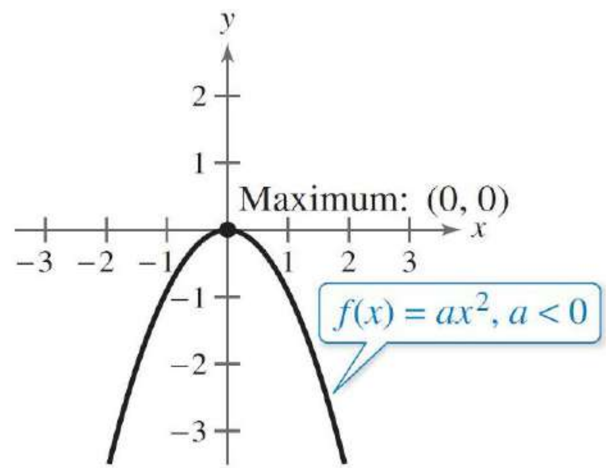
The Graph of a Quadratic Function

Basic Characteristics of Quadratic Functions

Graph of $f(x) = ax^2,\ a > 0$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Intercept: $(0, 0)$
Decreasing on $(-\infty, 0)$
Increasing on $(0, \infty)$
Even function
Axis of symmetry: $x = 0$
Relative minimum or vertex: $(0, 0)$



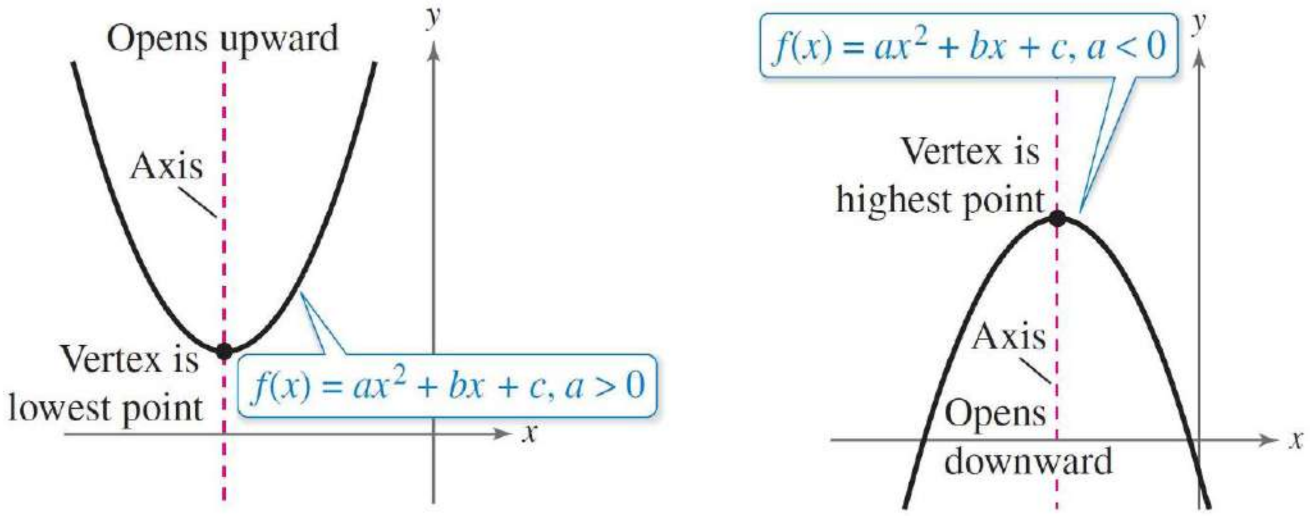
Graph of $f(x) = ax^2,\ a < 0$
Domain: $(-\infty, \infty)$
Range: $(-\infty, 0]$
Intercept: $(0, 0)$
Increasing on $(-\infty, 0)$
Decreasing on $(0, \infty)$
Even function
Axis of symmetry: $x = 0$
Relative maximum or vertex: $(0, 0)$



The Graph of a Quadratic Function

Basic Characteristics of Quadratic Functions

For the general quadratic form $f(x) = ax^2 + bx + c$, when the leading coefficient a is positive, the parabola opens upward; and when the leading coefficient a is negative, the parabola opens downward. Later in this section you will learn ways to find the coordinates of the vertex of a parabola.



Example 1 – *Library of Parent Functions: $f(x) = x^2$*

Sketch the graph of the function and describe how the graph is related to the graph of $f(x) = x^2$.

a. $g(x) = -x^2 + 1$

b. $h(x) = (x + 2)^2 - 3$

Example 1(a) – *Solution*

With respect to the graph of $f(x) = x^2$, the graph of g is obtained by a *reflection* in the x-axis and a vertical shift one unit *upward*.

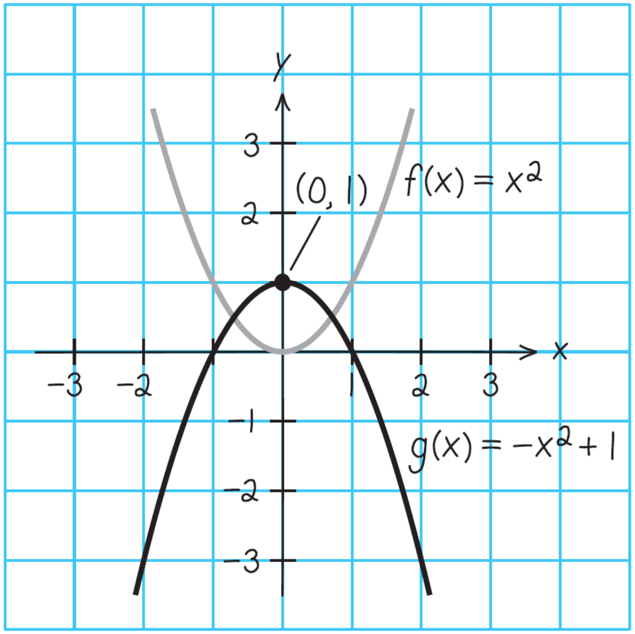


Figure 2.2

Example 1(b) – *Solution*

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The graph of *h* is obtained by a horizontal shift two units *to the left* and a vertical shift three units *downward*.

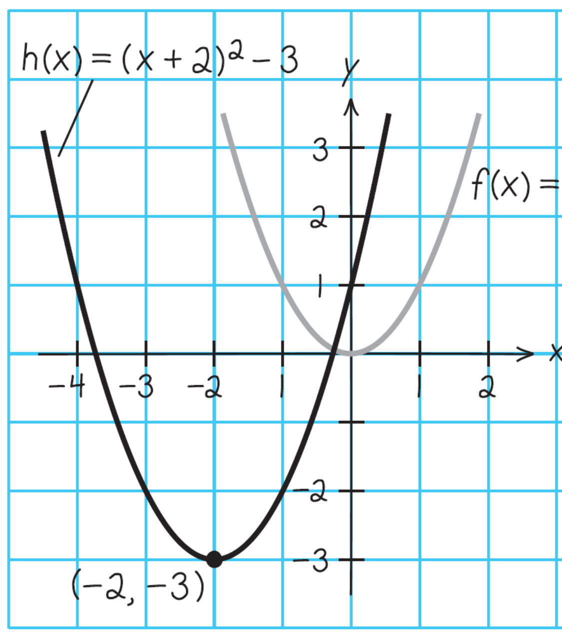


Figure 2.3

The Standard Form of a Quadratic Function

The Standard Form of a Quadratic Function

The equation in Example 1(b) is written in the **standard form or "graphing form"**

$f(x) = a(x - h)^2 + k$.

This form is especially convenient for sketching a parabola because it identifies the vertex of the parabola as (h, k).

Standard Form of a Quadratic Function

The quadratic function given by

$$f(x) = a(x - h)^2 + k, \quad a \neq 0$$

is in **standard form.** The graph of f is a parabola whose axis is the vertical line $x = h$ and whose vertex is the point (h, k). When $a > 0$, the parabola opens upward, and when $a < 0$, the parabola opens downward.

Example 2 – *Identifying the Vertex of a Quadratic Function*

Describe the graph of

$f(x) = 2x^2 + 8x + 7$

and identify the vertex.

Solution:

Write the quadratic function in standard form by completing the square. We know that the first step is to factor out any coefficient of x^2 that is not 1.

$f(x) = 2x^2 + 8x + 7$ *Write original function.*

Example 2 – *Solution*

cont'd

$= (2x^2 + 8x) + 7$ *Group x-terms.*

$= 2(x^2 + 4x) + 7$ *Factor 2 out of x-terms.*

$= 2(x^2 + 4x \underbrace{+ 4 - 4}_{\left(\frac{4}{2}\right)^2}) + 7$ *Add and subtract $(4/2)^2 = 4$ Within parentheses to complete the square*

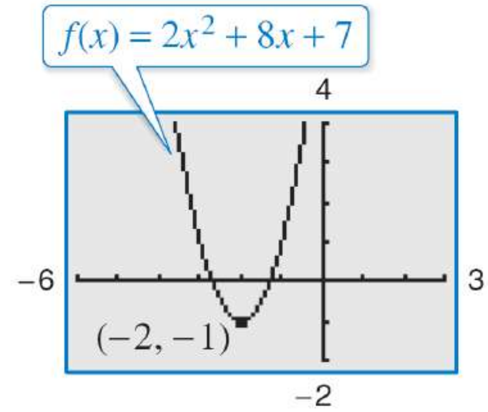
$= 2(x^2 + 4x + 4) - 2(4) + 7$ *Regroup terms.*

$= 2(x + 2)^2 - 1$ *Write in standard form.*

Example 2 – *Solution*

cont'd

From the standard form, you can see that the graph of *f* is a parabola that opens upward with vertex (–2, –1).



This corresponds to a left shift of two units and a downward shift of one unit relative to the graph of

$y = 2x^2$.

NOTE: If you have forgotten how to complete the square, or this method scares you ☺, just use –b/(2a) to find the x-coordinate of the vertex (h) and substitute it back into the function to find the y-coordinate of the vertex (k). The stretch factor /orientation (a) does not change.

Finding Minimum and Maximum Values

Finding Minimum and Maximum Values

Minimum and Maximum Values of Quadratic Functions

Consider the function $f(x) = ax^2 + bx + c$ with vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

1. If $a > 0$, then f has a *minimum* at $x = -\frac{b}{2a}$.

The minimum value is $f\left(-\frac{b}{2a}\right)$.

2. If $a < 0$, then f has a *maximum* at $x = -\frac{b}{2a}$.

The maximum value is $f\left(-\frac{b}{2a}\right)$.

Example 5 – *The Maximum Height of a Projectile*

A baseball is hit at a point 3 feet above the ground at a velocity of 100 feet per second and at an angle of 45° with respect to the ground. The path of the baseball is given by the function $f(x) = -0.0032x^2 + x + 3$, where $f(x)$ is the height of the baseball (in feet) and x is the horizontal distance from home plate (in feet). What is the maximum height reached by the baseball?

Solution:

For this quadratic function, you have

$$f(x) = ax^2 + bx + c = -0.0032x^2 + x + 3$$

which implies that $a = -0.0032$ and $b = 1$.

Example 5 – *Solution*

cont'd

Because the function has a maximum when $x = -b/(2a)$, you can conclude that the baseball reaches its maximum height when it is x feet from home plate, where x is

$$x = -\frac{b}{2a}$$

$$= -\frac{1}{2(-0.0032)}$$

$$= 156.25 \text{ feet}$$

At this distance, the maximum height is

$$f(156.25) = -0.0032(156.25)^2 + 156.25 + 3$$

$$= 81.125 \text{ feet.}$$