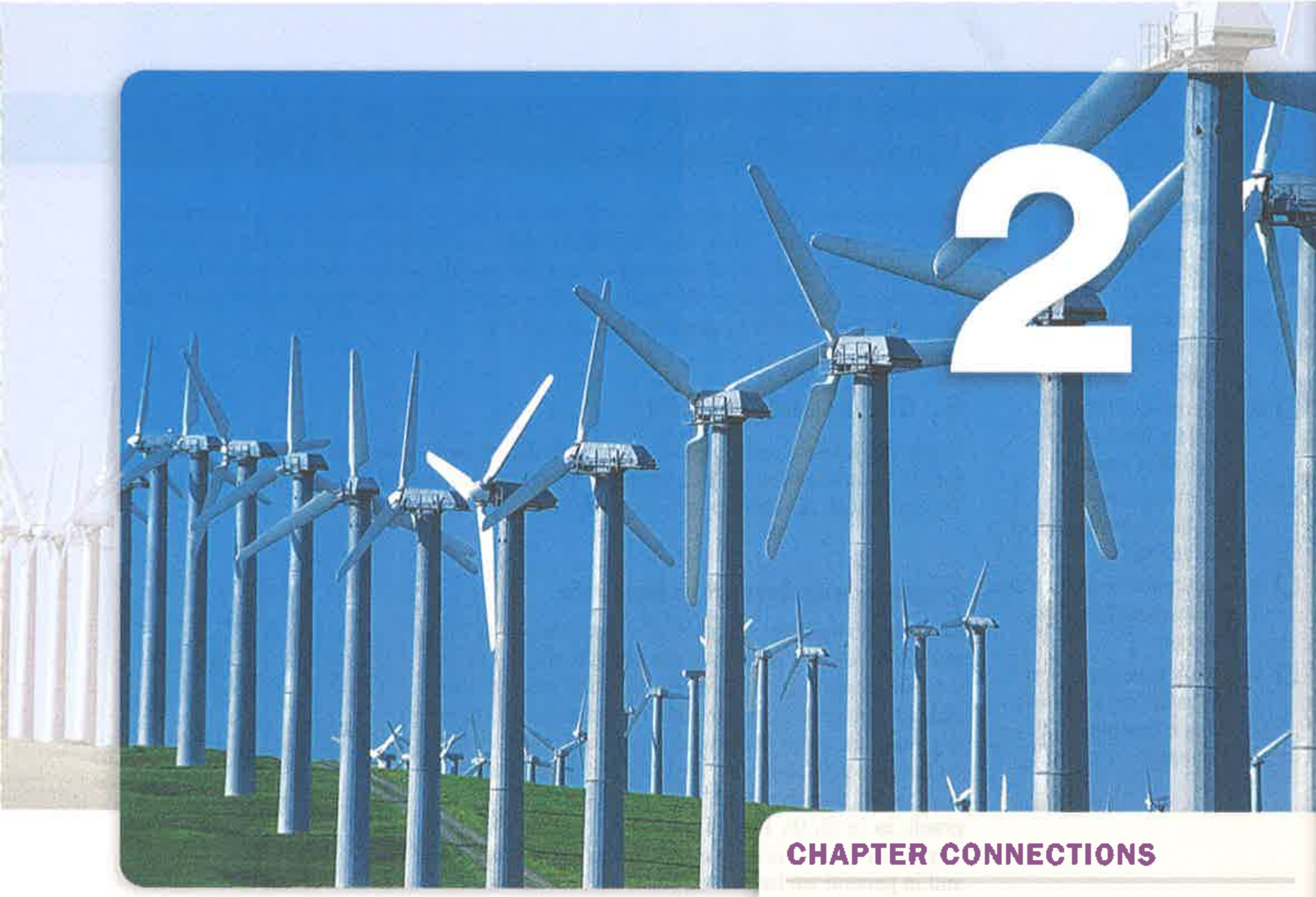


More on Functions

CHAPTER OUTLINE

2.1 Analyzing the Graph of a Function 106

2.2 The Toolbox Functions and Transformations 120

2.3 Absolute Value Functions, Equations, and Inequalities 136

2.4 Basic Rational Functions and Power Functions; More on the Domain 148

2.5 Piecewise-Defined Functions 163

2.6 Variation: The Toolbox Functions in Action 177

CHAPTER CONNECTIONS

Viewing a function in terms of an equation, a table of values, and the related graph, often brings a clearer understanding of the relationships involved. For example, the power generated by a wind turbine is often modeled by the function $P(v) = \frac{8v^3}{125}$, where P is the power in watts and v is the wind velocity in miles per hour. While the formula enables us to predict the power generated for a given wind speed, the graph offers a visual representation of this relationship, where we note a rapid growth in power output as the wind speed increases.

▶ This application appears as Exercise 107 in Section 2.2.

Connections to Calculus

The foundation and study of calculus involves using absolute value inequalities to analyze very small differences. The *Connections to Calculus* for Chapter 2 expands on the notation and language used in this analysis, and explores the need to solve a broad range of equation types.

2.1 Analyzing the Graph of a Function

LEARNING OBJECTIVES

In Section 2.1 you will see how we can

- ☐ **A.** Determine whether a function is even, odd, or neither
- ☐ **B.** Determine intervals where a function is positive or negative
- ☐ **C.** Determine where a function is increasing or decreasing
- ☐ **D.** Identify the maximum and minimum values of a function
- ☐ **E.** Locate local maximum and minimum values using a graphing calculator

In this section, we'll consolidate and refine many of the ideas we've encountered related to functions. When functions and graphs are applied as real-world models, we create numeric and visual representations that enable an informed response to questions involving *maximum* efficiency, *positive* returns, *increasing* costs, and other relationships that can have a great impact on our lives.

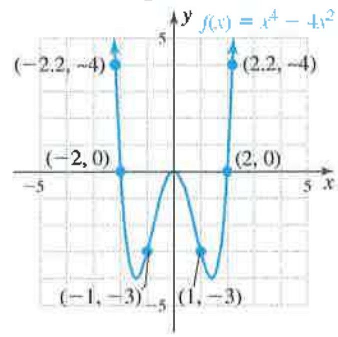
A. Graphs and Symmetry

While the domain and range of a function will remain dominant themes in our study, for the moment we turn our attention to other characteristics of a function's graph. We begin with the concept of symmetry.

Symmetry with Respect to the *y*-Axis

Consider the graph of $f(x) = x^4 - 4x^2$ shown in Figure 2.1, where the portion of the graph to the left of the y-axis appears to be a mirror image of the portion to the right. A function is **symmetric to the y-axis** if, given any point (x, y) on the graph, the point $(-x, y)$ is also on the graph. We note that $(-1, -3)$ is on the graph, as is $(1, -3)$, and that $(-2, 0)$ is an x-intercept of the graph, as is $(2, 0)$. Functions that are symmetric with respect to the y-axis are also known as **even functions** and in general we have:

Figure 2.1



Even Functions: *y*-Axis Symmetry

A function f is an *even function* if and only if, for each point (x, y) on the graph of f, the point $(-x, y)$ is also on the graph. In function notation

$$f(-x) = f(x)$$

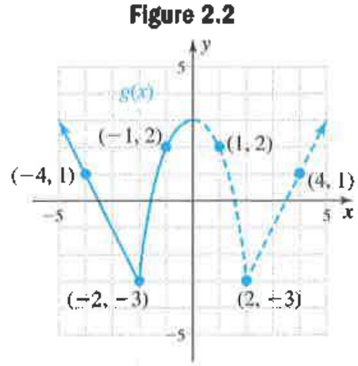
Symmetry can be a great help in graphing new functions, enabling us to plot fewer points and to complete the graph using properties of symmetry.

EXAMPLE 1 ▸ Graphing an Even Function Using Symmetry

a. The function $g(x)$ in Figure 2.2 (shown in solid blue) is known to be even. Draw the complete graph.

b. Show that $h(x) = x^{\frac{2}{3}}$ is an even function using the arbitrary value $x = k$ [show $h(-k) = h(k)$], then sketch the complete graph using $h(0)$, $h(1)$, $h(8)$, and y-axis symmetry.

Figure 2.2



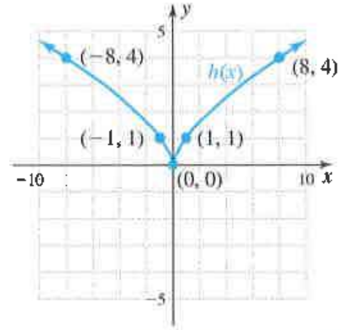
Solution ▸ **a.** To complete the graph of g (see Figure 2.2) use the points $(-4, 1)$, $(-2, -3)$, $(-1, 2)$, and y-axis symmetry to find additional points. The corresponding ordered pairs are $(4, 1)$, $(2, -3)$, and $(1, 2)$, which we use to help draw a "mirror image" of the partial graph given.

b. To prove that $h(x) = x^{\frac{2}{3}}$ is an even function, we must show $h(-k) = h(k)$ for any constant k. After writing $x^{\frac{2}{3}}$ as $[x^2]^{\frac{1}{3}}$, we have:

$$h(-k) \stackrel{?}{=} h(k) \quad \text{first step of proof}$$
$$[(-k)^2]^{\frac{1}{3}} \stackrel{?}{=} [(k)^2]^{\frac{1}{3}} \quad \text{evaluate } h(-k) \text{ and } h(k)$$
$$\sqrt[3]{(-k)^2} \stackrel{?}{=} \sqrt[3]{(k)^2} \quad \text{radical form}$$
$$\sqrt[3]{k^2} = \sqrt[3]{k^2}\checkmark \quad \text{result: } (-k)^2 = k^2$$

Using $h(0) = 0$, $h(1) = 1$, and $h(8) = 4$ with y-axis symmetry produces the graph shown in Figure 2.3.

Figure 2.3



WORTHY OF NOTE

The proof can also be demonstrated by writing $x^{\frac{2}{3}}$ as $(x^{\frac{1}{3}})^2$, and you are asked to complete this proof in **Exercise 69.**

Now try Exercises 7 through 12 ►

Symmetry with Respect to the Origin

Another common form of symmetry is known as **symmetry to the origin.** As the name implies, the graph is somehow "centered" at (0, 0). This form of symmetry is easy to see for closed figures with their center at (0, 0), like certain polygons, circles, and ellipses (these will exhibit both y-axis symmetry *and* symmetry with respect to the origin). Note the relation graphed in Figure 2.4 contains the points $(-3, 3)$ and $(3, -3)$, along with $(-1, -4)$ and $(1, 4)$. But the function $f(x)$ in Figure 2.5 also contains these points and is, in the same sense, symmetric to the origin (the paired points are on opposite sides of the x- and y-axes, and a like distance from the origin).

Figure 2.4

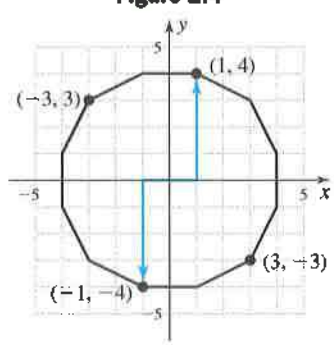
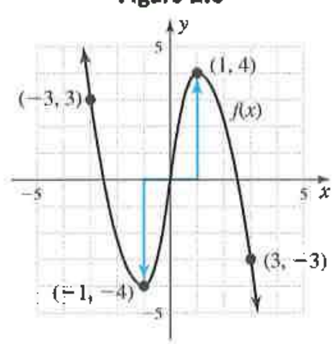


Figure 2.5



Functions symmetric to the origin are known as **odd functions** and in general we have:

Odd Functions: Symmetry About the Origin

A function f is an *odd function* if and only if, for each point (x, y) on the graph of f, the point $(-x, -y)$ is also on the graph. In function notation

$$f(-x) = -f(x)$$

EXAMPLE 2 ▶ **Graphing an Odd Function Using Symmetry**

a. In Figure 2.6, the function $g(x)$ given (shown in solid blue) is known to be *odd*. Draw the complete graph.

b. Show that $h(x) = x^3 - 4x$ is an odd function using the arbitrary value $x = k$ [show $h(-x) = -h(x)$], then sketch the graph using $h(-2)$, $h(-1)$, $h(0)$, and odd symmetry.

Solution ▶ **a.** To complete the graph of g, use the points $(-6, 3)$, $(-4, 0)$, and $(-2, 2)$ and odd symmetry to find additional points. The corresponding ordered pairs are $(6, -3)$, $(4, 0)$, and $(2, -2)$, which we use to help draw a "mirror image" of the partial graph given (see Figure 2.6).

Figure 2.6

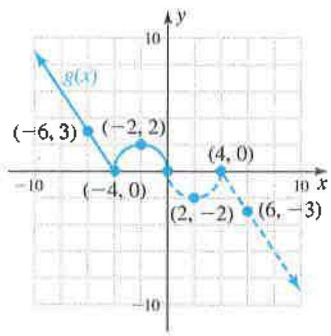
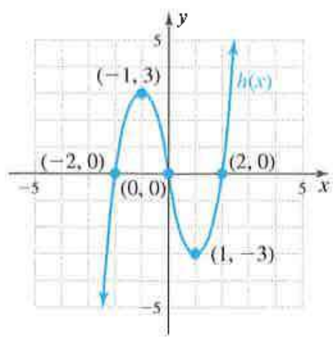


Figure 2.7



b. To prove that $h(x) = x^3 - 4x$ is an odd function, we must show that $h(-k) = -h(k)$.

$$h(-k) \stackrel{?}{=} -h(k)$$
$$(-k)^3 - 4(-k) \stackrel{?}{=} -[k^3 - 4k]$$
$$-k^3 + 4k = -k^3 + 4k \checkmark$$

Using $h(-2) = 0$, $h(-1) = 3$, and $h(0) = 0$ with symmetry about the origin produces the graph shown in Figure 2.7.

Now try Exercises 13 through 24 ▶

☑ **A. You've just seen how we can determine whether a function is even, odd, or neither**

Finally, some relations also exhibit a third form of symmetry, that of symmetry to the x-axis. If the graph of a circle is centered at the origin, the graph has both odd and even symmetry, and is also symmetric to the x-axis. Note that if a graph exhibits x-axis symmetry, *it cannot be the graph of a function.*

B. Intervals Where a Function Is Positive or Negative

Consider the graph of $f(x) = x^2 - 4$ shown in Figure 2.8, which has x-intercepts at $(-2, 0)$ and $(2, 0)$. As in Section 1.5, the x-intercepts have the form $(x, 0)$ and are called the **zeroes** of the function (the x-input causes an output of 0). Just as zero on the number line separates negative numbers from positive numbers, the zeroes of a function that crosses the x-axis separate x-intervals where a function is negative from x-intervals where the function is positive. Noting that outputs (y-values) are positive in Quadrants I and II, $f(x) > 0$ in intervals where its graph is *above the x-axis.* Conversely, $f(x) < 0$

in x-intervals where its graph is *below the x-axis*. To illustrate, compare the graph of f in Figure 2.8, with that of g in Figure 2.9.

Figure 2.8

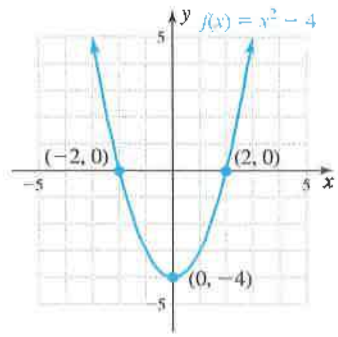
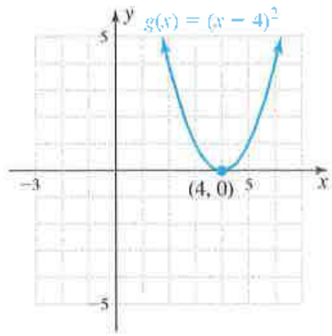


Figure 2.9



The graph of f is a parabola, with x-intercepts of $(-2, 0)$ and $(2, 0)$. Using our previous observations, we note $f(x) \geq 0$ for $x \in (-\infty, -2] \cup [2, \infty)$ since the graph is above the x-axis, and $f(x) < 0$ for $x \in (-2, 2)$. The graph of g is also a parabola, but is entirely above or on the x-axis, showing $g(x) \geq 0$ for $x \in \mathbb{R}$. The difference is that zeroes coming from factors of the form $(x - r)$ (with degree 1) allow the graph to cross the x-axis. The zeroes of f came from $(x + 2)(x - 2) = 0$. Zeroes that come from factors of the form $(x - r)^2$ (with degree 2) cause the graph to "bounce" off the x-axis (intersect without crossing) since all outputs must be nonnegative. The zero of g came from $(x - 4)^2 = 0$.

WORTHY OF NOTE

These observations form the basis for studying polynomials of higher degree in Chapter 4, where we extend the idea to factors of the form $(x - r)^n$ in a study of **roots of multiplicity.**

EXAMPLE 3 ▶ Solving an Inequality Using a Graph

Use the graph of $g(x) = x^3 - 2x^2 - 4x + 8$ given to solve the inequalities

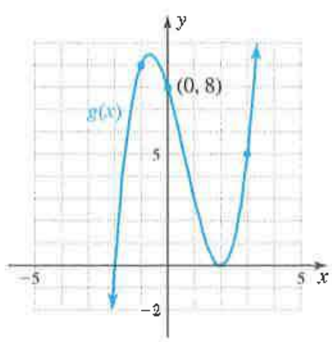
a. $g(x) \geq 0$ **b.** $g(x) < 0$

Solution ▶ From the graph, the zeroes of g (x-intercepts) occur at $(-2, 0)$ and $(2, 0)$.

a. For $g(x) \geq 0$, the graph must be on or above the x-axis, meaning the solution is $x \in [-2, \infty)$.

b. For $g(x) < 0$, the graph must be below the x-axis, and the solution is $x \in (-\infty, -2)$.

As we might have anticipated from the graph, factoring by grouping gives $g(x) = (x + 2)(x - 2)^2$, with the graph crossing the x-axis at -2, and bouncing off the x-axis (intersects without crossing) at $x = 2$.



Now try Exercises 25 through 28 ▶

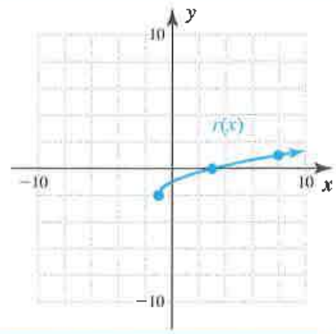
Even if the function is not a polynomial, the zeroes can still be used to find x-intervals where the function is positive or negative.

EXAMPLE 4 ▶ **Solving an Inequality Using a Graph**

For the graph of $r(x) = \sqrt{x+1} - 2$ shown, solve

a. $r(x) \le 0$

b. $r(x) > 0$



Solution ▶ a. The only zero of r is at (3, 0). The graph is on or below the x-axis for $x \in [-1, 3]$, so $r(x) \le 0$ in this interval.

b. The graph is above the x-axis for $x \in (3, \infty)$, and $r(x) > 0$ in this interval.

Now try Exercises 29 through 32 ▶

B. You've just seen how we can determine intervals where a function is positive or negative

This study of inequalities shows how the graphical solutions studied in Section 1.5 are easily extended to the graph of a general function. It also strengthens the foundation for the graphical solutions studied throughout this text.

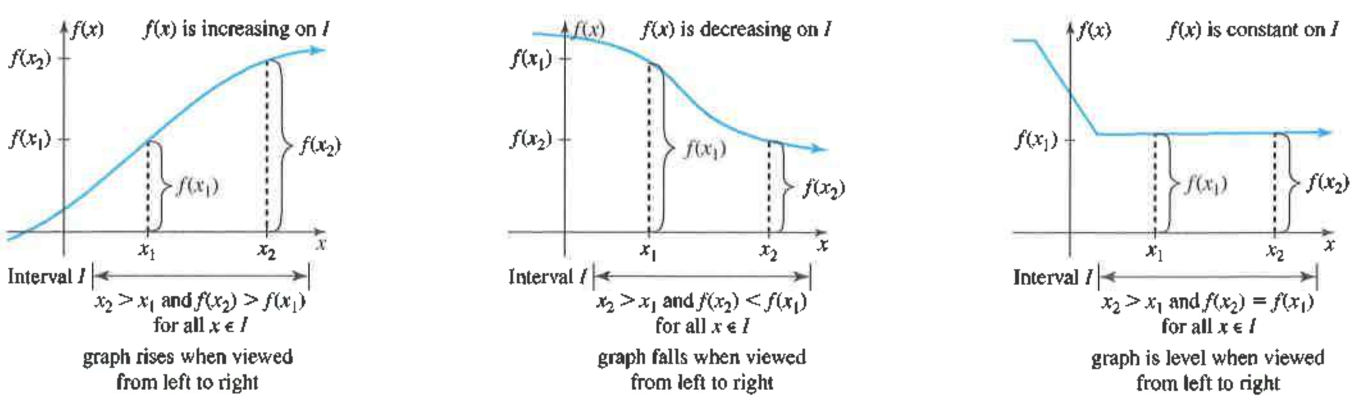
C. Intervals Where a Function Is Increasing or Decreasing

In our study of linear graphs, we said a graph was increasing if it "rose" when viewed from left to right. More generally, we say the graph of a function is increasing *on a given interval* if larger and larger x-values produce larger and larger y-values. This suggests the following tests for intervals where a function is increasing or decreasing.

Increasing and Decreasing Functions

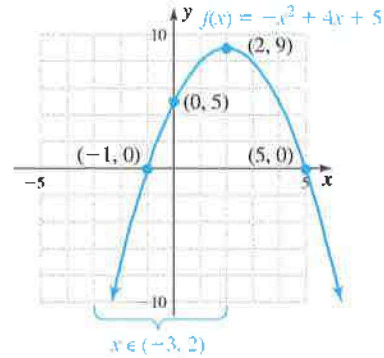
Given an interval I that is a subset of the domain, with x_1 and x_2 in I and $x_2 > x_1$,

1. A function is increasing on I if $f(x_2) > f(x_1)$ for all x_1 and x_2 in I (larger inputs produce larger outputs).
2. A function is decreasing on I if $f(x_2) < f(x_1)$ for all x_1 and x_2 in I (larger inputs produce smaller outputs).
3. A function is constant on I if $f(x_2) = f(x_1)$ for all x_1 and x_2 in I (larger inputs produce identical outputs).



Consider the graph of $f(x) = -x^2 + 4x + 5$ given in Figure 2.10. Since the parabola opens downward with the vertex at (2, 9), the function must increase until it reaches this peak at $x = 2$, and decrease thereafter. Notationally we'll write this as $f(x)\uparrow$ for $x \in (-\infty, 2)$ and $f(x)\downarrow$ for $x \in (2, \infty)$. Using the interval $(-3, 2)$ shown below the figure, we see that any larger input value from the interval will indeed produce a larger output value, and $f(x)\uparrow$ on the interval. For instance,

Figure 2.10



$$1 > -2 \qquad\qquad x_2 > x_1$$

and and

$$f(1) > f(-2) \qquad\qquad f(x_2) > f(x_1)$$

$$8 > -7$$

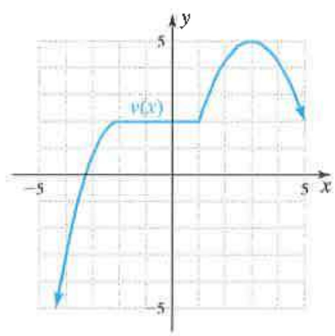
X	Y1
-3	-16
-2	-7
-1	0
0	5
1	8
2	9
3	8

Y1=8

A calculator check is shown in the figure. Note the outputs are increasing until $x = 2$, then they begin decreasing.

EXAMPLE 5 ▶ Finding Intervals Where a Function Is Increasing or Decreasing

Use the graph of $v(x)$ given to name the interval(s) where v is increasing, decreasing, or constant.



Solution ▶ From left to right, the graph of v increases until leveling off at $(-2, 2)$, then it remains constant until reaching (1, 2). The graph then increases once again until reaching a peak at (3, 5) and decreases thereafter. The result is $v(x)\uparrow$ for $x \in (-\infty, -2) \cup (1, 3)$, $v(x)\downarrow$ for $x \in (3, \infty)$, and $v(x)$ is constant for $x \in (-2, 1)$.

Now try Exercises 33 through 36 ▶

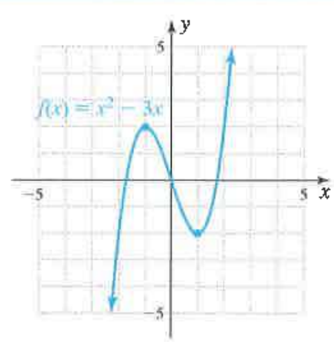
WORTHY OF NOTE

Questions about the behavior of a function are asked with respect to the y outputs: is the *function* positive, is the *function* increasing, etc. Due to the input/output, cause/effect nature of functions, the response is given in terms of x, that is, what is *causing* outputs to be positive, or to be increasing.

Notice the graph of f in Figure 2.10 and the graph of v in Example 5 have something in common. It appears that both the far left and far right branches of each graph point downward (in the negative y-direction). We say that the **end-behavior** of both graphs is identical, which is the term used to describe what happens to a graph as $|x|$ becomes very large. For $x > 0$, we say a graph is, "up on the right" or "down on the right," depending on the direction the "end" is pointing. For $x < 0$, we say the graph is "up on the left" or "down on the left," as the case may be.

EXAMPLE 6 ▶ Describing the End-Behavior of a Graph

The graph of $f(x) = x^3 - 3x$ is shown. Use the graph to name intervals where f is increasing or decreasing, and comment on the end-behavior of the graph.



Solution ▶ From the graph we observe that $f(x)\uparrow$ for $x \in (-\infty, -1) \cup (1, \infty)$, and $f(x)\downarrow$ for $x \in (-1, 1)$. The end-behavior of the graph is down on the left, and up on the right (down/up).

Now try Exercises 37 through 40 ▶

C. You've just seen how we can determine where a function is increasing or decreasing

D. Maximum and Minimum Values

The y-coordinate of the vertex of a parabola that opens downward, and the y-coordinate of "peaks" from other graphs are called **maximum values.** A **global maximum** (also called an *absolute* maximum) names the largest y-value over the entire domain. A **local maximum** (also called a *relative* maximum) gives the largest range value in a specified interval; and an **endpoint maximum** can occur at an endpoint of the domain. The same can be said for any corresponding minimum values.

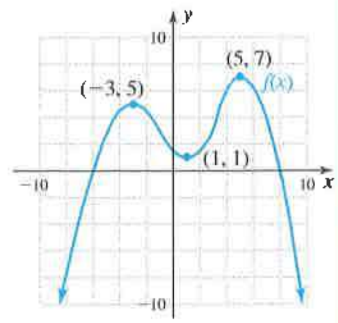
We will soon develop the ability to locate maximum and minimum values for quadratic and other functions. In future courses, methods are developed to help locate maximum and minimum values for almost *any* function. For now, our work will rely chiefly on a function's graph.

EXAMPLE 7 ▶ Analyzing Characteristics of a Graph

Analyze the graph of function f shown in Figure 2.11. Include specific mention of

a. domain and range,
b. intervals where f is increasing or decreasing,
c. maximum (max) and minimum (min) values,
d. intervals where $f(x) \geq 0$ and $f(x) < 0$, and
e. whether the function is even, odd, or neither.

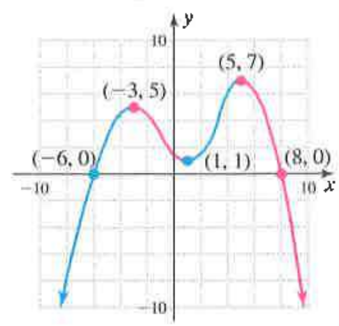
Figure 2.11



Solution ▶

a. Using vertical and horizontal boundary lines show the domain is $x \in \mathbb{R}$, with a range of: $y \in (-\infty, 7]$.
b. $f(x)\uparrow$ for $x \in (-\infty, -3) \cup (1, 5)$ shown in blue in Figure 2.12, and $f(x)\downarrow$ for $x \in (-3, 1) \cup (5, \infty)$ as shown in red.
c. From part (b) we find that $y = 5$ at $(-3, 5)$ and $y = 7$ at $(5, 7)$ are local maximums, with a local minimum of $y = 1$ at $(1, 1)$. The point $(5, 7)$ is also a global maximum (there is no global minimum).
d. $f(x) \geq 0$ for $x \in [-6, 8]$; $f(x) < 0$ for $x \in (-\infty, -6) \cup (8, \infty)$
e. The function is neither even nor odd.

Figure 2.12



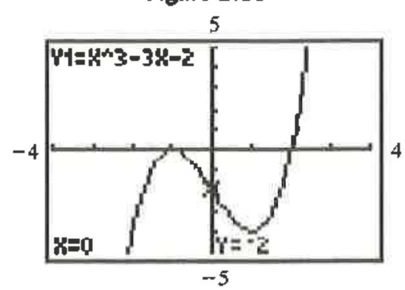
D. You've just seen how we can identify the maximum and minimum values of a function

Now try Exercises 41 through 48 ▶

E. Locating Maximum and Minimum Values Using Technology

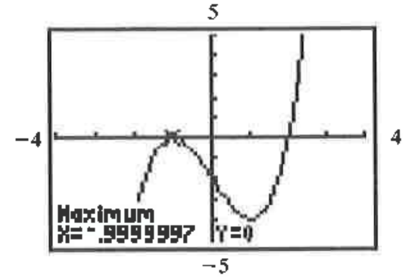
In Section 1.5, we used the 2nd TRACE **(CALC) 2:zero** option of a graphing calculator to locate the zeroes/x-intercepts of a function. The maximum or minimum values of a function are located in much the same way. To illustrate, enter the function $y = x^3 - 3x - 2$ as Y_1 on the Y= screen, then graph it in the window shown, where $x \in [-4, 4]$ and $y \in [-5, 5]$. As seen in Figure 2.13, it appears a local maximum occurs at $x = -1$ and a local minimum at $x = 1$. To actually find the local maximum, we access the 2nd TRACE **(CALC) 4:maximum** option, which returns you to the graph and asks for a **Left Bound?**, a **Right Bound?**, and a **Guess?** as before. Here, we entered a left bound of "-3," a right bound

Figure 2.13



of "0" and bypassed the guess option by pressing [ENTER] a third time (the calculator again sets the "▶" and "◀" markers to show the bounds chosen). The cursor will then be located at the local maximum in your selected interval, with the coordinates displayed at the bottom of the screen (Figure 2.14). Due to the algorithm the calculator uses to find these values, a decimal number very close to the expected value is sometimes displayed, even if the actual value is an integer (in Figure 2.14, -0.9999997 is displayed instead of -1). To check, we evaluate $f(-1)$ and find the local maximum is indeed 0.

Figure 2.14



EXAMPLE 8 ▶ Locating Local Maximum and Minimum Values on a Graphing Calculator

Find the maximum and minimum values of $f(x) = \frac{1}{2}(x^4 - 8x^2 + 7)$.

Solution ▶ Begin by entering $\frac{1}{2}(X^4 - 8X^2 + 7)$ as Y_1 on the [Y=] screen, and graph the function in the [ZOOM] **6:ZStandard** window. To locate the leftmost minimum value, we access the [2nd] [TRACE] **(CALC) 3:minimum** option, and enter a left bound of "−4," and a right bound of "−1" (Figure 2.15). After pressing [ENTER] once more, the cursor is located at the minimum in the interval we selected, and we find that a local minimum of -4.5 occurs at $x = -2$ (Figure 2.16). Repeating these steps using the appropriate options shows a local maximum of $y = 3.5$ occurs at $x = 0$, and a second local minimum of $y = -4.5$ occurs at $x = 2$. Note that $y = -4.5$ is also a global minimum.

Figure 2.15

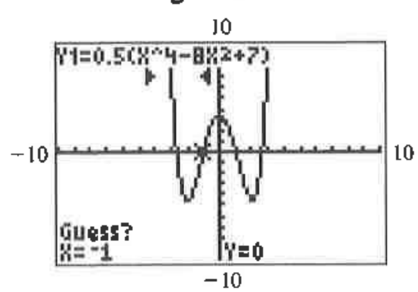
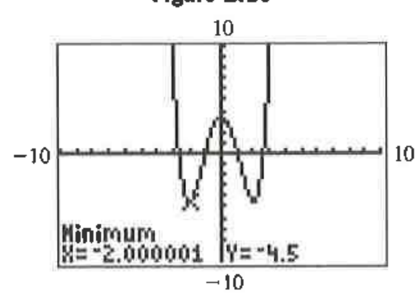


Figure 2.16



E. You've just seen how we can locate local maximum and minimum values using a graphing calculator

Now try Exercises 49 through 54 ▶

The ideas presented here can be applied to functions of all kinds, including rational functions, piecewise-defined functions, step functions, and so on. There is a wide variety of applications in **Exercises 57 through 64.**

2.1 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. The graph of a polynomial will cross through the x-axis at zeroes of ________ factors of degree 1, and ________ off the x-axis at the zeroes from linear factors of degree 2.

2. If $f(-x) = f(x)$ for all x in the domain, we say that f is an ________ function and symmetric to the ________ axis. If $f(-x) = -f(x)$, the function is ________ and symmetric to the ________.

3. If $f(x_2) > f(x_1)$ for $x_1 < x_2$ for all x in a given interval, the function is ________ in the interval.

4. If $f(c) \geq f(x)$ for all x in a specified interval, we say that $f(c)$ is a local ________ for this interval.

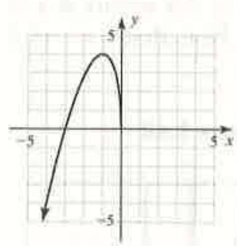
5. Discuss/Explain the following statement and give an example of the conclusion it makes. "If a function f is decreasing to the left of $(c, f(c))$ and increasing to the right of $(c, f(c))$, then $f(c)$ is either a local or a global minimum."

6. Without referring to notes or textbook, list as many features/attributes as you can that are related to analyzing the graph of a function. Include details on how to locate or determine each attribute.

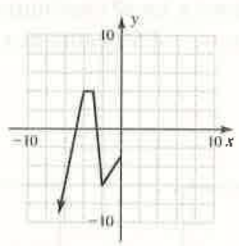
▶ DEVELOPING YOUR SKILLS

The following functions are known to be even. Complete each graph using symmetry.

7.



8.



Determine whether the following functions are even: $f(-k) = f(k)$.

9. $f(x) = -7|x| + 3x^2 + 5$

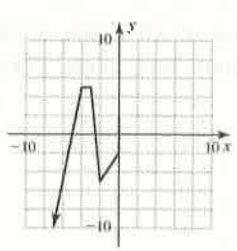
10. $p(x) = 2x^4 - 6x + 1$

11. $g(x) = \frac{1}{3}x^4 - 5x^2 + 1$

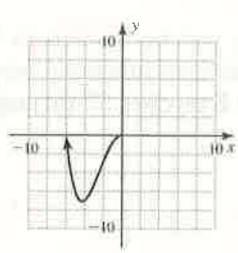
12. $q(x) = \frac{1}{x^2} - |x|$

The following functions are known to be odd. Complete each graph using symmetry.

13.



14.



Determine whether the following functions are odd: $f(-k) = -f(k)$.

15. $f(x) = 4\sqrt[3]{x} - x$

16. $g(x) = \frac{1}{2}x^3 - 6x$

17. $p(x) = 3x^3 - 5x^2 + 1$

18. $q(x) = \frac{1}{x} - x$

Determine whether the following functions are even, odd, or neither.

19. $w(x) = x^3 - x^2$

20. $q(x) = \frac{3}{4}x^2 + 3|x|$

21. $p(x) = 2\sqrt[3]{x} - \frac{1}{4}x^3$

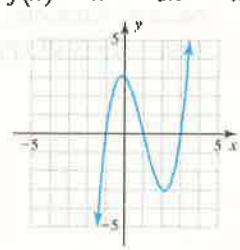
22. $g(x) = x^3 + 7x$

23. $v(x) = x^3 + 3|x|$

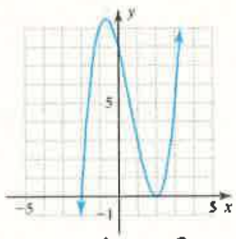
24. $f(x) = x^4 + 7x^2 - 30$

Use the graphs given to solve the inequalities indicated. Write all answers in interval notation.

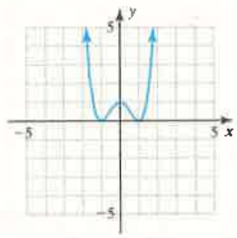
25. $f(x) = x^3 - 3x^2 - x + 3; f(x) \geq 0$



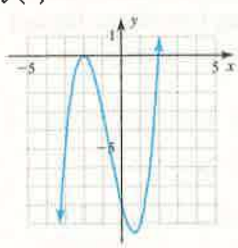
26. $f(x) = x^3 - 2x^2 - 4x + 8; f(x) > 0$



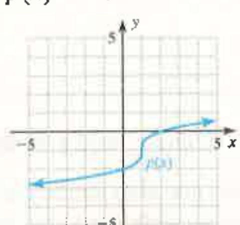
27. $f(x) = x^4 - 2x^2 + 1; f(x) > 0$



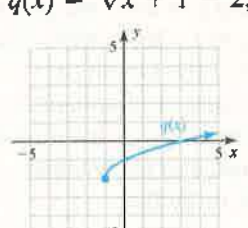
28. $f(x) = x^3 + 2x^2 - 4x - 8; f(x) \geq 0$



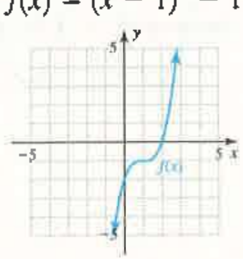
29. $p(x) = \sqrt[3]{x - 1} - 1; p(x) \geq 0$



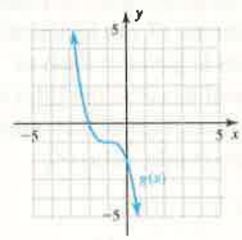
30. $q(x) = \sqrt{x + 1} - 2; q(x) > 0$



31. $f(x) = (x - 1)^3 - 1; f(x) \leq 0$

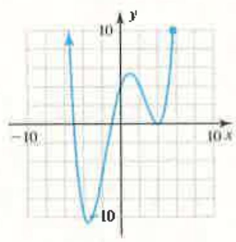


32. $g(x) = -(x + 1)^3 - 1; g(x) < 0$

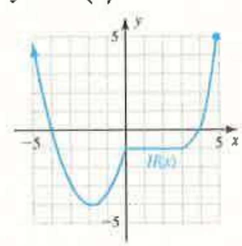


Name the interval(s) where the following functions are increasing, decreasing, or constant. Write answers using interval notation. Assume all endpoints have integer values.

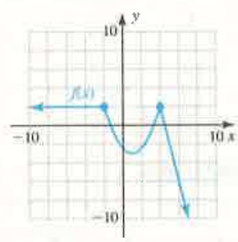
33. $y = V(x)$



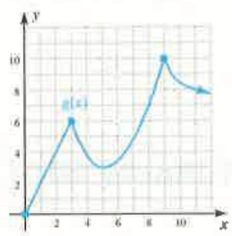
34. $y = H(x)$



35. $y = f(x)$

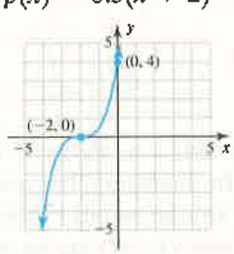


36. $y = g(x)$

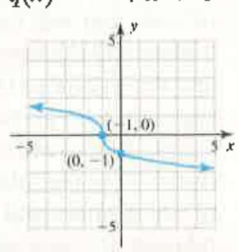


For Exercises 37 through 40, determine (a) interval(s) where the function is increasing, decreasing or constant, and (b) comment on the end-behavior.

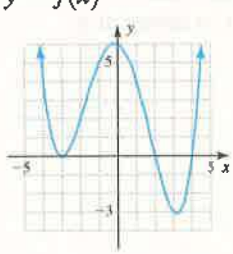
37. $p(x) = 0.5(x + 2)^3$



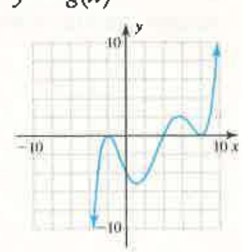
38. $q(x) = -\sqrt[3]{x + 1}$



39. $y = f(x)$

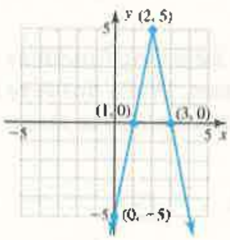


40. $y = g(x)$

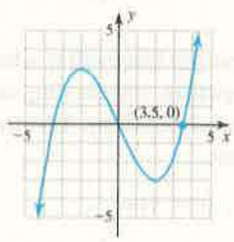


For Exercises 41 through 48, determine the following (answer in interval notation as appropriate): (a) domain and range of the function; (b) zeroes of the function; (c) interval(s) where the function is greater than or equal to zero, or less than or equal to zero; (d) interval(s) where the function is increasing, decreasing, or constant; and (e) location of any local max or min value(s).

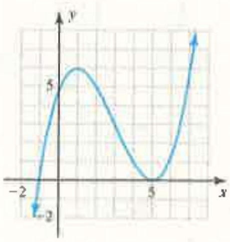
41. $y = H(x)$



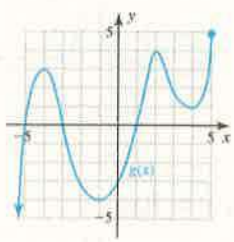
42. $y = f(x)$



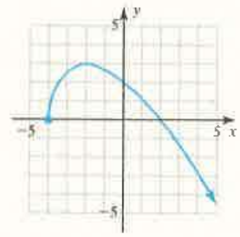
43. $y = g(x)$



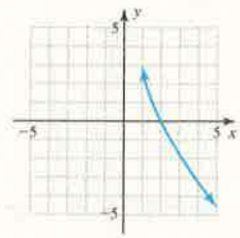
44. $y = h(x)$



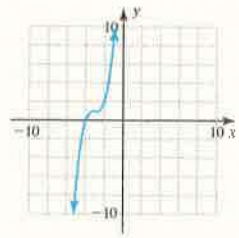
45. $y = Y_1$



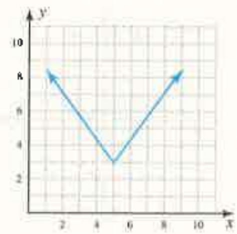
46. $y = Y_2$



47. $p(x) = (x + 3)^3 + 1$



48. $q(x) = |x - 5| + 3$



Use a graphing calculator to find the maximum and minimum values of the following functions. Round answers to nearest hundredth when necessary.

49. $y = \frac{3}{4}(x^3 - 5x^2 + 6x)$

50. $y = \frac{6}{5}(x^3 + 4x^2 + 3x)$

51. $y = 0.0016x^5 - 0.12x^3 + 2x$

52. $y = -0.01x^5 + 0.03x^4 + 0.25x^3 - 0.75x^2$

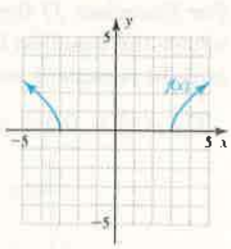
53. $y = x\sqrt{4 - x}$

54. $y = x^2\sqrt{x + 3} - 2$

▶ WORKING WITH FORMULAS

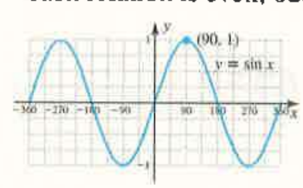
55. Conic sections—hyperbola: $y = \frac{1}{3}\sqrt{4x^2 - 36}$

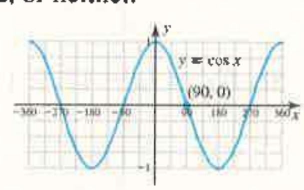
While the conic sections are not covered in detail until later in the course, we've already developed a number of tools that will help us understand these relations and their graphs. The equation here gives the "upper branches" of a hyperbola, as shown in the figure. Find the following by analyzing the equation: (a) the domain and range; (b) the zeroes of the relation; (c) interval(s) where y is increasing or decreasing; (d) whether the relation is even, odd, or neither, and (e) solve for x in terms of y.



56. Trigonometric graphs: $y = \sin(x)$ **and** $y = \cos(x)$

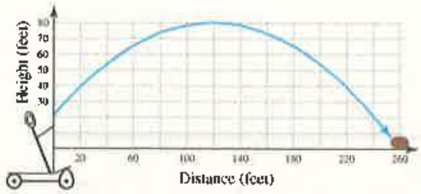
The trigonometric functions are also studied at some future time, but we can apply the same tools to analyze the graphs of these functions as well. The graphs of $y = \sin x$ and $y = \cos x$ are given, graphed over the interval $x \in [-360°, 360°]$. Use them to find (a) the range of the functions; (b) the zeroes of the functions; (c) interval(s) where y is increasing/decreasing; (d) location of minimum/maximum values; and (e) whether each relation is even, odd, or neither.





▶ APPLICATIONS

57. Catapults and projectiles: Catapults have a long and interesting history that dates back to ancient times, when they were used to launch javelins, rocks, and other projectiles. The diagram given illustrates the path of the projectile after release, which follows a parabolic arc. Use the graph to determine the following:



a. State the domain and range of the projectile.

b. What is the maximum height of the projectile?

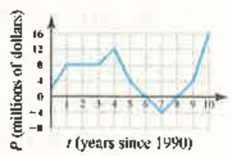
c. How far from the catapult did the projectile reach its maximum height?

d. Did the projectile clear the castle wall, which was 40 ft high and 210 ft away?

e. On what interval was the height of the projectile increasing?

f. On what interval was the height of the projectile decreasing?

58. Profit and loss: The profit of DeBartolo Construction Inc. is illustrated by the graph shown. Use the graph to estimate the point(s) or the interval(s) for which the profit P was:



a. increasing

b. decreasing

c. constant

d. a maximum

e. a minimum

f. positive

g. negative

h. zero

59. Functions and rational exponents: The graph of $f(x) = x^{\frac{2}{3}} - 1$ is shown. Use the graph to find:

a. domain and range of the function

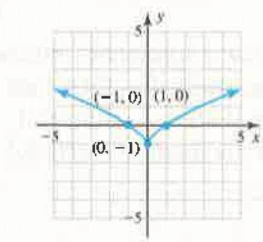
b. zeroes of the function

c. interval(s) where $f(x) \geq 0$ or $f(x) < 0$

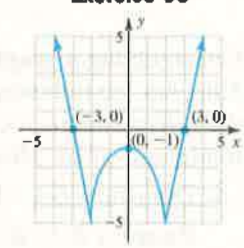
d. interval(s) where $f(x)$ is increasing, decreasing, or constant

e. location of any max or min value(s)

Exercise 59



Exercise 60



60. Analyzing a graph: Given $h(x) = |x^2 - 4| - 5$, whose graph is shown, use the graph to find:

a. domain and range of the function

b. zeroes of the function

c. interval(s) where $h(x) \geq 0$ or $h(x) < 0$

d. interval(s) where $f(x)$ is increasing, decreasing, or constant

e. location of any max or min value(s)

61. Analyzing interest rates: The graph shown approximates the average annual interest rates I on 30-yr fixed mortgages, rounded to the nearest $\frac{1}{4}$%. Use the graph to estimate the following (write all answers in interval notation).

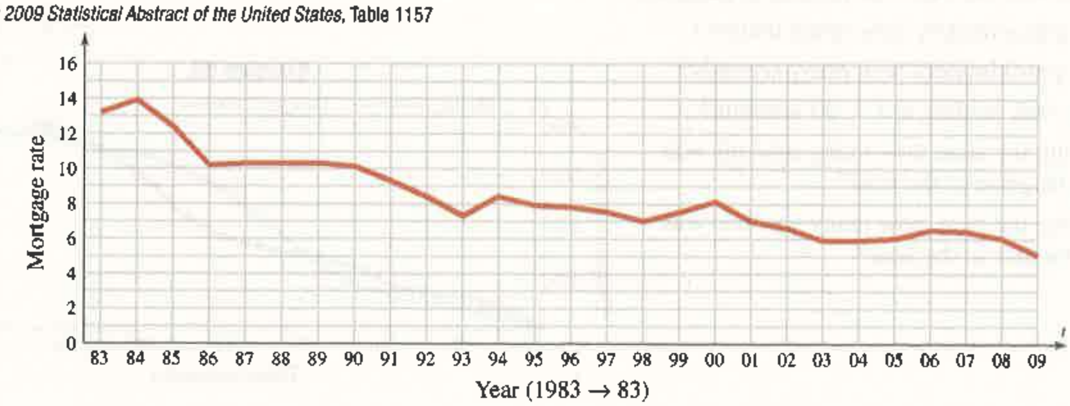
a. domain and range

b. interval(s) where $I(t)$ is increasing, decreasing, or constant

c. location of any global maximum or minimum values

d. the one-year period with the greatest rate of increase and the one-year period with the greatest rate of decrease

Source: 2009 Statistical Abstract of the United States, Table 1157



62. **Analyzing the surplus S:** The following graph approximates the federal surplus S of the United States. Use the graph to estimate the following. Write answers in interval notation and estimate all surplus values to the nearest \$10 billion.

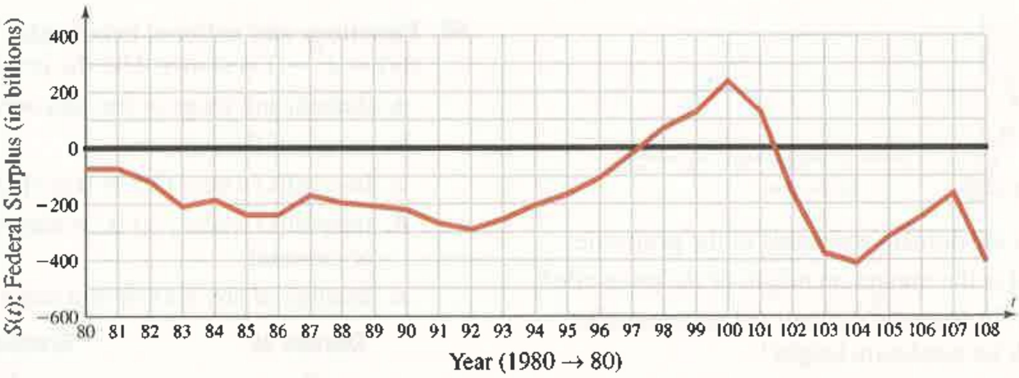
a. the domain and range

b. interval(s) where $S(t)$ is increasing, decreasing, or constant

c. the location of any global maximum and minimum values

d. the one-year period with the greatest rate of increase, and the one-year period with the greatest rate of decrease

Source: 2009 Statistical Abstract of the United States, Table 451



63. **Constructing a graph:** Draw a continuous function f that has the following characteristics, then state the zeroes and the location of all maximum and minimum values. [*Hint:* Write them as $(c, f(c))$.]

a. Domain: $x \in (-10, \infty)$
Range: $y \in (-6, \infty)$

b. $f(0) = 0; f(4) = 0$

c. $f(x)\uparrow$ for $x \in (-10, -6) \cup (-2, 2) \cup (4, \infty)$
$f(x)\downarrow$ for $x \in (-6, -2) \cup (2, 4)$

d. $f(x) \geq 0$ for $x \in [-8, -4] \cup [0, \infty)$
$f(x) < 0$ for $x \in (-\infty, -8) \cup (-4, 0)$

64. **Constructing a graph:** Draw a continuous function g that has the following characteristics, then state the zeroes and the location of all maximum and minimum values. [*Hint:* Write them as $(c, g(c))$.]

a. Domain: $x \in (-\infty, 8]$
Range: $y \in [-6, \infty)$

b. $g(0) = 4.5; g(6) = 0$

c. $g(x)\uparrow$ for $x \in (-6, 3) \cup (6, 8)$
$g(x)\downarrow$ for $x \in (-\infty, -6) \cup (3, 6)$

d. $g(x) \geq 0$ for $x \in (-\infty, -9] \cup [-3, 8]$
$g(x) < 0$ for $x \in (-9, -3)$

▶ EXTENDING THE CONCEPT

65. Does the function shown have a maximum value? Does it have a minimum value? Discuss/explain/justify why or why not.

Exercise 65

66. The graph drawn here depicts a 400-m race between a mother and her daughter. Analyze the graph to answer questions (a) through (f).

a. Who wins the race, the mother or daughter?

b. By approximately how many meters?

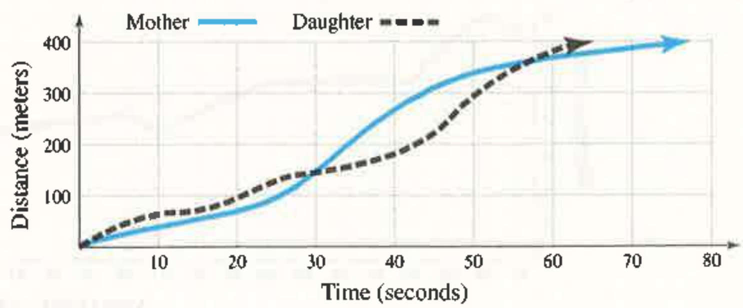
c. By approximately how many seconds?

d. Who was leading at $t = 40$ seconds?

e. During the race, how many seconds was the daughter in the lead?

f. During the race, how many seconds was the mother in the lead?

Exercise 66



67. The graph drawn here depicts the last 75 sec of the competition between Ian Thorpe (Australia) and Massimiliano Rosolino (Italy) in the men's 400-m freestyle at the 2000 Olympics, where a new Olympic record was set.

a. Who was in the lead at 180 sec? 210 sec?

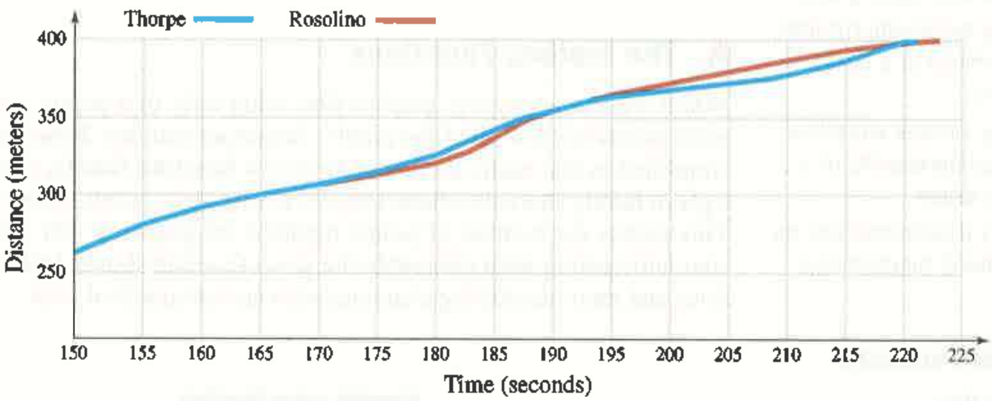
b. In the last 50 m, how many times were they tied, and when did the ties occur?

c. About how many seconds did Rosolino have the lead?

d. Which swimmer won the race?

e. By approximately how many seconds?

f. Use the graph to approximate the new Olympic record set in the year 2000.



68. Draw the graph of a general function $f(x)$ that has a local *maximum* at $(a, f(a))$ and a local *minimum* at $(b, f(b))$ but with $f(a) < f(b)$.

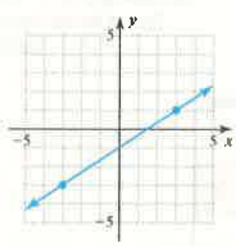
69. Verify that $h(x) = x^{\frac{2}{3}}$ is an even function, by first rewriting h as $h(x) = (x^{\frac{1}{3}})^2$.

▶ MAINTAINING YOUR SKILLS

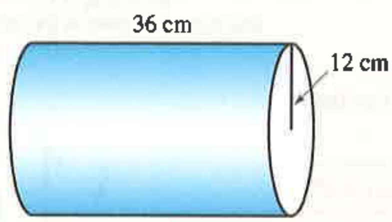
70. (Appendix A.4) Solve the given quadratic equation by factoring: $x^2 - 8x - 20 = 0$.

71. (Appendix A.5) Find the (a) sum and (b) product of the rational expressions $\frac{3}{x+2}$ and $\frac{3}{2-x}$.

72. (1.4) Write the equation of the line shown, in the form $y = mx + b$.



73. (Appendix A.2) Find the surface area and volume of the cylinder shown ($SA = 2\pi r^2 + \pi r^2 h$, $V = \pi r^2 h$).



2.2 The Toolbox Functions and Transformations

LEARNING OBJECTIVES

In Section 2.2 you will see how we can:

- ☐ **A.** Identify basic characteristics of the toolbox functions
- ☐ **B.** Apply vertical/horizontal shifts of a basic graph
- ☐ **C.** Apply vertical/horizontal reflections of a basic graph
- ☐ **D.** Apply vertical stretches and compressions of a basic graph
- ☐ **E.** Apply transformations on a general function $f(x)$

Many applications of mathematics require that we select a function known to fit the context, or build a function model from the information supplied. So far we've looked at linear functions. Here we'll introduce the absolute value, squaring, square root, cubing, and cube root functions. Together these are the six **toolbox functions,** so called because they give us a variety of "tools" to model the real world (see Section 2.6). In the same way a study of arithmetic depends heavily on the multiplication table, a study of algebra and mathematical modeling depends (in large part) on a solid working knowledge of these functions. More will be said about each function in later sections.

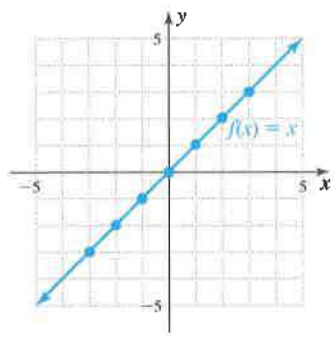
A. The Toolbox Functions

While we can accurately graph a line using only two points, most functions require more points to show all of the graph's important features. However, our work is greatly simplified in that each function belongs to a **function family,** in which all graphs from a given family share the characteristics of one basic graph, called the **parent function.** This means the number of points required for graphing will quickly decrease as we start anticipating what the graph of a given function should look like. The parent functions and their identifying characteristics are summarized here.

The Toolbox Functions

Identity function

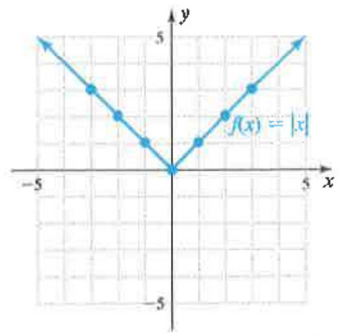
x	$f(x) = x$
−3	−3
−2	−2
−1	−1
0	0
1	1
2	2
3	3



Domain: $x \in (-\infty, \infty)$, Range: $y \in (-\infty, \infty)$
Symmetry: odd
Increasing: $x \in (-\infty, \infty)$
End-behavior: down on the left/up on the right

Absolute value function

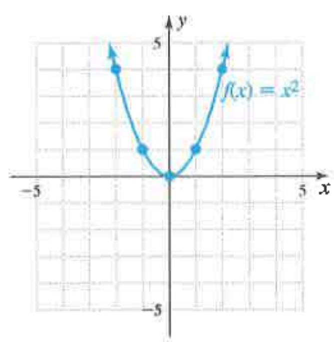
x	$f(x) = \lvert x \rvert$
−3	3
−2	2
−1	1
0	0
1	1
2	2
3	3



Domain: $x \in (-\infty, \infty)$, Range: $y \in [0, \infty)$
Symmetry: even
Decreasing: $x \in (-\infty, 0)$; Increasing: $x \in (0, \infty)$
End-behavior: up on the left/up on the right
Vertex at (0, 0)

Squaring function

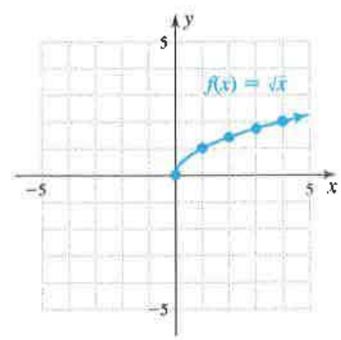
x	$f(x) = x^2$
−3	9
−2	4
−1	1
0	0
1	1
2	4
3	9



Domain: $x \in (-\infty, \infty)$, Range: $y \in [0, \infty)$
Symmetry: even
Decreasing: $x \in (-\infty, 0)$; Increasing: $x \in (0, \infty)$
End-behavior: up on the left/up on the right
Vertex at (0, 0)

Square root function

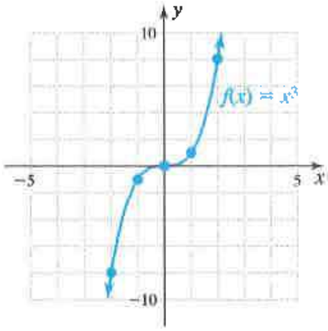
x	$f(x) = \sqrt{x}$
−2	—
−1	—
0	0
1	1
2	≈1.41
3	≈1.73
4	2



Domain: $x \in [0, \infty)$, Range: $y \in [0, \infty)$
Symmetry: neither even nor odd
Increasing: $x \in (0, \infty)$
End-behavior: up on the right
Initial point at (0, 0)

Cubing function

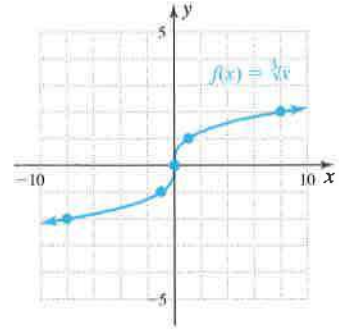
x	$f(x) = x^3$
−3	−27
−2	−8
−1	−1
0	0
1	1
2	8
3	27



Domain: $x \in (-\infty, \infty)$, Range: $y \in (-\infty, \infty)$
Symmetry: odd
Increasing: $x \in (-\infty, \infty)$
End-behavior: down on the left/up on the right
Point of inflection at (0, 0)

Cube root function

x	$f(x) = \sqrt[3]{x}$
−27	−3
−8	−2
−1	−1
0	0
1	1
8	2
27	3



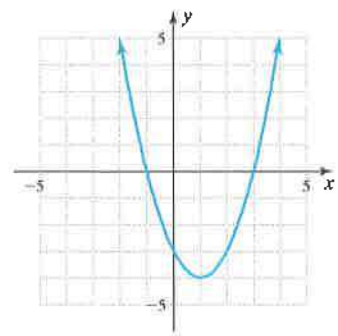
Domain: $x \in (-\infty, \infty)$, Range: $y \in (-\infty, \infty)$
Symmetry: odd
Increasing: $x \in (-\infty, \infty)$
End-behavior: down on the left/up on the right
Point of inflection at (0, 0)

In applications of the toolbox functions, the parent graph may be "morphed" and/or shifted from its original position, yet the graph will still retain its basic shape and features. The result is called a **transformation** of the parent graph.

EXAMPLE 1 ▶ Identifying the Characteristics of a Transformed Graph

The graph of $f(x) = x^2 - 2x - 3$ is given. Use the graph to identify each of the features or characteristics indicated.

a. function family
b. domain and range
c. vertex
d. max or min value(s)
e. intervals where f is increasing or decreasing
f. end-behavior
g. x- and y-intercept(s)



Solution ▶

a. The graph is a parabola, from the squaring function family.
b. domain: $x \in (-\infty, \infty)$; range: $y \in [-4, \infty)$
c. vertex: $(1, -4)$
d. minimum value $y = -4$ at $(1, -4)$
e. decreasing: $x \in (-\infty, 1)$, increasing: $x \in (1, \infty)$
f. end-behavior: up/up
g. y-intercept: $(0, -3)$; x-intercepts: $(-1, 0)$ and $(3, 0)$

Now try Exercises 7 through 34 ▶

Note that for Example 1(f), we can algebraically verify the x-intercepts by substituting 0 for $f(x)$ and solving the equation by factoring. This gives $0 = (x + 1)(x - 3)$, with solutions $x = -1$ and $x = 3$. It's also worth noting that while the parabola is no longer symmetric to the y-axis, it *is* symmetric to the vertical line $x = 1$. This line is called the **axis of symmetry** for the parabola, and for a vertical parabola, it will always be a vertical line that goes through the vertex.

☑ **A. You've just seen how we can identify basic characteristics of the toolbox functions**

B. Vertical and Horizontal Shifts

As we study specific transformations of a graph, try to develop a *global view* as the transformations can be applied to *any* function. When these are applied to the toolbox functions, we rely on characteristic features of the parent function to assist in completing the transformed graph.

Vertical Translations

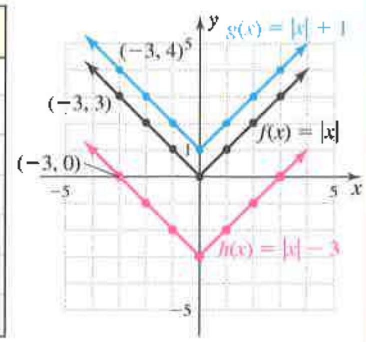
We'll first investigate vertical translations or vertical shifts of the toolbox functions, using the absolute value function to illustrate.

EXAMPLE 2 ▶ **Graphing Vertical Translations**

Construct a table of values for $f(x) = |x|$, $g(x) = |x| + 1$, and $h(x) = |x| - 3$ and graph the functions on the same coordinate grid. Then discuss what you observe.

Solution ▶ A table of values for all three functions is given, with the corresponding graphs shown in the figure.

x	$f(x) = \lvert x \rvert$	$g(x) = \lvert x \rvert + 1$	$h(x) = \lvert x \rvert - 3$
−3	3	4	0
−2	2	3	−1
−1	1	2	−2
0	0	1	−3
1	1	2	−2
2	2	3	−1
3	3	4	0



Note that outputs of $g(x)$ are one more than the outputs of $f(x)$, and that each point on the graph of f has been shifted *upward 1 unit* to form the graph of g. Similarly, each point on the graph of f has been shifted *downward 3 units* to form the graph of h, since $h(x) = f(x) - 3$.

Now try Exercises 35 through 42 ▶

We describe the transformations in Example 2 as a **vertical shift** or **vertical translation** of a basic graph. The graph of g is the graph of f *shifted up 1 unit,* and the graph of h, is the graph of *f shifted down 3 units.* In general, we have the following:

Vertical Translations of a Basic Graph

Given $k > 0$ and any function whose graph is determined by $y = f(x)$,

1. The graph of $y = f(x) + k$ is the graph of $f(x)$ shifted upward k units.
2. The graph of $y = f(x) - k$ is the graph of $f(x)$ shifted downward k units.

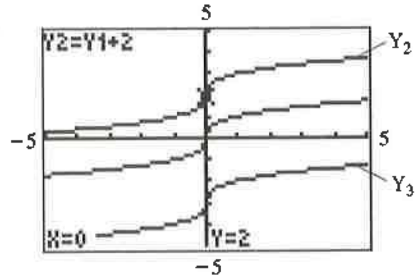
Graphing calculators are wonderful tools for exploring graphical transformations. To emphasize that a given graph is being shifted vertically as in Example 2, try entering $\sqrt[3]{X}$ as Y_1 on the Y= screen, then $Y_2 = Y_1 + 2$ and $Y_3 = Y_1 - 3$ (Figure 2.17 — recall the Y-variables are accessed using VARS ▶ **(Y-VARS)** ENTER). Using the Y-variables in this way enables us to study identical transformations on a variety of graphs, simply by changing the function in Y_1.

Figure 2.17

Using a window size of $x \in [-5, 5]$ and $y \in [-5, 5]$ for the cube root function, produces the graphs shown in Figure 2.18, which demonstrate the cube root graph has been shifted upward 2 units (Y_2), and downward 3 units (Y_3).

Try this exploration again using $Y_1 = \sqrt{X}$.

Figure 2.18



Horizontal Translations

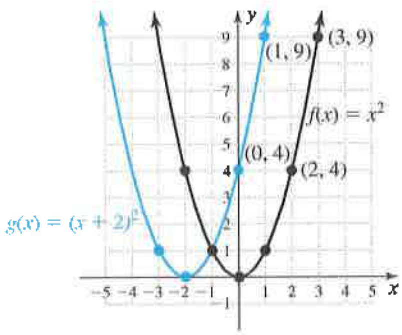
The graph of a parent function can also be shifted left or right. This happens when we *alter the inputs to the basic function,* as opposed to adding or subtracting something to the function itself. For $Y_1 = x^2 + 2$ note that we first square inputs, then add 2, which results in a vertical shift. For $Y_2 = (x + 2)^2$, we add 2 to x *prior to squaring* and since the input values are affected, we might anticipate the graph will shift along the x-axis—horizontally.

EXAMPLE 3 ▶ Graphing Horizontal Translations

Construct a table of values for $f(x) = x^2$ and $g(x) = (x + 2)^2$, then graph the functions on the same grid and discuss what you observe.

Solution ▶ Both f and g belong to the quadratic family and their graphs are parabolas. A table of values is shown along with the corresponding graphs.

x	$f(x) = x^2$	$g(x) = (x + 2)^2$
−3	9	1
−2	4	0
−1	1	1
0	0	4
1	1	9
2	4	16
3	9	25



It is apparent the graphs of g and f are identical, but the graph of g has been shifted horizontally 2 units left.

Now try Exercises 43 through 46 ▶

We describe the transformation in Example 3 as a **horizontal shift** or **horizontal translation** of a basic graph. The graph of g is the graph of f, *shifted 2 units to the left.* Once again it seems reasonable that since *input* values were altered, the shift must be horizontal rather than vertical. From this example, we also learn the direction of the shift is **opposite the sign:** $y = (x + 2)^2$ is 2 units *to the left* of $y = x^2$. Although it may seem counterintuitive, the shift *opposite the sign* can be "seen" by locating the new x-intercept, which in this case is also the vertex. Substituting 0 for y gives $0 = (x + 2)^2$ with $x = -2$, as shown in the graph. In general, we have

Horizontal Translations of a Basic Graph

Given $h > 0$ and any function whose graph is determined by $y = f(x)$,

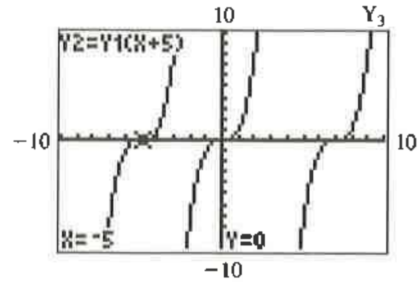
1. The graph of $y = f(x + h)$ is the graph of $f(x)$ shifted *to the left* h units.
2. The graph of $y = f(x - h)$ is the graph of $f(x)$ shifted *to the right* h units.

Figure 2.19

To explore horizontal translations on a graphing calculator, we input a basic function in Y_1 and indicate how we want the inputs altered in Y_2 and Y_3. Here we'll enter X^3 as Y_1 on the Y= screen, then $Y_2 = Y_1(X + 5)$ and $Y_3 = Y_1(X - 7)$ (Figure 2.19). Note how this duplicates the definition and notation for horizontal shifts in the orange box. Based on what we saw in Example 3, we expect the graph of $y = x^3$ will first be shifted 5 units left (Y_2), then 7 units right (Y_3). This in confirmed in Figure 2.20.

Try this exploration again using $Y_1 = \text{abs}(X)$.

Figure 2.20



EXAMPLE 4 ▶ Graphing Horizontal Translations

Sketch the graphs of $g(x) = |x - 2|$ and $h(x) = \sqrt{x + 3}$ using a horizontal shift of the parent function and a few characteristic points (not a table of values).

Solution ▶ The graph of $g(x) = |x - 2|$ (Figure 2.21) is the absolute value function shifted 2 units to the right (shift the vertex and two other points from $y = |x|$). The graph of $h(x) = \sqrt{x + 3}$ (Figure 2.22) is a square root function, shifted 3 units to the left (shift the initial point and one or two points from $y = \sqrt{x}$).

Figure 2.21

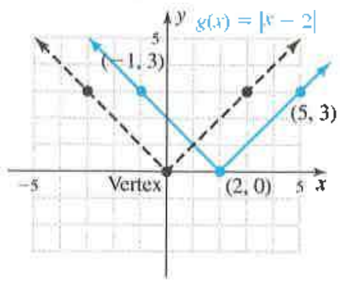
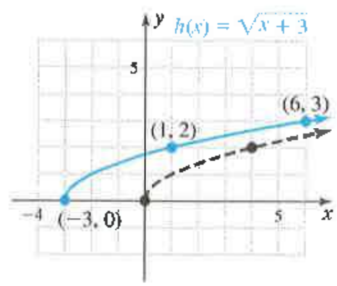


Figure 2.22



B. You've just seen how we can perform vertical/horizontal shifts of a basic graph

Now try Exercises 47 through 50 ▶

C. Vertical and Horizontal Reflections

The next transformation we investigate is called a **vertical reflection,** in which we compare the function $Y_1 = f(x)$ with the negative of the function: $Y_2 = -f(x)$.

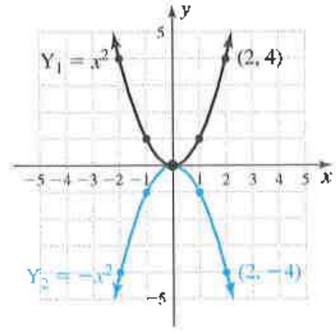
Vertical Reflections

EXAMPLE 5 ▶ Graphing Vertical Reflections

Construct a table of values for $Y_1 = x^2$ and $Y_2 = -x^2$, then graph the functions on the same grid and discuss what you observe.

Solution ▶ A table of values is given for both functions, along with the corresponding graphs.

x	$Y_1 = x^2$	$Y_2 = -x^2$
−2	4	−4
−1	1	−1
0	0	0
1	1	−1
2	4	−4



As you might have anticipated, the outputs for f and g differ only in sign. Each output is a **reflection** of the other, being an equal distance from the x-axis but on opposite sides.

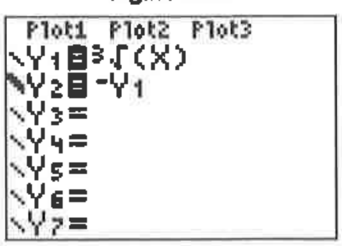
Now try Exercises 51 and 52 ▶

The vertical reflection in Example 5 is called a **reflection across the x-axis.** In general,

Vertical Reflections of a Basic Graph

For any function $y = f(x)$, the graph of $y = -f(x)$ is the graph of $f(x)$ reflected across the x-axis.

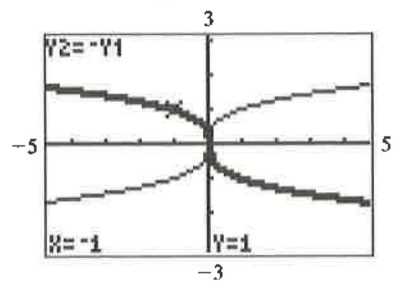
Figure 2.23



To view vertical reflections on a graphing calculator, we simply define $Y_2 = -Y_1$, as seen here using $\sqrt[3]{X}$ as Y_1 (Figure 2.23). As in Section 1.5, we can have the calculator graph Y_2 using a bolder line, to easily distinguish between the original graph and its reflection (Figure 2.24). To aid in the viewing, we have set a window size of $x \in [-5, 5]$ and $y \in [-3, 3]$.

Try this exploration again using $Y_1 = X^2 - 4$.

Figure 2.24



Horizontal Reflections

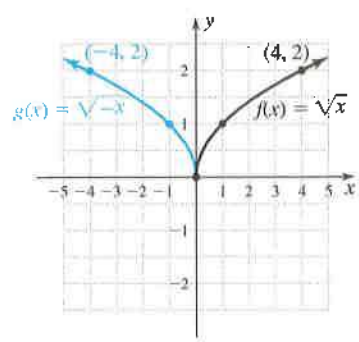
It's also possible for a graph to be reflected horizontally *across the y-axis.* Just as we noted that $f(x)$ versus $-f(x)$ resulted in a vertical reflection, $f(x)$ versus $f(-x)$ results in a horizontal reflection.

EXAMPLE 6 ▶ Graphing a Horizontal Reflection

Construct a table of values for $f(x) = \sqrt{x}$ and $g(x) = \sqrt{-x}$, then graph the functions on the same coordinate grid and discuss what you observe.

Solution ▶ A table of values is given here, along with the corresponding graphs.

x	$f(x) = \sqrt{x}$	$g(x) = \sqrt{-x}$
−4	not real	2
−2	not real	$\sqrt{2} \approx 1.41$
−1	not real	1
0	0	0
1	1	not real
2	$\sqrt{2} \approx 1.41$	not real
4	2	not real



The graph of g is the same as the graph of f, but it has been reflected across the y-axis. A study of the domain shows why—f represents a real number only for nonnegative inputs, so its graph occurs to the right of the y-axis, while g represents a real number for nonpositive inputs, so its graph occurs to the left.

Now try Exercises 53 and 54 ▶

The transformation in Example 6 is called a **horizontal reflection** of a basic graph. In general,

C. You've just seen how we can apply vertical/horizontal reflections of a basic graph

Horizontal Reflections of a Basic Graph

For any function $y = f(x)$, the graph of $y = f(-x)$ is the graph of $f(x)$ reflected across the y-axis.

D. Vertically Stretching/Compressing a Basic Graph

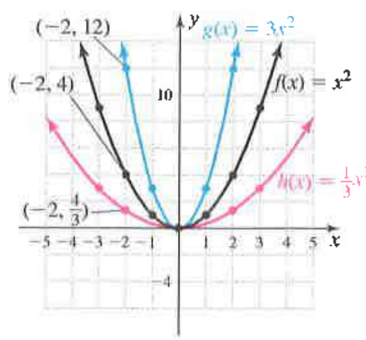
As the words "stretching" and "compressing" imply, the graph of a basic function can also become elongated or flattened after certain transformations are applied. However, even these transformations preserve the key characteristics of the graph.

EXAMPLE 7 ▶ Stretching and Compressing a Basic Graph

Construct a table of values for $f(x) = x^2$, $g(x) = 3x^2$, and $h(x) = \frac{1}{3}x^2$, then graph the functions on the same grid and discuss what you observe.

Solution ▶ A table of values is given for all three functions, along with the corresponding graphs.

x	$f(x) = x^2$	$g(x) = 3x^2$	$h(x) = \frac{1}{3}x^2$
−3	9	27	3
−2	4	12	$\frac{4}{3}$
−1	1	3	$\frac{1}{3}$
0	0	0	0
1	1	3	$\frac{1}{3}$
2	4	12	$\frac{4}{3}$
3	9	27	3



The outputs of g are triple those of f, making these outputs farther from the x-axis and *stretching* g upward (making the graph more narrow). The outputs of h are one-third those of f, and the graph of h is *compressed* downward, with its outputs closer to the x-axis (making the graph wider).

Now try Exercises 55 through 62 ▶

WORTHY OF NOTE

In a study of trigonometry, you'll find that a basic graph can also be stretched or compressed horizontally, a phenomenon known as *frequency variations*.

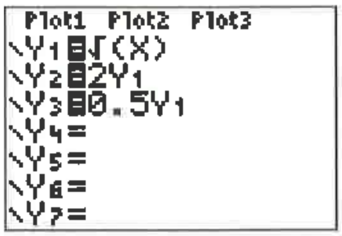
The transformations in Example 7 are called **vertical stretches** or **compressions** of a basic graph. Notice that while the outputs are increased or decreased by a constant factor (making the graph appear more narrow or more wide), the domain of the function remains unchanged. In general,

Stretches and Compressions of a Basic Graph

For any function $y = f(x)$, the graph of $y = af(x)$ is

1. the graph of $f(x)$ stretched vertically if $|a| > 1$,
2. the graph of $f(x)$ compressed vertically if $0 < |a| < 1$.

Figure 2.25

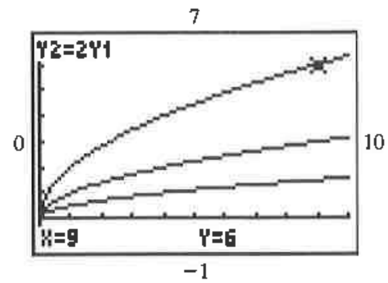


D. You've just seen how we can apply vertical stretches and compressions of a basic graph

To use a graphing calculator in a study of stretches and compressions, we simply define Y_2 and Y_3 as constant multiples of Y_1 (Figure 2.25). As seen in Example 7, if $|a| > 1$ the graph will be stretched vertically, if $0 < |a| < 1$, the graph will be vertically compressed. This is further illustrated here using $Y_1 = \sqrt{X}$, with $Y_2 = 2Y_1$ and $Y_3 = 0.5Y_1$. Since the domain of $y = \sqrt{x}$ is restricted to nonnegative values, a window size of $x \in [0, 10]$ and $y \in [-1, 7]$ was used (Figure 2.26).

Try this exploration again using $Y_1 = \text{abs}(X) - 4$.

Figure 2.26



E. Transformations of a General Function

If more than one transformation is applied to a basic graph, it's helpful to use the following sequence for graphing the new function.

General Transformations of a Basic Graph

Given a function $y = f(x)$, the graph of $y = af(x \pm h) \pm k$ can be obtained by applying the following sequence of transformations:

1. horizontal shifts **2.** reflections **3.** stretches/compressions **4.** vertical shifts

We generally use a few characteristic points to track the transformations involved, then draw the transformed graph through the new location of these points.

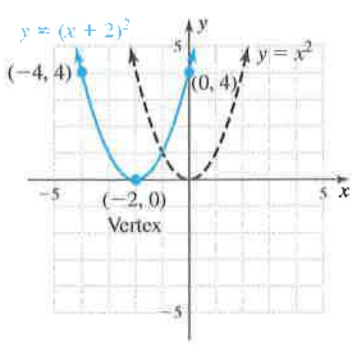
EXAMPLE 8 ▶ Graphing Functions Using Transformations

Use transformations of a parent function to sketch the graphs of

a. $g(x) = -(x + 2)^2 + 3$ **b.** $h(x) = 2\sqrt[3]{x - 2} - 1$

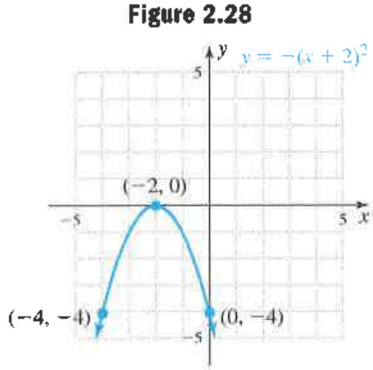
Solution ▶ a. The graph of g is a parabola, shifted left 2 units, reflected across the x-axis, and shifted up 3 units. This sequence of transformations is shown in Figures 2.27 through 2.29. Note that since the graph has been shifted 2 units left and 3 units up, the vertex of the parabola has likewise shifted from (0, 0) to (−2, 3).

Figure 2.27



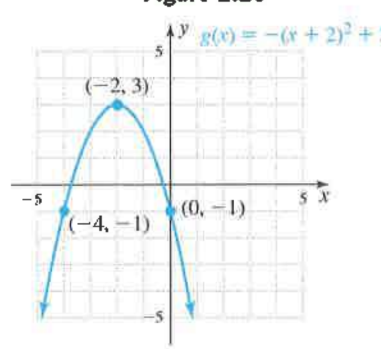
Shifted left 2 units

Figure 2.28



Reflected across the x-axis

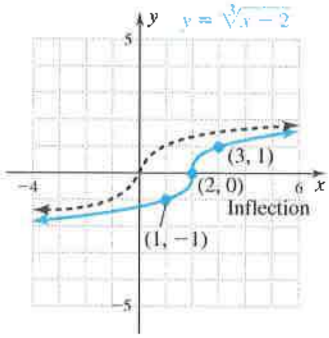
Figure 2.29



Shifted up 3 units

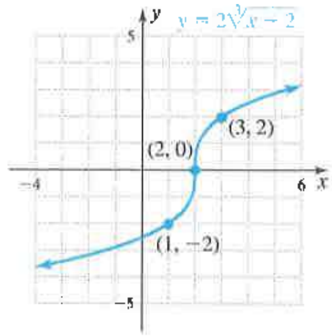
b. The graph of h is a cube root function, shifted right 2, stretched by a factor of 2, then shifted down 1. This sequence is shown in Figures 2.30 through 2.32 and illustrate how the inflection point has shifted from (0, 0) to (2, −1).

Figure 2.30



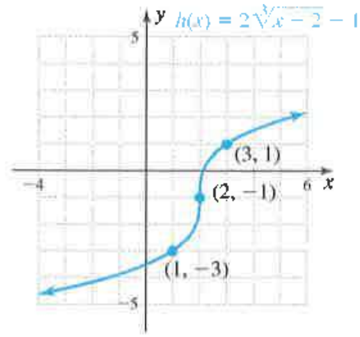
Shifted right 2

Figure 2.31



Stretched by a factor of 2

Figure 2.32



Shifted down 1

Now try Exercises 63 through 92 ▶

It's important to note that the transformations can actually be applied to *any function,* even those that are new and unfamiliar. Consider the following pattern:

Parent Function	Transformation of Parent Function
quadratic: $y = x^2$	$y = -2(x - 3)^2 + 1$
absolute value: $y = \|x\|$	$y = -2\|x - 3\| + 1$
cube root: $y = \sqrt[3]{x}$	$y = -2\sqrt[3]{x - 3} + 1$
general: $y = f(x)$	$y = -2f(x - 3) + 1$

In each case, the transformation involves a horizontal shift 3 units right, a vertical reflection, a vertical stretch, and a vertical shift up 1. Since the shifts are the same regardless of the initial function, we can generalize the results to any function $f(x)$.

General Function	Transformed Function
$y = f(x)$	$y = af(x \pm h) \pm k$

a: vertical reflections, vertical stretches and compressions
h: horizontal shift h units, opposite direction of sign
k: vertical shift k units, same direction as sign

WORTHY OF NOTE

Since the shape of the initial graph does not change when translations or reflections are applied, these are called **rigid transformations.** Stretches and compressions of a basic graph are called **nonrigid transformations,** as the graph is distended in some way.

Also bear in mind that the graph will be reflected across the y-axis (horizontally) if x is replaced with $-x$. This process is illustrated in Example 9 for selected transformations. Remember—if the graph of a function is shifted, the *individual points* on the graph are likewise shifted.

EXAMPLE 9 ▶ Graphing Transformations of a General Function

Given the graph of $f(x)$ shown in Figure 2.33, graph $g(x) = -f(x + 1) - 2$.

Solution ▶ For g, the graph of f is (1) shifted horizontally 1 unit left (Figure 2.34), (2) reflected across the x-axis (Figure 2.35), and (3) shifted vertically 2 units down (Figure 2.36). The final result is that in Figure 2.36.

Figure 2.33

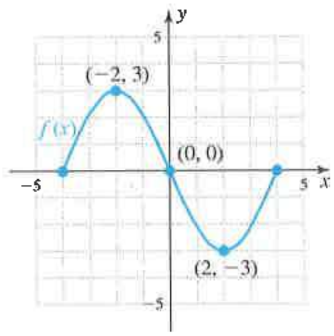


Figure 2.34

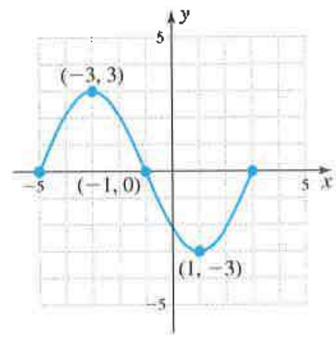


Figure 2.35

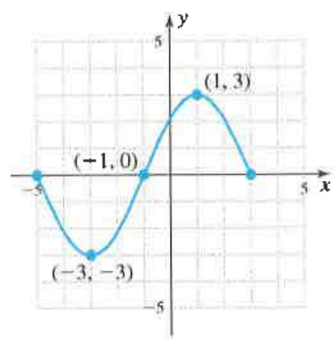
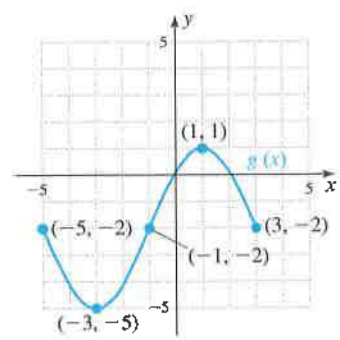


Figure 2.36



Now try Exercises 93 through 96 ▶

As noted in Example 9, these shifts and transformation are often combined—particularly when the toolbox functions are used as real-world models (Section 2.6). On a graphing calculator we again define Y_1 as needed, then define Y_2 as any desired combination of shifts, stretches, and/or reflections. For $Y_1 = X^2$, we'll define Y_2 as $-2\,Y_1(X + 5) + 3$ (Figure 2.37), and expect that the graph of Y_2 will be that of Y_1 shifted left 5 units, reflected across the x-axis, stretched vertically, and shifted up three units. This shows the new vertex should be at $(-5, 3)$, which is confirmed in Figure 2.38 along with the other transformations.

Figure 2.37

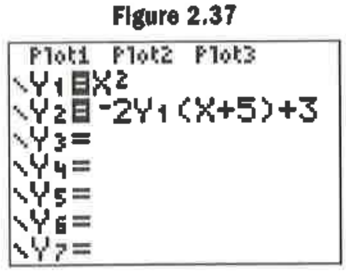
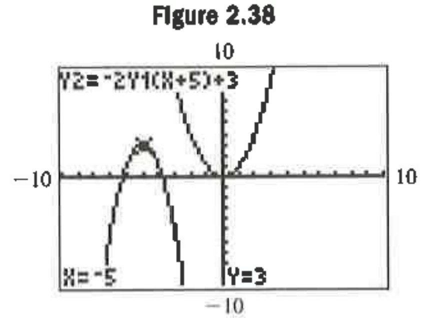


Figure 2.38

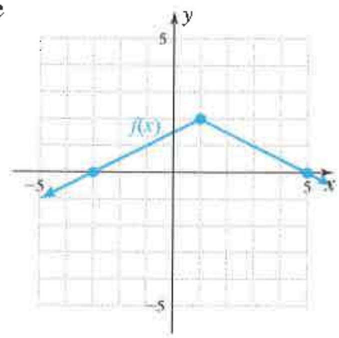


Try this exploration again using $Y_1 = \text{abs}(X)$.

Using the general equation $y = af(x \pm h) \pm k$, we can identify the vertex, initial point, or inflection point of any toolbox function and sketch its graph. Given the *graph* of a toolbox function, we can likewise identify these points and reconstruct its equation. We first identify the function family and the location (h, k) of any characteristic point. By selecting one other point (x, y) on the graph, we then use the general equation as a formula (substituting h, k, and the x- and y-values of the second point) to solve for a and complete the equation.

EXAMPLE 10 ▶ Writing the Equation of a Function Given Its Graph

Find the equation of the function $f(x)$ shown in the figure.



Solution ▶ The function f belongs to the absolute value family. The vertex (h, k) is at $(1, 2)$. For an additional point, choose the x-intercept $(-3, 0)$ and work as follows:

$y = a|x - h| + k$ — general equation (function is shifted right and up)

$0 = a|(-3) - 1| + 2$ — substitute 1 for h and 2 for k, substitute -3 for x and 0 for y

$0 = 4a + 2$ — simplify

$-2 = 4a$ — subtract 2

$-\frac{1}{2} = a$ — solve for a

The equation for f is $y = -\frac{1}{2}|x - 1| + 2$.

E. You've just seen how we can apply transformations on a general function $f(x)$

Now try Exercises 97 through 102 ▶

2.2 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. After a vertical _____, points on the graph are farther from the x-axis. After a vertical _______, points on the graph are closer to the x-axis.

2. Transformations that change only the location of a graph and not its shape or form, include ________ and ________.

3. The vertex of $h(x) = 3(x + 5)^2 - 9$ is at ______ and the graph opens ______.

4. The inflection point of $f(x) = -2(x - 4)^3 + 11$ is at _____ and the end-behavior is _____, _____.

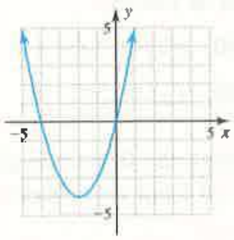
5. Given the graph of a general function $f(x)$, discuss/explain how the graph of $F(x) = -2f(x + 1) - 3$ can be obtained. If $(0, 5)$, $(6, 7)$, and $(-9, -4)$ are on the graph of f, where do they end up on the graph of F?

6. Discuss/Explain why the shift of $f(x) = x^2 + 3$ is a *vertical shift* of 3 units in the *positive* direction, while the shift of $g(x) = (x + 3)^2$ is a *horizontal shift* 3 units in the *negative* direction. Include several examples along with a table of values for each.

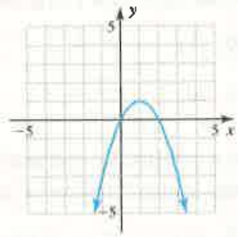
▶ DEVELOPING YOUR SKILLS

By carefully inspecting each graph given, (a) identify the function family; (b) describe or identify the end-behavior, vertex, intervals where the function is increasing or decreasing, maximum or minimum value(s) and x- and y-intercepts; and (c) determine the domain and range. Assume required features have integer values.

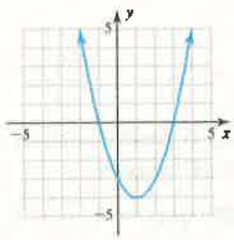
7. $f(x) = x^2 + 4x$



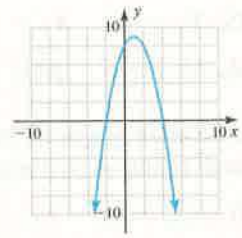
8. $g(x) = -x^2 + 2x$



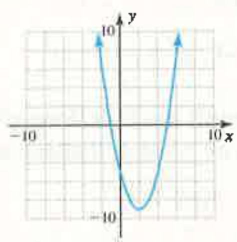
9. $p(x) = x^2 - 2x - 3$



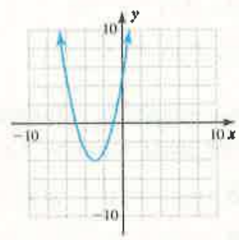
10. $q(x) = -x^2 + 2x + 8$



11. $f(x) = x^2 - 4x - 5$

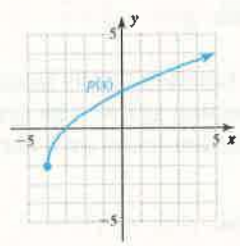


12. $g(x) = x^2 + 6x + 5$

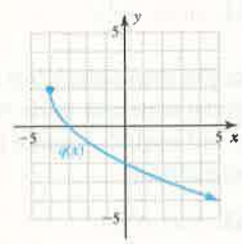


For each graph given, (a) identify the function family; (b) describe or identify the end-behavior, initial point, intervals where the function is increasing or decreasing, and x- and y-intercepts; and (c) determine the domain and range. Assume required features have integer values.

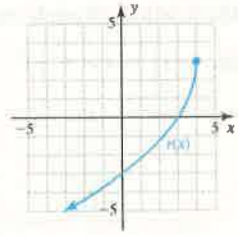
13. $p(x) = 2\sqrt{x + 4} - 2$



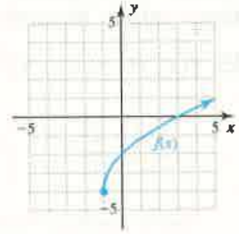
14. $q(x) = -2\sqrt{x + 4} + 2$



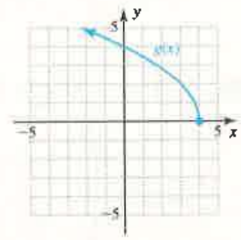
15. $r(x) = -3\sqrt{4 - x} + 3$



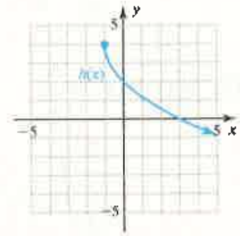
16. $f(x) = 2\sqrt{x + 1} - 4$



17. $g(x) = 2\sqrt{4 - x}$

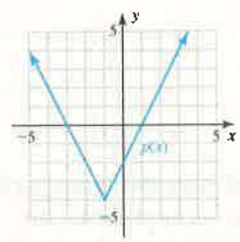


18. $h(x) = -2\sqrt{x + 1} + 4$

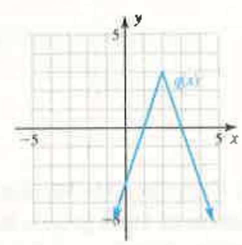


For each graph given, (a) identify the function family; (b) describe or identify the end-behavior, vertex, intervals where the function is increasing or decreasing, maximum or minimum value(s) and x- and y-intercepts; and (c) determine the domain and range. Assume required features have integer values.

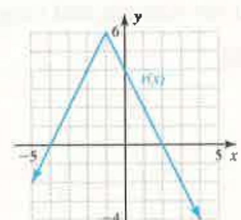
19. $p(x) = 2|x + 1| - 4$



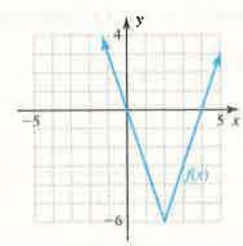
20. $q(x) = -3|x - 2| + 3$



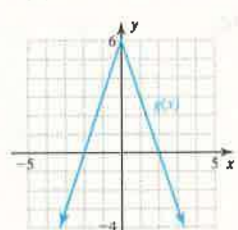
21. $r(x) = -2|x + 1| + 6$



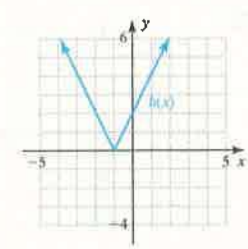
22. $f(x) = 3|x - 2| - 6$



23. $g(x) = -3|x| + 6$

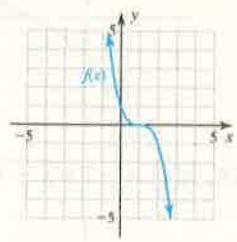


24. $h(x) = 2|x + 1|$

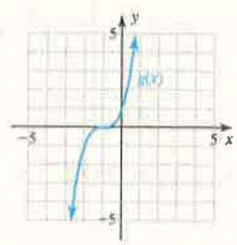


For each graph given, (a) identify the function family; (b) describe or identify the end-behavior, inflection point, and x- and y-intercepts; and (c) determine the domain and range. Assume required features have integer values. Be sure to note the scaling of each axis.

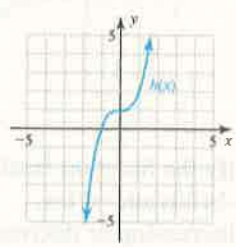
25. $f(x) = -(x - 1)^3$



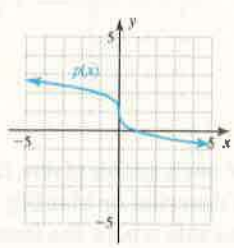
26. $g(x) = (x + 1)^3$



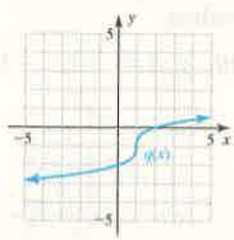
27. $h(x) = x^3 + 1$



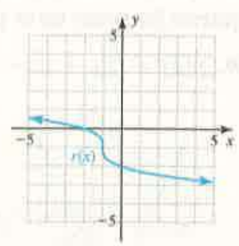
28. $p(x) = -\sqrt[3]{x} + 1$



29. $q(x) = \sqrt[3]{x - 1} - 1$

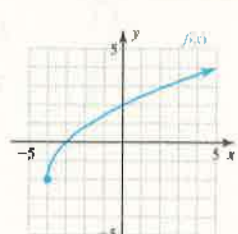


30. $r(x) = -\sqrt[3]{x + 1} - 1$

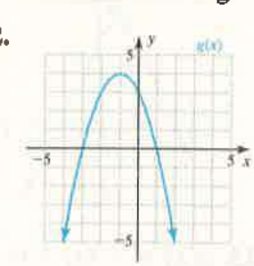


For Exercises 31–34, identify and state the characteristic features of each graph, including (as applicable) the function family, end-behavior, vertex, axis of symmetry, point of inflection, initial point, maximum and minimum value(s), x- and y-intercepts, and the domain and range.

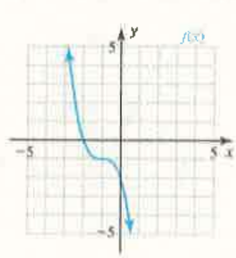
31.



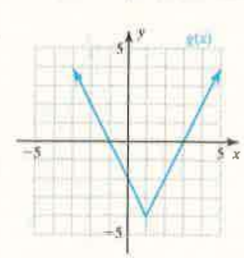
32.



33.



34.



Use a graphing calculator to graph the functions given in the same window. Comment on what you observe.

35. $f(x) = \sqrt{x}, \quad g(x) = \sqrt{x} + 2, \quad h(x) = \sqrt{x} - 3$

36. $f(x) = \sqrt[3]{x}, \quad g(x) = \sqrt[3]{x} - 3, \quad h(x) = \sqrt[3]{x} + 4$

37. $p(x) = |x|, \quad q(x) = |x| - 5, \quad r(x) = |x| + 2$

38. $p(x) = x^2, \quad q(x) = x^2 - 7, \quad r(x) = x^2 + 3$

Sketch each graph by hand using transformations of a parent function (without a table of values).

39. $f(x) = x^3 - 2$

40. $g(x) = \sqrt{x} - 4$

41. $h(x) = x^2 + 3$

42. $t(x) = |x| - 3$

Use a graphing calculator to graph the functions given in the same window. Comment on what you observe.

43. $p(x) = x^2, \quad q(x) = (x + 5)^2$

44. $f(x) = \sqrt{x}, \quad g(x) = \sqrt{x + 4}$

45. $Y_1 = |x|, \quad Y_2 = |x - 4|$

46. $h(x) = x^3, \quad H(x) = (x - 4)^3$

Sketch each graph by hand using transformations of a parent function (without a table of values).

47. $p(x) = (x - 3)^2$

48. $q(x) = \sqrt{x - 1}$

49. $h(x) = |x + 3|$

50. $f(x) = \sqrt[3]{x + 2}$

51. $g(x) = -|x|$

52. $j(x) = -\sqrt{x}$

53. $f(x) = \sqrt[3]{-x}$

54. $g(x) = (-x)^3$

Use a graphing calculator to graph the functions given in the same window. Comment on what you observe.

55. $p(x) = x^2, \quad q(x) = 3x^2, \quad r(x) = \frac{1}{5}x^2$

56. $f(x) = \sqrt{-x}, \quad g(x) = 4\sqrt{-x}, \quad h(x) = \frac{1}{4}\sqrt{-x}$

57. $Y_1 = |x|, \quad Y_2 = 3|x|, \quad Y_3 = \frac{1}{3}|x|$

58. $u(x) = x^3, \quad v(x) = 8x^3, \quad w(x) = \frac{1}{8}x^3$

Sketch each graph by hand using transformations of a parent function (without a table of values).

59. $f(x) = 4\sqrt[3]{x}$

60. $g(x) = -2|x|$

61. $p(x) = \frac{1}{3}x^3$

62. $q(x) = \frac{3}{4}\sqrt{x}$

Use the characteristics of each function family to match a given function to its corresponding graph. The graphs are not scaled—make your selection based on a careful comparison.

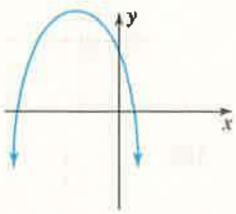
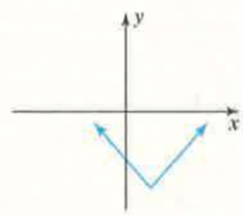
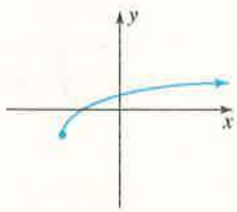
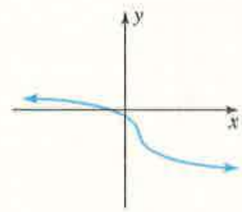
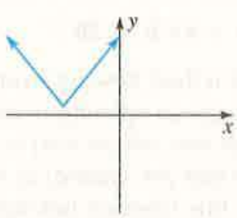
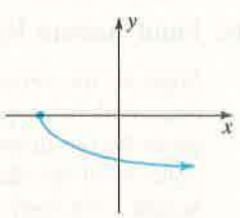
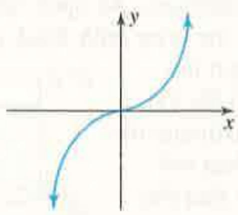
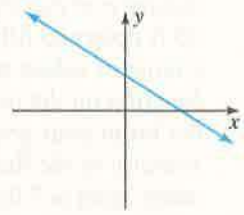
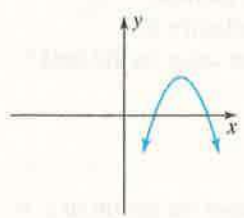
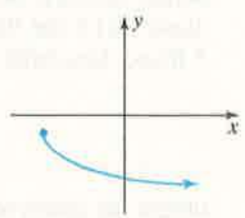
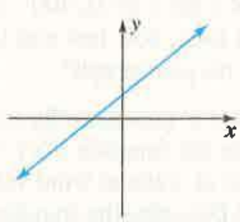
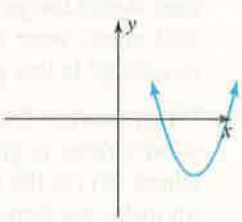
63. $f(x) = \frac{1}{2}x^3$

64. $f(x) = \frac{-2}{3}x + 2$

65. $f(x) = -(x - 3)^2 + 2$

66. $f(x) = -\sqrt[3]{x - 1} - 1$

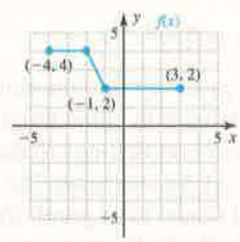
67. $f(x) = |x + 4| + 1$
68. $f(x) = -\sqrt{x + 6}$
69. $f(x) = -\sqrt{x + 6} - 1$
70. $f(x) = x + 1$
71. $f(x) = (x - 4)^2 - 3$
72. $f(x) = |x - 2| - 5$
73. $f(x) = \sqrt{x + 3} - 1$
74. $f(x) = -(x + 3)^2 + 5$

a.

b.

c.

d.

e.

f.

g.

h.

i.

j.

k.

l.


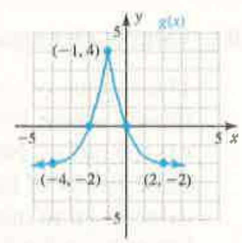
Graph each function using shifts of a parent function and a few characteristic points. *Clearly state and indicate the transformations used* and identify the location of all vertices, initial points, and/or inflection points.

75. $f(x) = \sqrt{x + 2} - 1$
76. $g(x) = \sqrt{x - 3} + 2$
77. $h(x) = -(x + 3)^2 - 2$
78. $H(x) = -(x - 2)^2 + 5$
79. $p(x) = (x + 3)^3 - 1$
80. $q(x) = (x - 2)^3 + 1$
81. $s(x) = \sqrt[3]{x + 1} - 2$
82. $t(x) = \sqrt[3]{x - 3} + 1$
83. $f(x) = -|x + 3| - 2$
84. $g(x) = -|x - 4| - 2$
85. $h(x) = -2(x + 1)^2 - 3$
86. $H(x) = \frac{1}{2}|x + 2| - 3$
87. $p(x) = -\frac{1}{3}(x + 2)^3 - 1$
88. $q(x) = 4\sqrt[3]{x + 1} + 2$
89. $u(x) = -2\sqrt{-x - 1} + 3$
90. $v(x) = 3\sqrt{-x + 2} - 1$
91. $h(x) = \frac{1}{3}(x - 3)^2 + 1$
92. $H(x) = -2|x - 3| + 4$

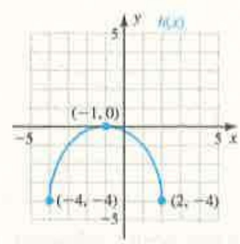
Apply the transformations indicated for the graph of the general functions given.

93.


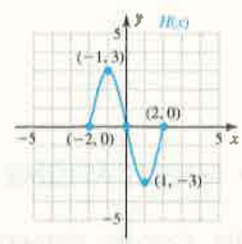
a. $f(x - 2)$
b. $-f(x) - 3$
c. $\frac{1}{2}f(x + 1)$
d. $f(-x) + 1$

94.


a. $g(x) - 2$
b. $-g(x) + 3$
c. $2g(x + 1)$
d. $\frac{1}{2}g(x - 1) + 2$

95.


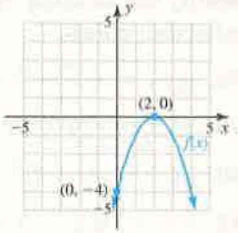
a. $h(x) + 3$
b. $-h(x - 2)$
c. $h(x - 2) - 1$
d. $\frac{1}{4}h(x) + 5$

96.


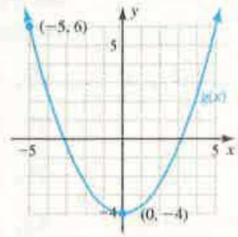
a. $H(x - 3)$
b. $-H(x) + 1$
c. $2H(x - 3)$
d. $\frac{1}{3}H(x - 2) + 1$

Use the graph given and the points indicated to determine the equation of the function shown using the general form $y = af(x \pm h) \pm k$.

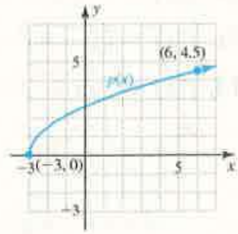
97.



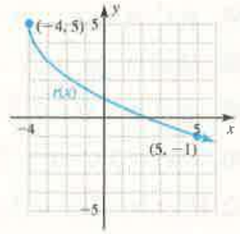
98.



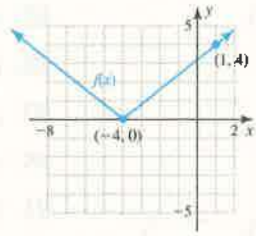
99.



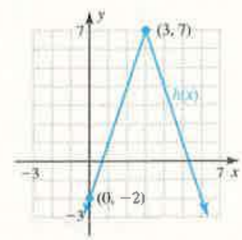
100.



101.



102.



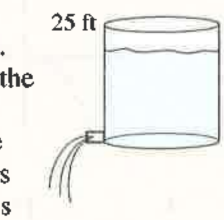
▶ WORKING WITH FORMULAS

103. Volume of a sphere: $V(r) = \frac{4}{3}\pi r^3$

The volume of a sphere is given by the function shown, where $V(r)$ is the volume in cubic units and r is the radius. Note this function belongs to the *cubic family* of functions. (a) Approximate the value of $\frac{4}{3}\pi$ to one decimal place, then graph the function on the interval [0, 3]. (b) From your *graph*, estimate the volume of a sphere with radius 2.5 in., then compute the actual volume. Are the results close? (c) For $V = \frac{4}{3}\pi r^3$, solve for r in terms of V.

104. Fluid motion: $V(h) = -4\sqrt{h} + 20$

Suppose the velocity of a fluid flowing from an open tank (no top) through an opening in its side is given by the function shown, where $V(h)$ is the velocity of the fluid (in feet per second) at water height h (in feet). Note this function belongs to the *square root family* of functions. An open tank is 25 ft deep and filled to the brim with fluid. (a) Use a table of values to graph the function on the interval [0, 25]. (b) From your graph, estimate the velocity of the fluid when the water level is 7 ft, then find the actual velocity. Are the answers close? (c) If the fluid velocity is 5 ft/sec, how high is the water in the tank?



▶ APPLICATIONS

105. Gravity, distance, time: After being released, the time it takes an object to fall x ft is given by the function $T(x) = \frac{1}{4}\sqrt{x}$, where $T(x)$ is in seconds. (a) Describe the transformation applied to obtain the graph of T from the graph of $y = \sqrt{x}$, then sketch the graph of T for $x \in [0, 100]$. (b) How long would it take an object to hit the ground if it were dropped from a height of 81 ft?

106. Stopping distance: In certain weather conditions, accident investigators will use the function $v(x) = 4.9\sqrt{x}$ to estimate the speed of a car (in miles per hour) that has been involved in an accident, based on the length of the skid marks x (in feet). (a) Describe the transformation applied to obtain the graph of v from the graph of $y = \sqrt{x}$, then sketch the graph of v for $x \in [0, 400]$. (b) If the skid marks were 225 ft long, how fast was the car traveling? Is this point on your graph?

107. Wind power: The power P generated by a certain wind turbine is given by the function $P(v) = \frac{8}{125}v^3$ where $P(v)$ is the power in watts at wind velocity v (in miles per hour). (a) Describe the transformation applied to obtain the graph of P from the graph of $y = v^3$, then sketch the graph of P for $v \in [0, 25]$ (scale the axes appropriately). (b) How much power is being generated when the wind is blowing at 15 mph?

108. Wind power: If the power P (in watts) being generated by a wind turbine is known, the velocity of the wind can be determined using the function $v(P) = \frac{5}{2}\sqrt[3]{P}$. (a) Describe the transformation applied to obtain the graph of v from the graph of $y = \sqrt[3]{P}$, then sketch the graph of v for $P \in [0, 512]$ (scale the axes appropriately). (b) How fast is the wind blowing if 343W of power is being generated? Is this point on your graph?

109. Distance rolled due to gravity: The *distance* a ball rolls down an inclined plane is given by the function $d(t) = 2t^2$, where $d(t)$ represents the distance in feet after t sec. (a) Describe the transformation applied to obtain the graph of d from the graph of $y = t^2$, then sketch the graph of d for $t \in [0, 3]$. (b) How far has the ball rolled after 2.5 sec?

110. Acceleration due to gravity: The *velocity* of a steel ball bearing as it rolls down an inclined plane is given by the function $v(t) = 4t$, where $v(t)$ represents the velocity in feet per second after t sec. (a) Describe the transformation applied to obtain the graph of v from the graph of $y = t$, then sketch the graph of v for $t \in [0, 3]$. (b) What is the velocity of the ball bearing after 2.5 sec? Is this point on your graph?

▶ EXTENDING THE CONCEPT

111. Carefully graph the functions $f(x) = |x|$ and $g(x) = 2\sqrt{x}$ on the same coordinate grid. From the graph, in what interval is the graph of $g(x)$ *above* the graph of $f(x)$? Pick a number (call it h) from this interval and substitute it in both functions. Is $g(h) > f(h)$? In what interval is the graph of $g(x)$ below the graph of $f(x)$? Pick a number from this interval (call it k) and substitute it in both functions. Is $g(k) < f(k)$?

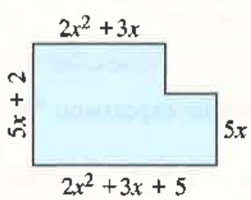
112. Sketch the graph of $f(x) = -2|x - 3| + 8$ using transformations of the parent function, then determine the area of the region in quadrant I that is beneath the graph and bounded by the vertical lines $x = 0$ and $x = 6$.

113. Sketch the graph of $f(x) = x^2 - 4$, then sketch the graph of $F(x) = |x^2 - 4|$ using your intuition and the meaning of absolute value (not a table of values). What happens to the graph?

▶ MAINTAINING YOUR SKILLS

114. (1.1) Find the distance between the points $(-13, 9)$ and $(7, -12)$, and the slope of the line containing these points.

115. (Appendix A.2) Find the perimeter of the figure shown.



116. (1.5) Solve for x: $\frac{2}{3}x + \frac{1}{4} = \frac{1}{2}x - \frac{7}{12}$.

117. (2.1) Without graphing, state intervals where $f(x)\uparrow$ and $f(x)\downarrow$ for $f(x) = (x - 4)^2 + 3$.

2.3 Absolute Value Functions, Equations, and Inequalities

LEARNING OBJECTIVES

In Section 2.3 you will see how we can:

- ☐ **A.** Solve absolute value equations
- ☐ **B.** Solve "less than" absolute value inequalities
- ☐ **C.** Solve "greater than" absolute value inequalities
- ☐ **D.** Solve absolute value equations and inequalities graphically
- ☐ **E.** Solve applications involving absolute value

While the equations $x + 1 = 5$ and $|x + 1| = 5$ are similar in many respects, note the first has only the solution $x = 4$, while either $x = 4$ or $x = -6$ will satisfy the second. The fact there are two solutions shouldn't surprise us, as it's a natural result of how absolute value is defined.

A. Solving Absolute Value Equations

The absolute value of a number x can be thought of as its distance from zero on the number line, regardless of direction. This means $|x| = 4$ will have *two solutions,* since there are two numbers that are four units from zero: $x = -4$ and $x = 4$ (see Figure 2.39).

Figure 2.39

This basic idea can be extended to include situations where the quantity within absolute value bars *is an algebraic expression,* and suggests the following property.

WORTHY OF NOTE

Note if $k < 0$, the equation $|X| = k$ has no solutions since the absolute value of any quantity is always positive or zero. On a related note, we can verify that if $k = 0$, the equation $|X| = 0$ has only the solution $X = 0$.

Property of Absolute Value Equations

If X represents an algebraic expression and k is a positive real number,

$$\text{then } |X| = k$$
$$\text{implies } X = -k \text{ or } X = k$$

As the statement of this property suggests, it can only be applied *after* the absolute value expression has been isolated on one side.

EXAMPLE 1 ▸ **Solving an Absolute Value Equation**

Solve: $-5|x - 7| + 2 = -13$.

Solution ▸ Begin by isolating the absolute value expression.

$$-5|x - 7| + 2 = -13 \quad \text{original equation}$$
$$-5|x - 7| = -15 \quad \text{subtract 2}$$
$$|x - 7| = 3 \quad \text{divide by } -5 \text{ (simplified form)}$$

Now consider $x - 7$ as the variable expression "X" in the property of absolute value equations, giving

$$x - 7 = -3 \quad \text{or} \quad x - 7 = 3 \quad \text{apply the property of absolute value equations}$$
$$x = 4 \quad \text{or} \quad x = 10 \quad \text{add 7}$$

Substituting into the original equation verifies the solution set is $\{4, 10\}$.

Now try Exercises 7 through 18 ▸

⚠ CAUTION ▸ For equations like those in Example 1, be careful not to treat the absolute value bars as simple grouping symbols. The equation $-5(x - 7) + 2 = -13$ has only the solution $x = 10$, and "misses" the second solution since it yields $x - 7 = 3$ in simplified form. The equation $-5|x - 7| + 2 = -13$ simplifies to $|x - 7| = 3$ and there are actually *two* solutions. Also note that $-5|x - 7| \neq |-5x + 35|$!

If an equation has more than one solution as in Example 1, they cannot be simultaneously stored using the, [X,T,θ,n] key to perform a calculator check (in function or "**Func**" mode, this is the variable X). While there are other ways to "get around" this (using Y_1 on the home screen, using a **TABLE** in ASK mode, enclosing the solutions in braces as in {4, 10}, etc.), we can also store solutions using the [ALPHA] keys. To illustrate, we'll place the solution $x = 4$ in storage location A, using 4 [STO▸] [ALPHA] [MATH] (A). Using this "[STO▸] [ALPHA]" sequence we'll next place the solution $x = 10$ in storage location B (Figure 2.40). We can then check both solutions in turn. Note that after we check the first solution, we can recall the expression using [2nd] [ENTER] and simply change the A to B (Figure 2.41).

Figure 2.40

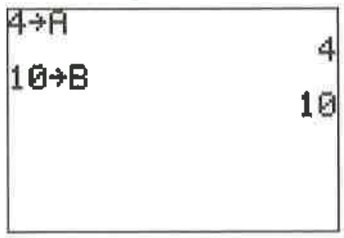
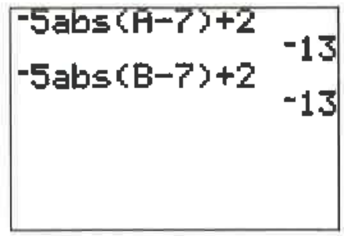


Figure 2.41



Absolute value equations come in many different forms. Always begin by isolating the absolute value expression, then apply the property of absolute value equations to solve.

EXAMPLE 2 ▶ Solving an Absolute Value Equation

Solve: $\left|5 - \frac{2}{3}x\right| - 9 = 8.$

Solution ▶

$\left|5 - \frac{2}{3}x\right| - 9 = 8$ — original equation

$\left|5 - \frac{2}{3}x\right| = 17$ — add 9

$5 - \frac{2}{3}x = -17$ or $5 - \frac{2}{3}x = 17$ — apply the property of absolute value equations

$-\frac{2}{3}x = -22$ or $-\frac{2}{3}x = 12$ — subtract 5

$x = 33$ or $x = -18$ — multiply by $-\frac{3}{2}$

Check ▶

For $x = 33$:

$$\begin{aligned}\left|5 - \tfrac{2}{3}(33)\right| - 9 &= 8\\ |5 - 2(11)| - 9 &= 8\\ |5 - 22| - 9 &= 8\\ |-17| - 9 &= 8\\ 17 - 9 &= 8\\ 8 &= 8 \checkmark\end{aligned}$$

For $x = -18$:

$$\begin{aligned}\left|5 - \tfrac{2}{3}(-18)\right| - 9 &= 8\\ |5 - 2(-6)| - 9 &= 8\\ |5 + 12| - 9 &= 8\\ |17| - 9 &= 8\\ 17 - 9 &= 8\\ 8 &= 8 \checkmark\end{aligned}$$

Both solutions check. The solution set is $\{-18, 33\}$.

WORTHY OF NOTE

As illustrated in both Examples 1 and 2, the property we use to solve absolute value equations can only be applied *after* the absolute value term has been isolated. As you will see, the same is true for the properties used to solve absolute value inequalities.

Now try Exercises 19 through 22 ▶

For some equations, it's helpful to apply the **multiplicative property of absolute value:**

Multiplicative Property of Absolute Value

If A and B represent algebraic expressions,

$$\text{then } |AB| = |A||B|.$$

Note that if $A = -1$ the property says $|-1 \cdot B| = |-1||B| = |B|$. More generally the property is applied where A is any constant.

EXAMPLE 3 ▶ Solving Equations Using the Multiplicative Property of Absolute Value

Solve: $|-2x| + 5 = 13$.

Solution ▶

$\|-2x\| + 5 = 13$	original equation
$\|-2x\| = 8$	subtract 5
$\|-2\|\|x\| = 8$	apply multiplicative property of absolute value
$2\|x\| = 8$	simplify
$\|x\| = 4$	divide by 2
$x = -4$ or $x = 4$	apply property of absolute value equations

Both solutions check. The solution set is $\{-4, 4\}$.

Now try Exercises 23 and 24 ▶

In some instances, we have one absolute value quantity equal to another, as in $|A| = |B|$. From this equation, four possible solutions are immediately apparent:

$$(1)\ A = B \qquad (2)\ A = -B \qquad (3)\ -A = B \qquad (4)\ -A = -B$$

However, basic properties of equality show that equations (1) and (4) are equivalent, as are equations (2) and (3), meaning all solutions can be found using only equations (1) and (2).

EXAMPLE 4 ▶ Solving Absolute Value Equations with Two Absolute Value Expressions

Solve the equation $|2x + 7| = |x - 1|$.

Solution ▶ This equation has the form $|A| = |B|$, where $A = 2x + 7$ and $B = x - 1$. From our previous discussion, all solutions can be found using $A = B$ and $A = -B$.

$A = B$	solution template	$A = -B$	solution template
$2x + 7 = x - 1$	substitute	$2x + 7 = -(x - 1)$	substitute
$2x = x - 8$	subtract 7	$2x + 7 = -x + 1$	distribute
$x = -8$	subtract x	$3x = -6$	add x, subtract 7
		$x = -2$	divide by 3

The solutions are $x = -8$ and $x = -2$. Verify the solutions by substituting them into the original equation.

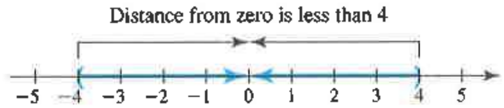
Now try Exercises 25 and 26 ▶

☑ **A.** You've just seen how we can solve absolute value equations

B. Solving "Less Than" Absolute Value Inequalities

Absolute value *inequalities* can be solved using the basic concept underlying the property of absolute value equalities. Whereas the equation $|x| = 4$ asks for all numbers x whose distance from zero is *equal* to 4, the inequality $|x| < 4$ asks for all numbers x whose distance from zero is *less than* 4.

Figure 2.42



As Figure 2.42 illustrates, the solutions are $x > -4$ and $x < 4$, which can be written as the joint inequality $-4 < x < 4$. This idea can likewise be extended to include the absolute value of an algebraic expression X as follows.

Property I: Absolute Value Inequalities (Less Than)

If X represents an algebraic expression and k is a positive real number,

$$\text{then } |X| < k$$
$$\text{implies } -k < X < k$$

Property I can also be applied when the "≤" symbol is used. Also notice that if $k < 0$, the solution is the empty set since the absolute value of any quantity is always positive or zero.

EXAMPLE 5 ▶ Solving "Less Than" Absolute Value Inequalities

Solve the inequalities:

a. $\frac{|3x + 2|}{4} \le 1$ **b.** $|2x - 7| < -5$

Solution ▶ **a.**

$$\frac{|3x + 2|}{4} \le 1 \quad \text{original inequality}$$
$$|3x + 2| \le 4 \quad \text{multiply by 4}$$
$$-4 \le 3x + 2 \le 4 \quad \text{apply Property I}$$
$$-6 \le 3x \le 2 \quad \text{subtract 2 from all three parts}$$
$$-2 \le x \le \frac{2}{3} \quad \text{divide all three parts by 3}$$

The solution interval is $[-2, \frac{2}{3}]$.

b. $|2x - 7| < -5$ original inequality

Since the absolute value of any quantity is always positive or zero, the solution for this inequality is the empty set: { }.

Now try Exercises 27 through 38 ▶

As with the inequalities from Section 1.5, solutions to absolute value inequalities can be checked using a test value. For Example 5(a), substituting $x = 0$ from the solution interval yields:

$$\frac{1}{2} \le 1 \checkmark$$

In addition to checking absolute value inequalities using a test value, the **TABLE** feature of a graphing calculator can be used, alone or in conjunction with a **relational test.** Relational tests have the calculator return a "1" if a given statement is true, and a "0" otherwise. To illustrate, consider the inequality $2|x - 3| + 1 \le 5$. Enter the expression on the left as Y_1, recalling the **"abs("** notation is accessed in the MATH menu: MATH ▶ **(NUM) "1:abs("** ENTER (note this option gives only the left parenthesis, you must supply the right). We can then simply inspect the Y_1 column of the **TABLE** to find outputs that are less than or equal to 5. To use a relational test, we enter $Y_1 \le 5$ as Y_2 (Figure 2.43), with the "less than or equal to" symbol accessed using 2nd MATH **6:≤.** Now the calculator will automatically check the truth of the statement for any value of x (but note we are only checking integer values), and display the result in the Y_2 column of the **TABLE** (Figure 2.44). After scrolling through the table, both approaches show that $2|x - 3| + 1 \le 5$ for $x \in [1, 5]$.

Figure 2.43

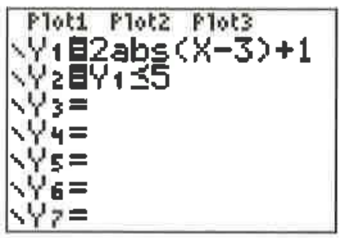
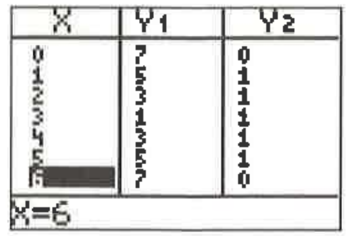


Figure 2.44



✓ **B. You've just seen how we can solve "less than" absolute value inequalities**

C. Solving "Greater Than" Absolute Value Inequalities

For "greater than" inequalities, consider $|x| > 4$. Now we're asked to find all numbers x whose distance from zero is *greater than* 4. As Figure 2.45 shows, solutions are found in the interval to the left of -4, or to the right of 4. The fact the intervals are disjoint (disconnected) is reflected in this graph, in the inequalities $x < -4$ **or** $x > 4$, as well as the interval notation $x \in (-\infty, -4) \cup (4, \infty)$.

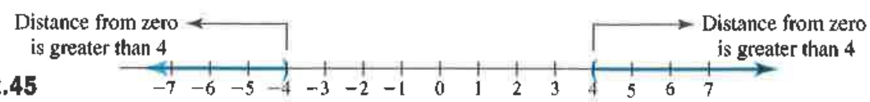


Figure 2.45

As before, we can extend this idea to include algebraic expressions, as follows:

Property II: Absolute Value Inequalities (Greater Than)

If X represents an algebraic expression and k is a positive real number,

then $|X| > k$

implies $X < -k$ or $X > k$

EXAMPLE 6 ▶ **Solving "Greater Than" Absolute Value Inequalities**

Solve the inequalities:

a. $-\frac{1}{3}\left|3 + \frac{x}{2}\right| < -2$ **b.** $|5x + 2| \ge -\frac{3}{2}$

Solution ▶ **a.** Note the exercise is given as a *less than* inequality, but as we multiply both sides by -3, we must *reverse the inequality symbol.*

$$-\frac{1}{3}\left|3+\frac{x}{2}\right| < -2 \qquad \text{original inequality}$$

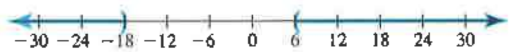
$$\left|3+\frac{x}{2}\right| > 6 \qquad \text{multiply by } -3, \textit{ reverse the symbol}$$

$$3+\frac{x}{2} < -6 \quad \text{or} \quad 3+\frac{x}{2} > 6 \qquad \text{apply Property II}$$

$$\frac{x}{2} < -9 \quad \text{or} \quad \frac{x}{2} > 3 \qquad \text{subtract 3}$$

$$x < -18 \quad \text{or} \quad x > 6 \qquad \text{multiply by 2}$$

Property II yields the disjoint intervals $x \in (-\infty, -18) \cup (6, \infty)$ as the solution.



b. $|5x+2| \geq -\frac{3}{2}$ original inequality

Since the absolute value of any quantity is always positive or zero, the solution for this inequality is all real numbers: $x \in \mathbb{R}$.

Now try Exercises 39 through 54 ▶

A calculator check is shown for part (a) in Figures 2.46 through 2.48.

Figure 2.46

Figure 2.47

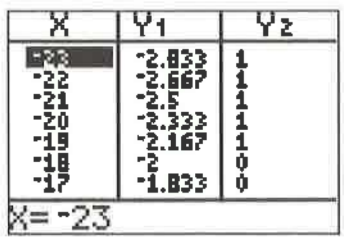
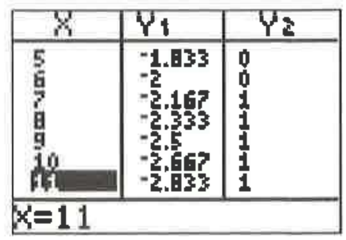


Figure 2.48



This helps to verify the solution interval is $x \in (-\infty, -18) \cup (6, \infty)$.

Due to the nature of absolute value functions, there are times when an absolute value relation cannot be satisfied. For instance the equation $|x-4| = -2$ has no solutions, as the left-hand expression will always represent a nonnegative value. The inequality $|2x+3| < -1$ has no solutions for the same reason. On the other hand, the inequality $|9-x| \geq 0$ is true for all real numbers, since any value substituted for x will result in a nonnegative value. We can generalize many of these special cases as follows.

C. You've just seen how we can solve "greater than" absolute value inequalities

Absolute Value Functions—Special Cases

Given k is a positive real number and A represents an algebraic expression,

$\lvert A\rvert = -k$	$\lvert A\rvert < -k$	$\lvert A\rvert > -k$
has no solutions	has no solutions	is true for all real numbers

See **Exercises 51 through 54.**

CAUTION ▶ Be sure you note the difference between the individual solutions of an absolute value equation, and the solution intervals that often result from solving absolute value inequalities. The solution $\{-2, 5\}$ indicates that both $x = -2$ and $x = 5$ are solutions, while the solution $[-2, 5)$ indicates that all numbers between -2 and 5, including -2, are solutions.

D. Solving Absolute Value Equations and Inequalities Graphically

The concepts studied in Section 1.5 (solving linear equations and inequalities graphically) are easily extended to other kinds of relations. Essentially, we treat each expression forming the equation or inequality as a *separate function,* then graph both functions to find points of intersection (equations) or where one graph is above or

Figure 2.49

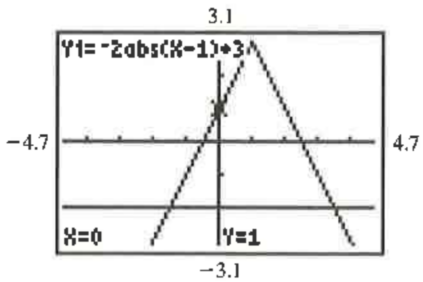
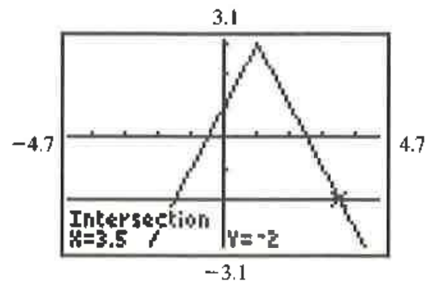


Figure 2.50



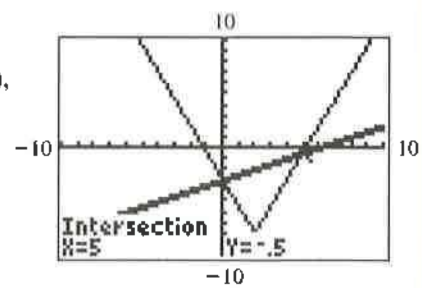
below the other (inequalities). For $-2|x - 1| + 3 < -2$, enter the expression $-2|X - 1| + 3$ as Y_1 on the Y= screen, and -2 as Y_2. Using ZOOM **4:ZDecimal** produces the graph shown in Figure 2.49. Using 2nd TRACE **(CALC) 5:intersect,** we find the graphs intersect at $x = -1.5$ and $x = 3.5$ (Figure 2.50), and the graph of Y_1 is above the graph of Y_2 in this interval. Since this is a "less than" inequality, the solutions are *outside* of this interval, which gives $x \in (-\infty, -1.5) \cup (3.5, \infty)$ as the solution interval. Note that the zeroes/x-intercept method could also have been used.

EXAMPLE 7 ▶ Solving Absolute Equations and Inequalities Graphically

For $f(x) = 2.5|x - 2| - 8$ and $g(x) = \frac{1}{2}x - 3$, solve

a. $f(x) = g(x)$ **b.** $f(x) \le g(x)$ **c.** $f(x) > g(x)$

Solution ▶ **a.** With $f(x) = 2.5|x - 2| - 8$ as Y_1 and $g(x) = \frac{1}{2}x - 3$ as Y_2 (set to graph in **bold**), using 2nd TRACE **(CALC) 5:intersect** shows the graphs intersect ($Y_1 = Y_2$) at $x = 0$ and $x = 5$ (see figure). These are the solutions to $2.5|x - 2| - 8 = \frac{1}{2}x - 3$.



b. The graph of Y_1 is *below* the graph of Y_2 ($Y_1 < Y_2$) between these points of intersection, so the solution interval for $2.5|x - 2| - 8 \le \frac{1}{2}x - 3$ is $x \in [0, 5]$.

c. The graph of Y_1 is *above* the graph of Y_2 ($Y_1 > Y_2$) outside this interval, giving a solution of $x \in (-\infty, 0) \cup (5, \infty)$ for $2.5|x - 2| - 8 > \frac{1}{2}x - 3$.

Now try Exercises 55 through 58 ▶

☑ D. You've just seen how we can solve absolute value equations and inequalities graphically

E. Applications Involving Absolute Value

Applications of absolute value often involve finding a range of values for which a given statement is true. Many times, the equation or inequality used must be modeled after a given description or from given information, as in Example 8.

EXAMPLE 8 ► **Solving Applications Involving Absolute Value Inequalities**

For new cars, the number of miles per gallon (mpg) a car will get is heavily dependent on whether it is used mainly for short trips and city driving, or primarily on the highway for longer trips. For a certain car, the number of miles per gallon that a driver can expect varies by no more than 6.5 mpg above or below its field tested average of 28.4 mpg. What range of mileage values can a driver expect for this car?

Solution ► Field tested average: 28.4 mpg — gather information

mileage varies by no more than 6.5 mpg — highlight key phrases

make the problem visual

Let m represent the miles per gallon a driver can expect. Then the difference between m and 28.4 can be no more than 6.5, or $|m - 28.4| \le 6.5$. — assign a variable; write an equation model

$$|m - 28.4| \le 6.5 \quad \text{equation model}$$
$$-6.5 \le m - 28.4 \le 6.5 \quad \text{apply Property I}$$
$$21.9 \le m \le 34.9 \quad \text{add 28.4 to all three parts}$$

The mileage that a driver can expect ranges from a low of 21.9 mpg to a high of 34.9 mpg.

E. You've just seen how we can solve applications involving absolute value

Now try Exercises 61 through 70 ►

2.3 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

1. When multiplying or dividing by a negative quantity, we ________ the inequality symbol to maintain a true statement.

2. To write an absolute value equation or inequality in simplified form, we ________ the absolute value expression on one side.

3. The absolute value equation $|2x + 3| = 7$ is true when $2x + 3 =$ ____ or when $2x + 3 =$ ____.

4. The absolute value inequality $|3x - 6| < 12$ is true when $3x - 6 >$ ____ and $3x - 6 <$ ____.

Describe the solution set for each inequality (assume $k > 0$). Justify your answer.

5. $|ax + b| < -k$

6. $|ax + b| > -k$

▶ DEVELOPING YOUR SKILLS

Solve each absolute value equation. Write the solution in set notation. For Exercises 7 to 18, verify solutions by substituting into the original equation. For Exercises 19–26 verify solutions using a calculator.

7. $2|m - 1| - 7 = 3$
8. $3|n - 5| - 14 = -2$
9. $-3|x + 5| + 6 = -15$
10. $-2|y + 3| - 4 = -14$
11. $2|4v + 5| - 6.5 = 10.3$
12. $7|2w + 5| + 6.3 = 11.2$
13. $-|7p - 3| + 6 = -5$
14. $-|3q + 4| + 3 = -5$
15. $-2|b| - 3 = -4$
16. $-3|c| - 5 = -6$
17. $-2|3x| - 17 = -5$
18. $-5|2y| - 14 = 6$
19. $-3\left|\frac{w}{2} + 4\right| - 1 = -4$
20. $-2\left|3 - \frac{v}{3}\right| + 1 = -5$
21. $8.7|p - 7.5| - 26.6 = 8.2$
22. $5.3|q + 9.2| + 6.7 = 43.8$
23. $8.7|-2.5x| - 26.6 = 8.2$
24. $5.3|1.25n| + 6.7 = 43.8$
25. $|x - 2| = |3x + 4|$
26. $|2x - 1| = |x + 3|$

Solve each absolute value inequality. Write solutions in interval notation. Check solutions by back substitution, or using a calculator.

27. $3|p + 4| + 5 < 8$
28. $5|q - 2| - 7 \le 8$
29. $-3|m| - 2 > 4$
30. $-2|n| + 3 > 7$
31. $|3b - 11| + 6 \le 9$
32. $|2c + 3| - 5 < 1$
33. $|4 - 3z| + 12 < 7$
34. $|2 - 3u| + 5 \le 4$
35. $\frac{|5v + 1|}{4} + 8 < 9$
36. $\frac{|3w - 2|}{2} + 6 < 8$
37. $\left|\frac{4x + 5}{3} - \frac{1}{2}\right| \le \frac{7}{6}$
38. $\left|\frac{2y - 3}{4} - \frac{3}{8}\right| \le \frac{15}{16}$
39. $|n + 3| > 7$
40. $|m - 1| > 5$
41. $-2|w| - 5 \le -11$
42. $-5|v| - 3 \le -23$
43. $\frac{|q|}{2} - \frac{5}{6} \ge \frac{1}{3}$
44. $\frac{|p|}{5} + \frac{3}{2} \ge \frac{9}{4}$
45. $3|5 - 7d| + 9 \ge 15$
46. $5|2c + 7| + 1 \ge 11$
47. $2 < \left|-3m + \frac{4}{5}\right| - \frac{1}{5}$
48. $4 \le \left|\frac{5}{4} - 2n\right| - \frac{3}{4}$
49. $4|5 - 2h| - 9 > 11$
50. $3|7 + 2k| - 11 > 10$
51. $3.9|4q - 5| + 8.7 \le -22.5$
52. $0.9|2p + 7| + 16.11 \le 10.89$
53. $|4z - 9| + 6 \ge 4$
54. $|5u - 3| + 8 > 6$

Use the intersect command on a graphing calculator and the given functions to solve (a) $f(x) = g(x)$, (b) $f(x) \ge g(x)$, and (c) $f(x) < g(x)$.

55. $f(x) = |x - 3| + 2$, $g(x) = \frac{1}{2}x + 2$
56. $f(x) = -|x + 2| - 1$, $g(x) = -\frac{3}{2}x - 9$
57. $f(x) = 0.5|x + 3| + 1$, $g(x) = -2|x + 1| + 5$
58. $f(x) = 2|x - 3| + 2$, $g(x) = |x - 4| + 6$

▶ WORKING WITH FORMULAS

59. Spring Oscillation: $|d - x| \le L$

A weight attached to a spring hangs at rest a distance of x in. off the ground. If the weight is pulled down (stretched) a distance of L inches and released, the weight begins to bounce and its distance d off the ground must satisfy the indicated formula. (a) If x equals 4 ft and the spring is stretched 3 in. and released, solve the inequality to find what distances from the ground the weight will oscillate between. (b) Solve for x in terms of L and d.

60. A "Fair" Coin: $\left|\frac{h - 50}{5}\right| < 1.645$

If we flipped a coin 100 times, we expect "heads" to come up about 50 times if the coin is "fair." In a study of probability, it can be shown that the number of heads h that appears in such an experiment should satisfy the given inequality to be considered "fair." (a) Solve this inequality for h. (b) If you flipped a coin 100 times and obtained 40 heads, is the coin "fair"?

▶ APPLICATIONS

Solve each application of absolute value.

61. Altitude of jet stream: To take advantage of the jet stream, an airplane must fly at a height h (in feet) that satisfies the inequality $|h - 35{,}050| \le 2550$. Solve the inequality and determine if an altitude of 34,000 ft will place the plane in the jet stream.

62. Quality control tests: In order to satisfy quality control, the marble columns a company produces must earn a stress test score S that satisfies the inequality $|S - 17{,}750| \le 275$. Solve the inequality and determine if a score of 17,500 is in the passing range.

63. Submarine depth: The sonar operator on a submarine detects an old World War II submarine net and must decide to detour over or under the net. The computer gives him a depth model $|d - 394| - 20 > 164$, where d is the depth in feet that represents safe passage. At what depth should the submarine travel to go under or over the net? Answer using simple inequalities.

64. Optimal fishing depth: When deep-sea fishing, the optimal depths d (in feet) for catching a certain type of fish satisfy the inequality $28|d - 350| - 1400 < 0$. Find the range of depths that offer the best fishing. Answer using simple inequalities.

For Exercises 65 through 68, (a) develop a model that uses an absolute value inequality, and (b) solve.

65. Stock value: My stock in MMM Corporation fluctuated a great deal in 2009, but never by more than \$3.35 from its current value. If the stock is worth \$37.58 today, what was its range in 2009?

66. Traffic studies: On a given day, the volume of traffic at a busy intersection averages 726 cars per hour (cph). During rush hour the volume is much higher, during "off hours" much lower. Find the range of this volume if it never varies by more than 235 cph from the average.

67. Physical training for recruits: For all recruits in the 3rd Armored Battalion, the average number of sit-ups is 125. For an individual recruit, the amount varies by no more than 23 sit-ups from the battalion average. Find the range of sit-ups for this battalion.

68. Computer consultant salaries: The national average salary for a computer consultant is \$53,336. For a large computer firm, the salaries offered to their employees vary by no more than \$11,994 from this national average. Find the range of salaries offered by this company.

69. Tolerances for sport balls: According to the official rules for golf, baseball, pool, and bowling, (a) golf balls must be within 0.03 mm of $d = 42.7$ mm, (b) baseballs must be within 1.01 mm of $d = 73.78$ mm, (c) billiard balls must be within 0.127 mm of $d = 57.150$ mm, and (d) bowling balls must be within 12.05 mm of $d = 2171.05$ mm. Write each statement using an absolute value inequality, then (e) determine which sport gives the least tolerance t $\left(t = \frac{\text{width of interval}}{\textit{average value}}\right)$ for the diameter of the ball.

70. **Automated packaging:** The machines that fill boxes of breakfast cereal are programmed to fill each box within a certain tolerance. If the box is overfilled, the company loses money. If it is underfilled, it is considered unsuitable for sale. Suppose that boxes marked "14 ounces" of cereal must be filled to within 0.1 oz. Find the acceptable range of weights for this cereal.

▶ EXTENDING THE CONCEPT

71. Determine the value or values (if any) that will make the equation or inequality true.

   a. $|x| + x = 8$ b. $|x - 2| \le \frac{x}{2}$

   c. $x - |x| = x + |x|$ d. $|x + 3| \ge 6x$

   e. $|2x + 1| = x - 3$

72. The equation $|5 - 2x| = |3 + 2x|$ has only one solution. Find it and explain why there is only one.

73. In many cases, it can be helpful to view the solutions to absolute value equations and inequalities as follows. For any algebraic expression X and positive constant k, the equation $|X| = k$ has solutions $X = k$ and $-X = k$, since the absolute value of either quantity on the left will indeed yield the positive constant k. Likewise, $|X| < k$ has solutions $X < k$ and $-X < k$. Note the inequality symbol has not been reversed as yet, but will naturally be reversed as part of the solution process. Solve the following equations or inequalities using this idea.

   a. $|x - 3| = 5$

   b. $|x - 7| > 4$

   c. $3|x + 2| \le 12$

   d. $-3|x - 4| + 7 = -11$

▶ MAINTAINING YOUR SKILLS

74. (Appendix A.4) Factor the expression completely: $18x^3 + 21x^2 - 60x$.

75. (1.5) Solve $V^2 = \frac{2W}{C\rho A}$ for ρ (physics).

76. (Appendix A.6) Simplify $\frac{-1}{3 + \sqrt{3}}$ by rationalizing the denominator. State the result in exact form and approximate form (to hundredths).

77. (Appendix A.3) Solve the inequality, then write the solution set in interval notation:

   $-3(2x - 5) > 2(x + 1) - 7.$

MID-CHAPTER CHECK

1. Determine whether the following function is even, odd, or neither. $f(x) = x^2 + \frac{|x|}{4x}$

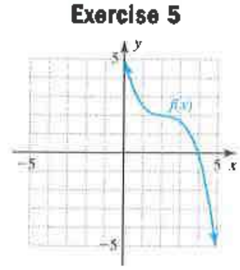
2. Use a graphing calculator to find the maximum and minimum values of $f(x) = -1.9(x^4 - 2.3x^3 + 2.2x - 5.1)$. Round to the nearest hundredth.

3. Use interval notation to identify the interval(s) where the function from Exercise 2 is increasing, decreasing, or constant. Round to the nearest hundredth.

4. Write the equation of the function that has the same graph of $f(x) = \sqrt{x}$, shifted left 4 units and up 2 units.

5. For the graph given, (a) identify the function family, (b) describe or identify the end-behavior, inflection point, and x- and y-intercepts, (c) determine the domain and range, and (d) determine the value of k if $f(k) = 2.5$. Assume required features have integer values.

Exercise 5



6. Use a graphing calculator to graph the given functions in the same window and comment on what you observe.
$p(x) = (x - 3)^2 \qquad q(x) = -(x - 3)^2$
$r(x) = -\frac{1}{2}(x - 3)^2$

7. Solve the following absolute value equations. Write the solution in set notation.
a. $\frac{2}{3}|d - 5| + 1 = 7$ **b.** $5 - |s + 3| = \frac{11}{2}$

8. Solve the following absolute value inequalities. Write solutions in interval notation.
a. $3|q + 4| - 2 < 10$ **b.** $\left|\frac{x}{3} + 2\right| + 5 \le 5$

9. Solve the following absolute value inequalities. Write solutions in interval notation.
a. $3.1|d - 2| + 1.1 \ge 7.3$
b. $\frac{|1 - y|}{3} + 2 > \frac{11}{2}$
c. $-5|k - 2| + 3 < 4$

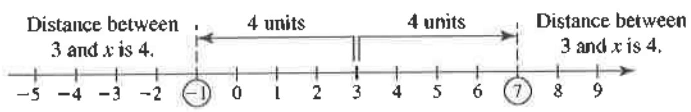
10. **Kiteboarding:** With the correct sized kite, a person can kiteboard when the wind is blowing at a speed w (in mph) that satisfies the inequality $|w - 17| \le 9$. Solve the inequality and determine if a person can kiteboard with a windspeed of (a) 5 mph? (b) 12 mph?

REINFORCING BASIC CONCEPTS

Using Distance to Understand Absolute Value Equations and Inequalities

For any two numbers a and b on the number line, *the distance between a and b* can be written $|a - b|$ or $|b - a|$. In exactly the same way, the equation $|x - 3| = 4$ can be read, "the distance between 3 and an unknown number is equal to 4." The advantage of reading it in this way (instead of "the absolute value of x minus 3 is 4"), is that a much clearer *visualization* is formed, giving a constant reminder there are two solutions. In diagram form we have Figure 2.51.

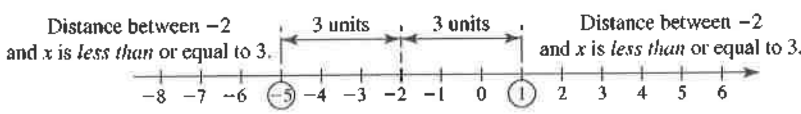
Figure 2.51



From this we note the solutions are $x = -1$ and $x = 7$.

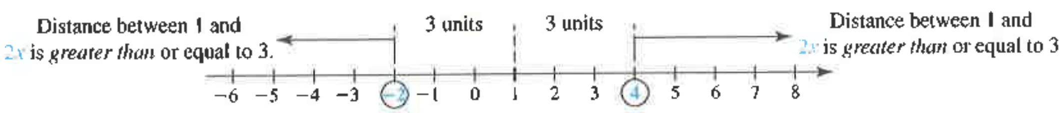
In the case of an inequality such as $|x + 2| \le 3$, we rewrite the inequality as $|x - (-2)| \le 3$ and read it, "the distance between -2 and an unknown number is less than or equal to 3." With some practice, visualizing this relationship mentally enables a quick statement of the solution: $x \in [-5, 1]$. In diagram form we have Figure 2.52.

Figure 2.52



Equations and inequalities where the coefficient of x is not 1 still lend themselves to this form of conceptual understanding. For $|2x - 1| \ge 3$ we read, "the distance between 1 and twice an unknown number is greater than or equal to 3." On the number line (Figure 2.53), the number 3 units to the right of 1 is 4, and the number 3 units to the left of 1 is -2.

Figure 2.53



For $2x \le -2$, $x \le -1$, and for $2x \ge 4$, $x \ge 2$, and the solution set is $x \in (-\infty, -1] \cup [2, \infty)$.

Attempt to solve the following equations and inequalities by visualizing a number line. Check all results algebraically.

Exercise 1: $|x - 2| = 5$ **Exercise 2:** $|x + 1| \le 4$ **Exercise 3:** $|2x - 3| \ge 5$

2.4 Basic Rational Functions and Power Functions; More on the Domain

LEARNING OBJECTIVES

In Section 2.4 you will see how we can:

- ❑ **A.** Graph basic rational functions, identify vertical and horizontal asymptotes, and describe end-behavior
- ❑ **B.** Use transformations to graph basic rational functions and write the equation for a given graph
- ❑ **C.** Graph basic power functions and state their domains
- ❑ **D.** Solve applications involving basic rational and power functions

In this section, we introduce two new kinds of relations, **rational functions** and **power functions.** While we've already studied a variety of functions, we still lack the ability to model a large number of important situations. For example, functions that model the amount of medication remaining in the bloodstream over time, the relationship between altitude and weightlessness, and the equations modeling planetary motion come from these two families.

A. Rational Functions and Asymptotes

Just as a rational number is the ratio of two integers, a **rational function** is the ratio of two polynomials. In general,

Rational Functions

A rational function $V(x)$ is one of the form

$$V(x) = \frac{p(x)}{d(x)},$$

where p and d are polynomials and $d(x) \neq 0$.
The domain of $V(x)$ is all real numbers, *except the zeroes of d.*

The simplest rational functions are the reciprocal function $y = \frac{1}{x}$ and the reciprocal square function $y = \frac{1}{x^2}$, as both have a constant numerator and a single term in the denominator. Since division by zero is undefined, the domain of both excludes $x = 0$. A preliminary study of these two functions will provide a strong foundation for our study of general rational functions in Chapter 4.

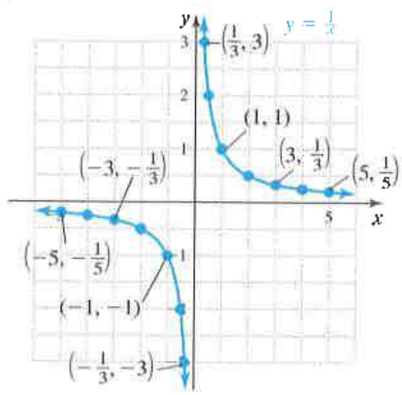
The Reciprocal Function: $y = \frac{1}{x}$

The reciprocal function takes any input (other than zero) and gives its reciprocal as the output. This means large inputs produce small outputs and vice versa. A table of values (Table 2.1) and the resulting graph (Figure 2.54) are shown.

Table 2.1

x	y	x	y
−1000	−1/1000	1/1000	1000
−5	−1/5	1/3	3
−4	−1/4	1/2	2
−3	−1/3	1	1
−2	−1/2	2	1/2
−1	−1	3	1/3
−1/2	−2	4	1/4
−1/3	−3	5	1/5
−1/1000	−1000	1000	1/1000
0	undefined		

Figure 2.54



WORTHY OF NOTE

The notation used for graphical behavior always begins by describing what is happening to the x-values, and the resulting effect on the y-values. Using Figure 2.55, visualize that for a point (x, y) on the graph of $y = \frac{1}{x}$, as x gets larger, y must become smaller, particularly since their product must always be 1 ($y = \frac{1}{x} \Rightarrow xy = 1$).

Figure 2.55

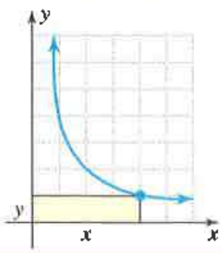


Table 2.1 and Figure 2.54 reveal some interesting features. First, the graph passes the vertical line test, verifying $y = \frac{1}{x}$ is indeed a function. Second, since division by zero is undefined, there can be no corresponding point on the graph, *creating a break at* $x = 0$. In line with our definition of rational functions, the domain is $x \in (-\infty, 0) \cup (0, \infty)$. Third, this is an odd function, with a "branch" of the graph in the first quadrant and one in the third quadrant, as the reciprocal of any input maintains its sign. Finally, we note in QI that as x becomes an infinitely large positive number, y becomes infinitely small and closer to zero. It seems convenient to symbolize this end-behavior using the following notation:

as $x \to \infty$, $\quad y \to 0$

as x becomes an infinitely large positive number $\quad$ y approaches 0

Graphically, the curve becomes very close to, or *approaches the x-axis.*

We also note that as x approaches zero from the right, y becomes an infinitely large positive number: as $x \to 0^+$, $y \to \infty$. Note a superscript + or − sign is used to indicate the *direction of the approach,* meaning *from the positive side* (right) or *from the negative side* (left).

EXAMPLE 1 ▶ Describing the End-Behavior of Rational Functions

For $y = \frac{1}{x}$ in QIII (Figure 2.54),

a. Describe the end-behavior of the graph.

b. Describe what happens as x approaches zero.

Solution ▶ Similar to the graph's behavior in QI, we have

a. In words: As x becomes an infinitely large negative number, y approaches zero. In notation: As $x \to -\infty$, $y \to 0$.

b. In words: As x approaches zero from the left, y becomes an infinitely large negative number. In notation: As $x \to 0^-$, $y \to -\infty$.

Now try Exercises 7 and 8 ▶

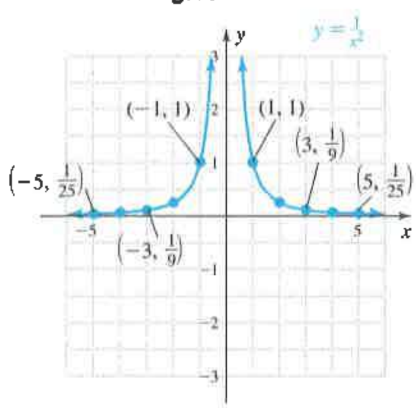
The Reciprocal Square Function: $y = \frac{1}{x^2}$

From our previous work, we anticipate this graph will also have a break at $x = 0$. But since the square of any negative number is positive, the branches of the **reciprocal square function** are both *above the x-axis.* Note the result is the graph of an even function. See Table 2.2 and Figure 2.56.

Table 2.2

x	y	x	y
−1000	1/1,000,000	1/1000	1,000,000
−5	1/25	1/3	9
−4	1/16	1/2	4
−3	1/9	1	1
−2	1/4	2	1/4
−1	1	3	1/9
−1/2	4	4	1/16
−1/3	9	5	1/25
−1/1000	1,000,000	1000	1/1,000,000
0	undefined		

Figure 2.56



Similar to $y = \frac{1}{x}$, large positive inputs generate small, positive outputs: as $x \to \infty$, $y \to 0$. This is one indication of **asymptotic behavior** in the *horizontal* direction, and we say the line $y = 0$ (the x-axis) is a **horizontal asymptote** for the reciprocal and reciprocal square functions. In general,

Horizontal Asymptotes

Given a constant k, the line $y = k$ is a horizontal asymptote for V if, as x increases or decreases without bound, $V(x)$ approaches k:

as $x \to -\infty$, $V(x) \to k$ or as $x \to \infty$, $V(x) \to k$

As shown in Figures 2.57 and 2.58, asymptotes are represented graphically as dashed lines that seem to "guide" the branches of the graph. Figure 2.57 shows a horizontal asymptote at $y = 1$, which suggests the graph of $f(x)$ is the graph of $y = \frac{1}{x}$ shifted up 1 unit. Figure 2.58 shows a horizontal asymptote at $y = -2$, which suggests the graph of $g(x)$ is the graph of $y = \frac{1}{x^2}$ shifted down 2 units.

Figure 2.57

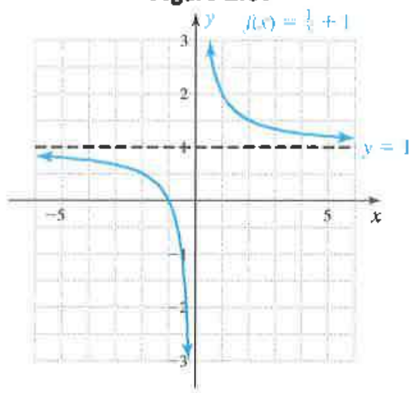
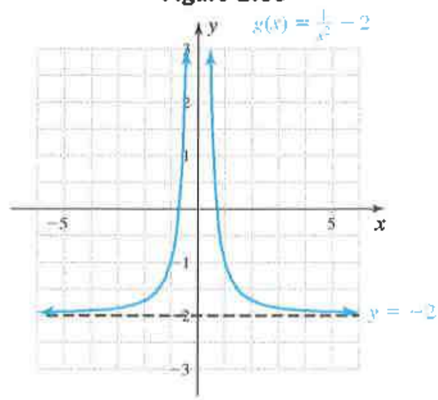


Figure 2.58



EXAMPLE 2 ▸ **Describing the End-Behavior of Rational Functions**

For the graph in Figure 2.58, use mathematical notation to

a. Describe the end-behavior of the graph and name the horizontal asymptote.

b. Describe what happens as x approaches zero.

Solution ▸ **a.** as $x \to -\infty$, $g(x) \to -2$,
as $x \to \infty$, $g(x) \to -2$,
$y = -2$ is a horizontal asymptote

b. as $x \to 0^-$, $g(x) \to \infty$,
as $x \to 0^+$, $g(x) \to \infty$

Now try Exercises 9 and 10 ▸

While the graphical view of Example 2(a) (Figure 2.58) makes these concepts believable, a numerical view of this end-behavior can be even more compelling. Try entering $\frac{1}{x^2} - 2$ as Y_1 on the Y= screen, then go to the **TABLE** feature (TblStart $= -3$, ΔTbl $= 1$; Figure 2.59). Scrolling in either direction shows that as $|x|$ becomes very large, Y_1 becomes closer and closer to -2, but will never be equal to -2 (Figure 2.60).

Figure 2.59

X	Y1
-3	-1.889
-2	-1.75
-1	-1
0	ERR:
1	-1
2	-1.75
3	-1.889

X= -3

Figure 2.60

X	Y1
14	-1.995
15	-1.996
16	-1.996
17	-1.997
18	-1.997
19	-1.997
20	-1.998

X=20

From Example 2(b), we note that as x becomes *smaller and close to 0*, g becomes very large and *increases without bound*. This is one indication of asymptotic behavior in the *vertical* direction, and we say the line $x = 0$ (the y-axis) is a **vertical asymptote** for g ($x = 0$ is also a vertical asymptote for f in Figure 2.57). In general,

Vertical Asymptotes

Given a constant h, the vertical line $x = h$ is a vertical asymptote for a function V if, as x approaches h, $V(x)$ increases or decreases without bound:

as $x \to h^+$, $V(x) \to \pm\infty$ or as $x \to h^-$, $V(x) \to \pm\infty$

Here is a brief summary:

A. You've just seen how we can graph basic rational functions, identify vertical and horizontal asymptotes, and describe end-behavior

Reciprocal Function	Reciprocal Quadratic Function
$f(x) = \frac{1}{x}$	$g(x) = \frac{1}{x^2}$
Domain: $x \in (-\infty, 0) \cup (0, \infty)$	Domain: $x \in (-\infty, 0) \cup (0, \infty)$
Range: $y \in (-\infty, 0) \cup (0, \infty)$	Range: $y \in (0, \infty)$
Horizontal asymptote: $y = 0$	Horizontal asymptote: $y = 0$
Vertical asymptote: $x = 0$	Vertical asymptote: $x = 0$

B. Using Asymptotes to Graph Basic Rational Functions

Identifying these asymptotes is useful because the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$ can be transformed *in exactly the same way as the toolbox functions.* When their graphs shift—the vertical and horizontal asymptotes shift with them and can be used as guides to redraw the graph. In shifted form,

$$f(x) = \frac{a}{x \pm h} \pm k \text{ for the reciprocal function, and}$$

$$g(x) = \frac{a}{(x \pm h)^2} \pm k \text{ for the reciprocal square function.}$$

When horizontal and/or vertical shifts are applied to simple rational functions, we first apply them to the asymptotes, then calculate the x- and y-intercepts as before. An additional point or two can be computed as needed to round out the graph.

EXAMPLE 3 ▸ Graphing Transformations of the Reciprocal Function

Sketch the graph of $g(x) = \frac{1}{x - 2} + 1$ using transformations of the parent function.

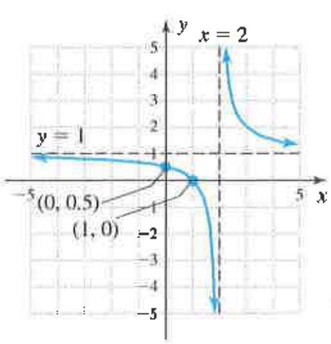
Solution ▸ The graph of g is the same as that of $y = \frac{1}{x}$, but shifted 2 units right and 1 unit upward. This means the vertical asymptote is also shifted 2 units right, and the horizontal asymptote is shifted 1 unit up. The y-intercept is $g(0) = \frac{1}{2}$. For the x-intercept:

$$0 = \frac{1}{x - 2} + 1 \quad \text{substitute 0 for } g(x)$$

$$-1 = \frac{1}{x - 2} \quad \text{subtract 1}$$

$$-1(x - 2) = 1 \quad \text{multiply by } (x - 2)$$

$$x = 1 \quad \text{solve}$$



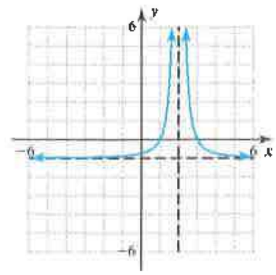
The x-intercept is (1, 0). Knowing the graph is from the reciprocal function family and shifting the asymptotes and intercepts yields the graph shown.

Now try Exercises 11 through 26 ▸

These ideas can be "used in reverse" to determine the equation of a basic rational function from its given graph, as in Example 4.

EXAMPLE 4 ▶ Writing the Equation of a Basic Rational Function, Given Its Graph

Identify the function family for the graph given, then use the graph to write the equation of the function in "shifted form." Assume $|a| = 1$.



Solution ▶ The graph appears to be from the reciprocal square family, and has been shifted 2 units right (the vertical asymptote is at $x = 2$), and 1 unit down (the horizontal asymptote is at $y = -1$). From $y = \frac{1}{x^2}$, we obtain $f(x) = \frac{1}{(x-2)^2} - 1$ as the shifted form.

Now try Exercises 27 through 38 ▶

B. You've just seen how we can use asymptotes and transformations to graph basic rational functions and write the equation for a given graph

Using the definition of negative exponents, the basic reciprocal and reciprocal square functions can be written as $y = x^{-1}$ and $y = x^{-2}$, respectively. In this form, we note that these functions also belong to a family of functions known as the *power functions* **(see Exercise 80).**

C. Graphs of Basic Power Functions

Italian physicist and astronomer Galileo Galilei (1564–1642) made numerous contributions to astronomy, physics, and other fields. But perhaps he is best known for his experiments with gravity, in which he dropped objects of different weights from the Leaning Tower of Pisa. Due in large part to his work, we know that the velocity of an object after it has fallen a certain distance is $v = \sqrt{2gs}$, where g is the acceleration due to gravity (32 ft/sec^2), s is the distance in feet the object has fallen, and v is the velocity of the object in feet per second **(see Exercise 71).** As you will see, this is an example of a formula that uses a power function.

From previous coursework or a review of radicals and rational exponents (Appendix A.6), we know that $\sqrt{x}$ can be written as $x^{\frac{1}{2}}$, and $\sqrt[3]{x}$ as $x^{\frac{1}{3}}$, enabling us to write these functions in *exponential form:* $f(x) = x^{\frac{1}{2}}$ and $g(x) = x^{\frac{1}{3}}$. In this form, we see that these actually belong to a larger family of functions, where x is raised to some power, called the **power functions.**

Power Functions and Root Functions

For any constant real number p and variable x, functions of the form

$$f(x) = x^p$$

are called *power functions* in x. If p is of the form $\frac{1}{n}$ for integers $n \geq 2$, the functions

$$f(x) = x^{\frac{1}{n}} \Leftrightarrow f(x) = \sqrt[n]{x}$$

are called *root functions* in x.

The functions $y = x^2$, $y = x^{\frac{1}{4}}$, $y = x^3$, $y = \sqrt[5]{x}$, and $y = x^{\frac{3}{2}}$ are all power functions, but only $y = x^{\frac{1}{4}}$ and $y = \sqrt[5]{x}$ are also root functions. Initially we will focus on power functions where $p > 0$.

EXAMPLE 5 ▶ **Comparing the Graphs of Power Functions**

Use a graphing calculator to graph the power functions $f(x) = x^{\frac{1}{4}}$, $g(x) = x^{\frac{2}{3}}$, $h(x) = x^1$, $p(x) = x^{\frac{3}{2}}$, and $q(x) = x^2$ in the standard viewing window. Make an observation in QI regarding the effect of the exponent on each function, then discuss what the graphs of $y = x^{\frac{1}{6}}$ and $y = x^{\frac{7}{2}}$ would look like.

Solution ▶ First we enter the functions in sequence as Y_1 through Y_5 on the Y= screen (Figure 2.61). Using ZOOM **6:ZStandard** produces the graphs shown in Figure 2.62. Narrowing the window to focus on QI (Figure 2.63: $x \in [-4, 10]$, $y \in [-4, 10]$), we quickly see that for $x \geq 1$, larger values of p cause the graph of $y = x^p$ to increase at a faster rate, and smaller values at a slower rate. In other words (for $x \geq 1$), since $\frac{1}{6} < \frac{1}{4}$, the graph of $y = x^{\frac{1}{6}}$ would increase slower and appear to be "under" the graph of $Y_1 = X^{\frac{1}{4}}$. Since $\frac{7}{2} > 2$, the graph of $y = x^{\frac{7}{2}}$ would increase faster and appear to be "more narrow" than the graph of $Y_5 = X^2$ (verify this).

Figure 2.61, 2.62

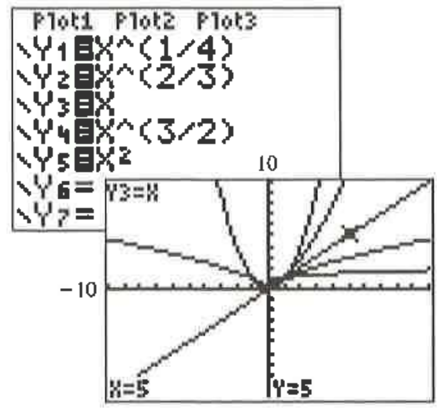
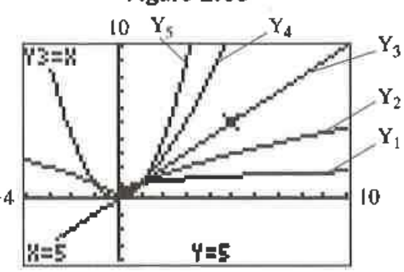


Figure 2.63



Now try Exercises 39 through 48 ▶

The Domain of a Power Function

In addition to the observations made in Example 5, we can make other important notes, particularly regarding the *domains* of power functions. When the exponent on a power function is a rational number $\frac{m}{n} > 0$ in simplest form, it appears the domain is all real numbers if n is odd, as seen in the graphs of $g(x) = x^{\frac{2}{3}}$, $h(x) = x^1 = x^{\frac{1}{1}}$, and $q(x) = x^2 = x^{\frac{2}{1}}$. If n is an even number, the domain is all nonnegative real numbers as seen in the graphs of $f(x) = x^{\frac{1}{4}}$ and $p(x) = x^{\frac{3}{2}}$. Further exploration will show that if p is irrational, as in $y = x^{\pi}$, the domain is also all nonnegative real numbers and we have the following:

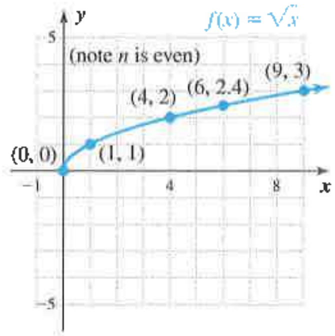
The Domain of a Power Function

Given a power function $f(x) = x^p$ with $p > 0$.

1. If $p = \frac{m}{n}$ is a rational number in simplest form,
   - a. the domain of f is all real numbers if n is odd: $x \in (-\infty, \infty)$,
   - b. the domain of f is all nonnegative real numbers if n is even: $x \in [0, \infty)$.
2. If p is an irrational number, the domain of f is all nonnegative real numbers: $x \in [0, \infty)$.

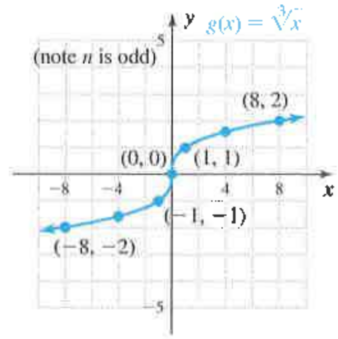
Further confirmation of statement 1 can be found by recalling the graphs of $y = \sqrt{x} = x^{\frac{1}{2}}$ and $y = \sqrt[3]{x} = x^{\frac{1}{3}}$ from Section 2.2 (Figures 2.64 and 2.65).

Figure 2.64



Domain: $x \in [0, \infty)$
Range: $y \in [0, \infty)$

Figure 2.65



Domain: $x \in [-\infty, \infty)$
Range: $y \in [-\infty, \infty)$

EXAMPLE 6 ▶ Determining the Domains of Power Functions

State the domain of the following power functions, and identity whether each is also a root function.

a. $f(x) = x^{\frac{4}{3}}$ **b.** $g(x) = x^{\frac{1}{10}}$ **c.** $h(x) = \sqrt[8]{x}$ **d.** $q(x) = x^{\frac{2}{3}}$ **e.** $r(x) = x^{\sqrt{5}}$

Solution ▶

a. Since n is odd, the domain of f is all real numbers; f is not a root function.

b. Since n is even, the domain of g is $x \in [0, \infty)$; g is a root function.

c. In exponential form $h(x) = x^{\frac{1}{8}}$. Since n is even, the domain of h is $x \in [0, \infty)$; h is a root function.

d. Since n is odd, the domain of q is all real numbers; q is not a root function

e. Since p is irrational, the domain of r is $x \in [0, \infty)$; r is not a root function

Now try Exercises 49 through 58 ▶

Transformations of Power and Root Functions

As we saw in Section 2.2 (Toolbox Functions and Transformations), the graphs of the root functions $y = \sqrt{x}$ and $y = \sqrt[3]{x}$ can be transformed using shifts, stretches, reflections, and so on. In Example 8(b) (Section 2.2) we noted the graph of $h(x) = 2\sqrt[3]{x - 2} - 1$ was the graph of $y = \sqrt[3]{x}$ shifted 2 units right, stretched by a factor of 2, and shifted 1 unit down. Graphs of other power functions can be transformed in exactly the same way.

EXAMPLE 7 ▶ **Graphing Transformations of Power Functions**

Based on our previous observations,

a. Determine the domain of $f(x) = x^{\frac{2}{3}}$ and $g(x) = x^{\frac{3}{2}}$, then verify by graphing them on a graphing calculator.

b. Next, discuss what the graphs of $F(x) = (x - 2)^{\frac{2}{3}} - 3$ and $G(x) = -x^{\frac{3}{2}} + 2$ will look like, then graph each on a graphing calculator to verify.

Solution ▶ **a.** Both f and g are power functions of the form $y = x^{\frac{m}{n}}$. For f, n is odd so its domain is all real numbers. For g, n is even and the domain is $x \in [0, \infty)$. Their graphs support this conclusion (Figures 2.66 and 2.67).

Figure 2.66

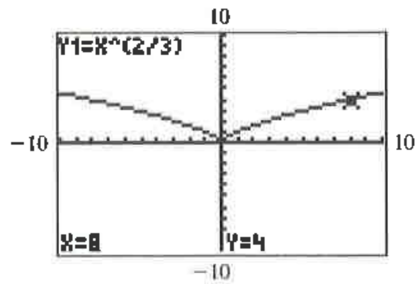
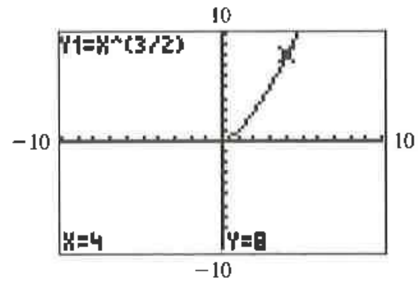


Figure 2.67



b. The graph of F will be the same as the graph of f, but shifted two units right and three units down, moving the vertex to $(2, -3)$. The graph of G will be the same as the graph of g, but reflected across the x-axis, and shifted 2 units up (Figures 2.68 and 2.69).

Figure 2.68

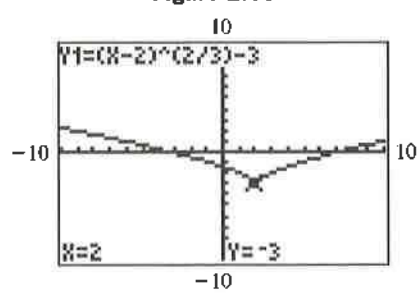
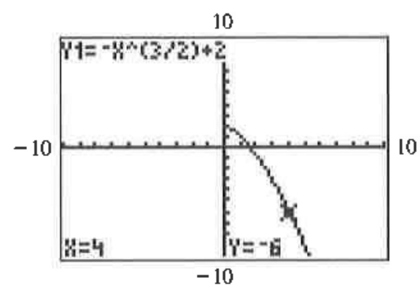


Figure 2.69



C. You've just seen how we can graph basic power functions and state their domains

Now try Exercises 59 through 62 ▶

D. Applications of Rational and Power Functions

These new functions have a variety of interesting and significant applications in the real world. Examples 8 through 10 provide a small sample, and there are a number of additional applications in the Exercise Set. In many applications, the coefficients may be rather large, and the axes should be scaled accordingly.

EXAMPLE 8 ▶ **Modeling the Cost to Remove Waste**

For a large urban-centered county, the cost to remove chemical waste and other pollutants from a local river is given by the function $C(p) = \frac{-18{,}000}{p - 100} - 180$, where $C(p)$ represents the cost (in thousands of dollars) to remove p percent of the pollutants.

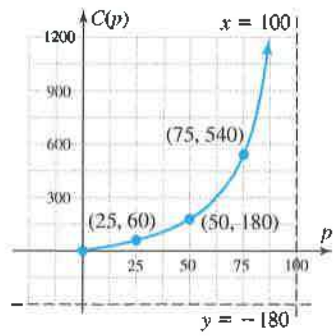
a. Find the cost to remove 25%, 50%, and 75% of the pollutants and comment on the results.

b. Graph the function using an appropriate scale.

c. Use mathematical notation to state what happens as the county attempts to remove 100% of the pollutants.

Solution ▶ **a.** We evaluate the function as indicated, finding that $C(25) = 60$, $C(50) = 180$, and $C(75) = 540$. The cost is escalating rapidly. The change from 25% to 50% brought a \$120,000 increase, but the change from 50% to 75% brought *a \$360,000 increase!*

b. From the context, we need only graph the portion from $0 \leq p < 100$. For the C-intercept we substitute $p = 0$ and find $C(0) = 0$, which seems reasonable as 0% would be removed if \$0 were spent. We also note there must be a vertical asymptote at $x = 100$, since this x-value causes a denominator of 0. Using this information and the points from part (a) produces the graph shown.



c. As the percentage of pollutants removed approaches 100%, the cost of the cleanup skyrockets. Using notation: as $p \to 100^-$, $C \to \infty$.

Now try Exercises 65 through 70 ▶

While not obvious at first, the function $C(p)$ in Example 8 is from the family of reciprocal functions $y = \frac{1}{x}$. A closer inspection shows it has the form $y = \frac{-a}{x - h} - k \to \frac{-18{,}000}{x - 100} - 180$, showing the graph of $y = \frac{1}{x}$ is shifted right 100 units, reflected across the x-axis, stretched by a factor of 18,000 and shifted 180 units down (the horizontal asymptote is $y = -180$). As sometimes occurs in real-world applications, portions of the graph were ignored due to the context. To see the full graph, we reason that the second branch occurs on the opposite side of the vertical and horizontal asymptotes, and set the window as shown in Figure 2.70. After entering $C(p)$ as Y_1 on the Y= screen and pressing GRAPH, the full graph appears as shown in Figure 2.71 (for effect, the vertical and horizontal asymptotes were drawn separately using the 2nd PRGM **(DRAW)** options).

Figure 2.70

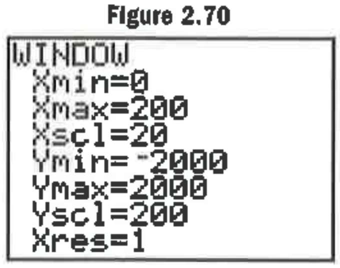
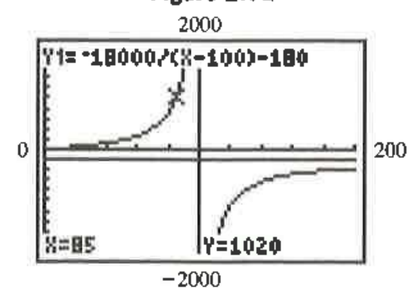


Figure 2.71



Next, we'll use a root function to model the distance to the horizon from a given height.

EXAMPLE 9 ▶ The Distance to the Horizon

On a clear day, the distance a person can see from a certain height (the distance to the horizon) is closely approximated by the root function $d(h) = 3.57\sqrt{h}$, where $d(h)$ represents the viewing distance (in kilometers) from a height of h meters above sea level.

a. To the nearest kilometer, how far can a person see when standing on the observation level of the John Hancock building in Chicago, Illinois, about 335 m high?

b. To the nearest meter, how high is the observer's eyes, if the viewing distance is 130 km?

Solution ▶ **a.** Substituting 335 for h we have

$$d(h) = 3.57\sqrt{h} \quad \text{original function}$$
$$d(335) = 3.57\sqrt{335} \quad \text{substitute 335 for } h$$
$$\approx 65.34 \quad \text{result}$$

On a clear day, a person can see about 65 kilometers.

b. We substitute 130 for $d(h)$:

$$d(h) = 3.57\sqrt{h} \quad \text{original function}$$
$$130 = 3.57\sqrt{h} \quad \text{substitute 130 for } d(h)$$
$$36.415 \approx \sqrt{h} \quad \text{divide by 3.57}$$
$$1326.052 \approx h \quad \text{square both sides}$$

If the distance to the horizon is 130 km, the observer's eyes are at a height of approximately 1326 m. Check the answer to part (b) by solving graphically.

Now try Exercises 71 through 74 ▶

One area where power functions and modeling with regression are used extensively is **allometric studies.** This area of inquiry studies the relative growth of a part of an animal in relation to the growth of the whole, like the wingspan of a bird compared to its weight, or the daily food intake of a mammal or bird compared to its size.

EXAMPLE 10 ▶ Modeling the Food Requirements of Certain Bird Species

To study the relationship between the weight of a nonpasserine bird and its daily food intake, the data shown in the table was collected (nonpasserine: nonsinging, nonperching birds).

Bird	Average weight (g)	Daily food intake (g)
Common pigeon	350	25
Ring-necked duck	725	50
Ring-necked pheasant	1400	70
Canadian goose	4525	165
White swan	9075	240

a. On a graphing calculator, enter the data in L1 and L2, then set an appropriate window to view a scatterplot of the data. Does a power regression STAT **CALC, A:PwrReg** seem appropriate?

b. Use a graphing calculator to find an equation model using a power regression on the data, and enter the equation in Y_1 (round values to three decimal places).

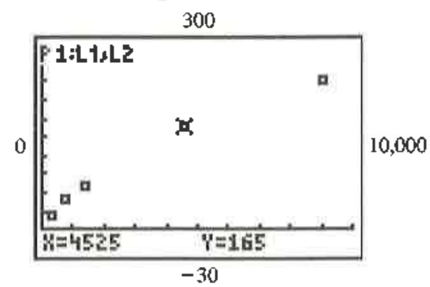
c. Use the equation to estimate the daily food intake required by a barn owl (470 g), and a gray-headed albatross (6800 g).

d. Use the intersection of graphs method to find the weight of a Great-Spotted Kiwi, given the daily food requirement is 130 g.

Solution ▶

a. After entering the weights in L1 and food intake in L2, we set a window that will comfortably fit the data. Using $x \in [0, 10{,}000]$ and $y \in [-30, 300]$ produces the scatterplot shown (Figure 2.72). The data does not appear linear, and based on our work in Example 5, a power function seems appropriate.

Figure 2.72



b. To access the power regression option, use STAT ▶ **(CALC) A:PwrReg.** To three decimal places the equation for Y_1 would be $0.493X^{0.685}$ (Figure 2.73).

Figure 2.73

c. For the barn owl, $x = 470$ and we find the estimated food requirement is about 33.4 g per day (Figure 2.74). For the gray-headed albatross $x = 6800$ and the model estimates about 208.0 g of food daily is required.

d. Here we're given the food intake of the Great-Spotted Kiwi (the output value), and want to know what input value (weight) was used. Entering $Y_2 = 130$, we'll attempt to find where the graphs of Y_1 and Y_2 intersect (it will help to deactivate **Plot1** on the Y= screen, so that only the graphs of Y_1 and Y_2 appear). Using 2nd TRACE **(CALC) 5:Intersect** shows the graphs intersect at about (3423.3, 130) (Figure 2.75), indicating the average weight of a Great-Spotted Kiwi is near 3423.3 g (about 7.5 lb).

Figure 2.74

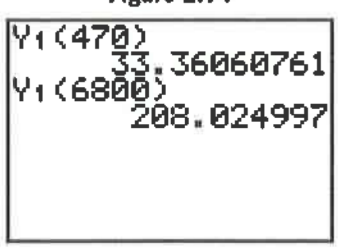
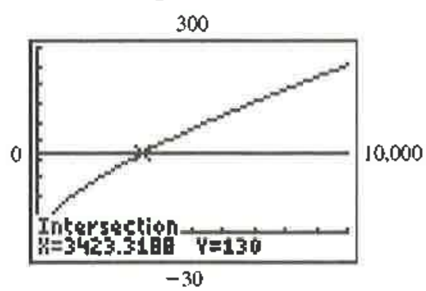


Figure 2.75



D. You've just seen how we can solve applications involving basic rational and power functions

Now try Exercises 75 through 78 ▶

2.4 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. Write the following in notational form. *As x becomes an infinitely large negative number, y approaches 2.* ________

2. For any constant k, the notation "as $|x| \to +\infty, y \to k$" is an indication of a ________ asymptote, while "$x \to k, |y| \to +\infty$" indicates a ________ asymptote.

3. Given the function $g(x) = \dfrac{1}{(x-3)^2} + 2$, a ________ asymptote occurs at $x = 3$ and a horizontal asymptote at ________.

4. The graph of $Y_1 = \dfrac{1}{x}$ has branches in Quadrants I and III. The graph of $Y_2 = -\dfrac{1}{x}$ has branches in Quadrants ______ and ______.

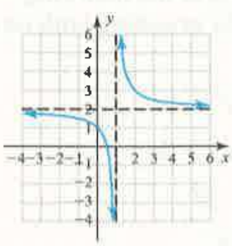
5. Discuss/Explain how and why the range of the reciprocal function differs from the range of the reciprocal quadratic function. In the reciprocal quadratic function, all range values are positive.

6. If the graphs of $Y_1 = \dfrac{1}{x}$ and $Y_2 = \dfrac{1}{x^2}$ were drawn on the same grid, where would they intersect? In what interval(s) is $Y_1 > Y_2$?

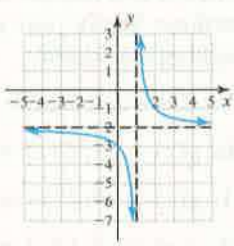
▶ DEVELOPING YOUR SKILLS

For each graph given, (a) use mathematical notation to describe the end-behavior of each graph and (b) describe what happens as *x* approaches 1.

7. $V(x) = \dfrac{1}{x-1} + 2$

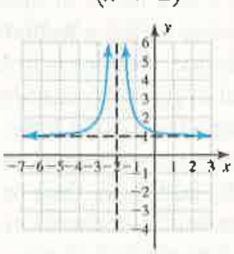


8. $v(x) = \dfrac{1}{x-1} - 2$

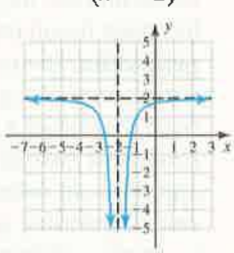


For each graph given, (a) use mathematical notation to describe the end-behavior of each graph, (b) name the horizontal asymptote, and (c) describe what happens as *x* approaches −2.

9. $Q(x) = \dfrac{1}{(x+2)^2} + 1$



10. $q(x) = \dfrac{-1}{(x+2)^2} + 2$



Sketch the graph of each function using transformations of the parent function (not by plotting points). Clearly state the transformations used, and label the horizontal and vertical asymptotes as well as the *x*- and *y*-intercepts (if they exist). Also state the domain and range of each function.

11. $f(x) = \dfrac{1}{x} - 1$

12. $g(x) = \dfrac{1}{x} + 2$

13. $h(x) = \dfrac{1}{x+2}$

14. $f(x) = \dfrac{1}{x-3}$

15. $g(x) = \dfrac{-1}{x-2}$

16. $h(x) = \dfrac{-1}{x} - 2$

17. $f(x) = \dfrac{1}{x+2} - 1$

18. $g(x) = \dfrac{1}{x-3} + 2$

19. $h(x) = \dfrac{1}{(x-1)^2}$

20. $f(x) = \dfrac{1}{(x+5)^2}$

21. $g(x) = \dfrac{-1}{(x+2)^2}$

22. $h(x) = \dfrac{-1}{x^2} - 2$

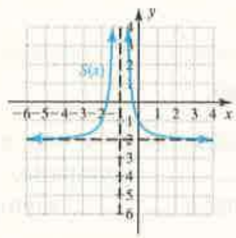
23. $f(x) = \dfrac{1}{x^2} - 2$

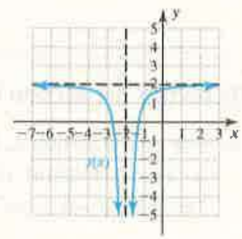
24. $g(x) = \dfrac{1}{x^2} + 3$

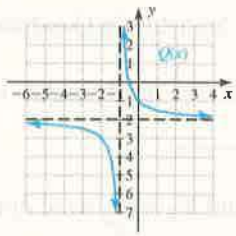
25. $h(x) = 1 + \dfrac{1}{(x+2)^2}$

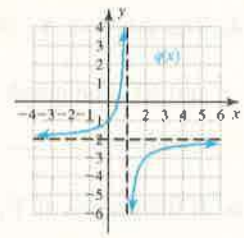
26. $g(x) = -2 + \dfrac{1}{(x-1)^2}$

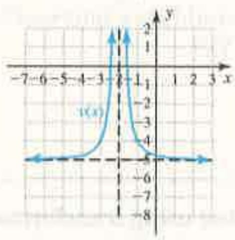
Identify the parent function for each graph given, then use the graph to construct the equation of the function in shifted form. Assume $|a| = 1$.

27. 

28. 

29. 

30. 

31. 

32. 

Use the graph shown to complete each statement using the direction/approach notation.

Exercises 33 through 38

33. As $x \to -\infty$, y ______.

34. As $x \to \infty$, y ______.

35. As $x \to -1^+$, y ______.

36. As $x \to -1^-$, y ______.

37. The line $x = -1$ is a vertical asymptote, since: as $x \to$ ______, $y \to$ ______.

38. The line $y = -2$ is a horizontal asymptote, since: as $x \to$ ______, $y \to$ ______.

For each pair of functions given, state which function increases faster for $x > 1$, then use the INTERSECT command of a graphing calculator to find where (a) $f(x) = g(x)$, (b) $f(x) > g(x)$, and (c) $f(x) < g(x)$.

39. $f(x) = x^2, g(x) = x^3$
40. $f(x) = x^4, g(x) = x^5$
41. $f(x) = x^4, g(x) = x^2$
42. $f(x) = x^3, g(x) = x^5$
43. $f(x) = x^{\frac{2}{3}}, g(x) = x^{\frac{4}{3}}$
44. $f(x) = x^{\frac{7}{4}}, g(x) = x^{\frac{3}{2}}$
45. $f(x) = \sqrt[6]{x}, g(x) = \sqrt[3]{x}$
46. $f(x) = \sqrt[5]{x}, g(x) = \sqrt[4]{x}$
47. $f(x) = \sqrt[3]{x^2}, g(x) = x^{\frac{5}{4}}$
48. $f(x) = x^{\frac{3}{2}}, g(x) = \sqrt[4]{x^3}$

State the domain of the following functions.

49. $f(x) = x^{\frac{7}{8}}$
50. $g(x) = x^{\frac{6}{7}}$
51. $h(x) = x^{\frac{6}{5}}$
52. $q(x) = x^{\frac{5}{6}}$
53. $r(x) = \sqrt[7]{x}$
54. $s(x) = x^{\frac{1}{5}}$

Using the functions from Exercises 49–54, identify which of the following are defined and which are not. Do not use a calculator or evaluate.

55. a. $f(-2)$ b. $f(2)$ c. $g(-2)$ d. $g(2)$

56. a. $h(0.3)$ b. $h(-0.3)$ c. $q(0.3)$ d. $q(-0.3)$

57. a. $h(-1.2)$ b. $r(-7)$ c. $s(-\pi)$ d. $s(0)$

58. a. $f\left(-\frac{7}{8}\right)$ b. $g\left(-\frac{8}{7}\right)$ c. $q(-1.9)$ d. $q(0)$

Compare and discuss the graphs of the following functions. Verify your answer by graphing both on a graphing calculator.

59. $f(x) = x^{\frac{7}{8}}$; $F(x) = (x + 1)^{\frac{7}{8}} - 2$

60. $g(x) = x^{\frac{8}{7}}$; $G(x) = (x - 3)^{\frac{8}{7}} + 2$

61. $p(x) = x^{\frac{6}{5}}$; $P(x) = -(x - 2)^{\frac{6}{5}}$

62. $q(x) = x^{\frac{5}{6}}$; $Q(x) = 2x^{\frac{5}{6}} - 5$

▶ WORKING WITH FORMULAS

63. **Gravitational attraction: $F = \dfrac{km_1m_2}{d^2}$**

The gravitational force F between two objects with masses m_1 and m_2 depends on the distance d between them and some constant k. (a) If the masses of the two objects are constant while the distance between them gets larger and larger, what happens to F? (b) Let m_1 and m_2 equal 1 mass unit with $k = 1$ as well, and investigate using a table of values. What family does this function belong to? (c) Solve for m_2 in terms of k, m_1, d and F.

64. **Velocity of a bullet: $v = \dfrac{m + M}{m}\sqrt{2gh}$**

For centuries, the velocity v of a bullet of mass m has been found using a device called a **ballistic pendulum.** In one such device, a bullet is fired into a stationary block of wood of mass M, suspended from the end of a pendulum. The height h the pendulum swings after impact is measured, and the approximate velocity of the bullet can then be calculated using $g = 9.8$ m/sec^2 (acceleration due to gravity). When a .22-caliber bullet of mass 2.6 g is fired into a wood block of mass 400 g, their combined mass swings to a height of 0.23 m. To the nearest meter per second, find the velocity of the bullet the moment it struck the wood.

▶ APPLICATIONS

65. Deer and predators: By banding deer over a period of 10 yr, a capture-and-release project determines the number of deer per square mile in the Mark Twain National Forest can be modeled by the function $D(p) = \frac{75}{p}$, where p is the number of predators present and D is the number of deer. Use this model to answer the following.

a. As the number of predators increases, what will happen to the population of deer? Evaluate the function at $D(1)$, $D(3)$, and $D(5)$ to verify.

b. What happens to the deer population if the number of predators becomes very large?

c. Graph the function using an appropriate scale. Judging from the graph, use mathematical notation to describe what happens to the deer population if the number of predators becomes very small (less than 1 per square mile).

66. Balance of nature: A marine biology research group finds that in a certain reef area, the number of fish present depends on the number of sharks in the area. The relationship can be modeled by the function $F(s) = \frac{20{,}000}{s}$, where $F(s)$ is the fish population when s sharks are present.

a. As the number of sharks increases, what will happen to the population of fish? Evaluate the function at $F(10)$, $F(50)$, and $F(200)$ to verify.

b. What happens to the fish population if the number of sharks becomes very large?

c. Graph the function using an appropriate scale. Judging from the graph, use mathematical notation to describe what happens to the fish population if the number of sharks becomes very small.

67. Intensity of light: The intensity I of a light source depends on the distance of the observer from the source. If the intensity is 100 W/m^2 at a distance of 5 m, the relationship can be modeled by the function $I(d) = \frac{2500}{d^2}$. Use the model to answer the following.

a. As the distance from the lightbulb increases, what happens to the intensity of the light? Evaluate the function at $I(5)$, $I(10)$, and $I(15)$ to verify.

b. If the intensity is increasing, is the observer moving away or toward the light source?

c. Graph the function using an appropriate scale. Judging from the graph, use mathematical notation to describe what happens to the intensity if the distance from the lightbulb becomes very small.

68. Electrical resistance: The resistance R (in ohms) to the flow of electricity is related to the length of the wire and its gauge (diameter in fractions of an inch). For a certain wire with fixed length, this relationship can be modeled by the function $R(d) = \frac{0.2}{d^2}$, where $R(d)$ represents the resistance in a wire with diameter d.

a. As the diameter of the wire increases, what happens to the resistance? Evaluate the function at $R(0.05)$, $R(0.25)$, and $R(0.5)$ to verify.

b. If the resistance is increasing, is the diameter of the wire getting larger or smaller?

c. Graph the function using an appropriate scale. Judging from the graph, use mathematical notation to describe what happens to the resistance in the wire as the diameter gets larger and larger.

69. Pollutant removal: For a certain coal-burning power plant, the cost to remove pollutants from plant emissions can be modeled by $C(p) = \frac{-8000}{p - 100} - 80$, where $C(p)$ represents the cost (in thousands of dollars) to remove p percent of the pollutants. (a) Find the cost to remove 20%, 50%, and 80% of the pollutants, then comment on the results; (b) graph the function using an appropriate scale; and (c) use mathematical notation to state what happens if the power company attempts to remove 100% of the pollutants.

70. City-wide recycling: A large city has initiated a new recycling effort, and wants to distribute recycling bins for use in separating various recyclable materials. City planners anticipate the cost of the program can be modeled by the function $C(p) = \frac{-22{,}000}{p - 100} - 220$, where $C(p)$ represents the cost (in \$10,000) to distribute the bins to p percent of the population. (a) Find the cost to distribute bins to 25%, 50%, and 75% of the population, then comment on the results; (b) graph the function using an appropriate scale; and (c) use mathematical notation to state what happens if the city attempts to give recycling bins to 100% of the population.

71. **Hot air ballooning:** If air resistance is neglected, the velocity (in ft/s) of a falling object can be closely approximated by the function $V(s) = 8\sqrt{s}$, where s is the distance the object has fallen (in feet). A balloonist suddenly finds it necessary to release some ballast in order to quickly gain altitude. (a) If she were flying at an altitude of 1000 ft, with what velocity will the ballast strike the ground? (b) If the ballast strikes the ground with a velocity of 225 ft/sec, what was the altitude of the balloon?

72. **River velocities:** The ability of a river or stream to move sand, dirt, or other particles depends on the size of the particle and the velocity of the river. This relationship can be used to approximate the velocity (in mph) of the river using the function $V(d) = 1.77\sqrt{d}$, where d is the diameter (in inches) of the particle being moved. (a) If a creek can move a particle of diameter 0.095 in., how fast is it moving? (b) What is the largest particle that can be moved by a stream flowing 1.1 mph?

73. **Shoe sizes:** Although there may be some notable exceptions, the size of shoe worn by the average man is related to his height. This relationship is modeled by the function $S(h) = 0.75h^{\frac{3}{2}}$, where h is the person's height in feet and S is the U.S. shoe size. (a) Approximate Denzel Washington's shoe size given he is 6 ft, 0 in. tall. (b) Approximate Dustin Hoffman's height given his shoe size is 9.5.

74. **Whale weight:** For a certain species of whale, the relationship between the length of the whale and the weight of the whale can be modeled by the function $W(l) = 0.03l^{\frac{27}{11}}$, where l is the length of the whale in meters and W is the weight of the whale in metric tons (1 metric ton ≈ 2205 pounds). (a) Estimate the weight of a newborn calf that is 6 m long. (b) At 81 metric tons, how long is an average adult?

75. **Gestation periods:** The data shown in the table can be used to study the relationship between the weight of mammal and its length of pregnancy. Use a graphing calculator to (a) graph a scatterplot of the data and (b) find an equation model using a power regression (round to three decimal places). Use the equation to estimate (c) the length of pregnancy of a racoon (15.5 kg) and (d) the weight of a fox, given the length of pregnancy is 52 days.

Mammal	Average Weight (kg)	Gestation (days)
Rat	0.4	24
Rabbit	3.5	50
Armadillo	6.0	51
Coyote	13.1	62
Dog	24.0	64

76. **Bird wingspans:** The data in the table explores the relationship between a bird's weight and its wingspan. Use a graphing calculator to (a) graph a scatterplot of the data and (b) find an equation model using a power regression (round to three decimal places). Use the equation to estimate (c) the wingspan of a Bald Eagle (16 lb) and (d) the weight of a Bobwhite Quail with a wingspan of 0.9 ft.

Bird	Weight (lb)	Wingspan (ft)
Golden Eagle	10.5	6.5
Horned Owl	3.1	2.6
Peregrine Falcon	3.3	4.0
Whooping Crane	17.0	7.5
Raven	1.5	2.0

77. **Species-area relationship:** To study the relationship between the number of species of birds on islands in the Caribbean, the data shown in the table was collected. Use a graphing calculator to (a) graph a scatterplot of the data and (b) find an equation model using a power regression (round to three decimal places). Use the equation to estimate (c) the number of species of birds on Andros (2300 mi^2) and (d) the area of Cuba, given there are 98 such species.

Island	Area (mi^2)	Species
Great Inagua	600	16
Trinidad	2000	41
Puerto Rico	3400	47
Jamaica	4500	38
Hispaniola	30,000	82

78. **Planetary orbits:** The table shown gives the time required for the first five planets to make one complete revolution around the Sun (in years), along with the average orbital radius of the planet in astronomical units (1 AU = 92.96 million miles). Use a graphing calculator to (a) graph a scatterplot of the data and (b) find an equation model using a power regression (round to four decimal places). Use the equation to estimate (c) the average orbital radius of Saturn, given it orbits the Sun every 29.46 yr, and (d) estimate how many years it takes Uranus to orbit the Sun, given it has an average orbital radius of 19.2 AU.

Planet	Years	Radius
Mercury	0.24	0.39
Venus	0.62	0.72
Earth	1.00	1.00
Mars	1.88	1.52
Jupiter	11.86	5.20

▶ EXTENDING THE CONCEPT

79. Consider the graph of $f(x) = \frac{1}{x}$ once again, and the x by $f(x)$ rectangles mentioned in the Worthy of Note on page 149. Calculate the area of each rectangle formed for $x \in \{1, 2, 3, 4, 5, 6\}$. What do you notice? Repeat the exercise for $g(x) = \frac{1}{x^2}$ and the x by $g(x)$ rectangles. Can you detect the pattern formed here?

80. All of the power functions presented in this section had positive exponents, but the definition of these types of functions does allow for negative exponents as well. In addition to the reciprocal and reciprocal square functions ($y = x^{-1}$ and $y = x^{-2}$), these types of power functions have significant applications. For example, the temperature of ocean water depends on several factors, including salinity, latitude, depth, and density. However, between depths of 125 m and 2000 m, ocean temperatures are relatively predictable, as indicated by the data shown for tropical oceans in the table. Use a graphing calculator to find the power regression model and use it to estimate the water temperature at a depth of 2850 m.

Depth (meters)	Temp (°C)
125	13.0
250	9.0
500	6.0
750	5.0
1000	4.4
1250	3.8
1500	3.1
1750	2.8
2000	2.5

▶ MAINTAINING YOUR SKILLS

81. (1.4) Solve the equation for y, then sketch its graph using the slope/intercept method: $2x + 3y = 15$.

82. (1.3) Using a scale from 1 (lousy) to 10 (great), Charlie gave the following ratings: {(The Beatles, 9.5), (The Stones, 9.6), (The Who, 9.5), (Queen, 9.2), (The Monkees, 6.1), (CCR, 9.5), (Aerosmith, 9.2), (Lynyrd Skynyrd, 9.0), (The Eagles, 9.3), (Led Zeppelin, 9.4), (The Stones, 9.8)}. Is the relation (group, rating) as given, also a function? State why or why not.

83. (1.5) Solve for c: $E = mc^2$.

84. (2.3) Use a graphing calculator to solve $|x - 2| + 1 \geq -2|x + 1| + 3$.

2.5 Piecewise-Defined Functions

LEARNING OBJECTIVES

In Section 2.5 you will see how we can:

- ☐ **A.** State the equation, domain, and range of a piecewise-defined function from its graph
- ☐ **B.** Graph functions that are piecewise-defined
- ☐ **C.** Solve applications involving piecewise-defined functions

Most of the functions we've studied thus far have been smooth and continuous. Although "smooth" and "continuous" are defined more formally in advanced courses, for our purposes *smooth* simply means the graph has no sharp turns or jagged edges, and *continuous* means you can draw the entire graph without lifting your pencil. In this section, we study a special class of functions, called **piecewise-defined functions**, whose graphs may be various combinations of smooth/not smooth and continuous/not continuous. The absolute value function is one example (**see Exercise 31**). Such functions have a tremendous number of applications in the real world.

A. The Domain of a Piecewise-Defined Function

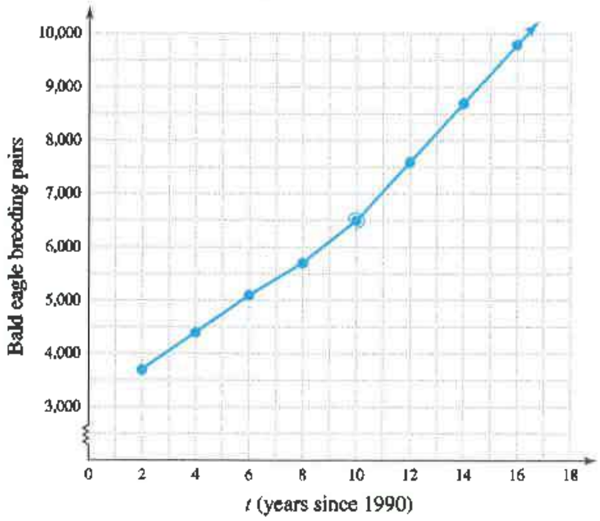
For the years 1990 to 2000, the American bald eagle remained on the nation's endangered species list, although the number of breeding pairs was growing slowly. After 2000, the population of eagles grew at a much faster rate, and they were removed from the list soon afterward. From Table 2.3 and plotted points modeling this growth (see Figure 2.76), we observe that a linear model would fit the period from 1992 to 2000 very well, but a line with greater slope would be needed for the years 2000 to 2006 and (perhaps) beyond.

Table 2.3

Year (1990 → 0)	Bald Eagle Breeding Pairs	Year (1990 → 0)	Bald Eagle Breeding Pairs
2	3700	10	6500
4	4400	12	7600
6	5100	14	8700
8	5700	16	9800

Source: www.fws.gov/midwest/eagle/population

Figure 2.76



WORTHY OF NOTE

For the years 1992 to 2000, we can estimate the growth in breeding pairs $\frac{\Delta \text{pairs}}{\Delta \text{time}}$ using the points (2, 3700) and (10, 6500) in the slope formula. The result is $\frac{350}{1}$, or 350 pairs per year. For 2000 to 2006, using (10, 6500) and (16, 9800) shows the rate of growth is significantly larger: $\frac{\Delta \text{pairs}}{\Delta \text{years}} = \frac{550}{1}$ or 550 pairs per year.

The combination of these two lines would be a single function that modeled the population of breeding pairs from 1990 to 2006, but it would be *defined in two pieces.* This is an example of a **piecewise-defined function.**

The notation for these functions is a large "left brace" indicating the equations it groups are part of a single function. Using selected data points and techniques from Section 1.4, we find equations that could represent each piece are $p(t) = 350t + 3000$ for $0 \le t \le 10$ and $p(t) = 550t + 1000$ for $t > 10$, where $p(t)$ is the number of breeding pairs in year t. The complete function is then written:

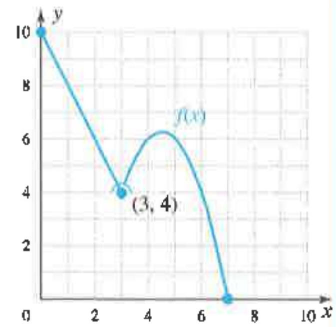
function name — function pieces — domain of each piece

$$p(t) = \begin{cases} 350t + 3000, & 2 \le t \le 10 \\ 550t + 1000, & t > 10 \end{cases}$$

In Figure 2.76, note that we indicated the exclusion of $t = 10$ from the second piece of the function using an open half-circle.

EXAMPLE 1 ▶ Writing the Equation and Domain of a Piecewise-Defined Function

The linear piece of the function shown has an equation of $y = -2x + 10$. The equation of the quadratic piece is $y = -x^2 + 9x - 14$.



a. Use the correct notation to write them as a single piecewise-defined function and state the domain of each piece by inspecting the graph.

b. State the range of the function.

Solution ▶ **a.** From the graph we note the linear portion is defined between 0 and 3, with these endpoints included as indicated by the closed dots. The domain here is $0 \le x \le 3$. The quadratic portion begins at $x = 3$ *but does not include 3*, as indicated by the half-circle notation. The equation is

function name function pieces domain

$$f(x) = \begin{cases} -2x + 10, & 0 \le x \le 3 \\ -x^2 + 9x - 14, & 3 < x \le 7 \end{cases}$$

b. The largest y-value is 10 and the smallest is zero. The range is $y \in [0, 10]$.

A. You've just seen how we can state the equation, domain, and range of a piecewise-defined function from its graph

Now try Exercises 7 and 8 ▶

Piecewise-defined functions can be composed of more than two pieces, and can involve functions of many kinds.

B. Graphing Piecewise-Defined Functions

As with other functions, piecewise-defined functions can be graphed by simply plotting points. Careful attention must be paid to the domain of each piece, both to evaluate the function correctly and to consider the inclusion/exclusion of endpoints. In addition, try to keep the transformations of a basic function in mind, as this will often help graph the function more efficiently.

EXAMPLE 2 ▶ Graphing a Piecewise-Defined Function

Evaluate the piecewise-defined function by noting the effective domain of each piece, then graph by plotting these points and using your knowledge of basic functions.

$$h(x) = \begin{cases} -x - 2, & -5 \le x < -1 \\ 2\sqrt{x+1} - 1, & x \ge -1 \end{cases}$$

Solution ▶ The first piece of h is a line with negative slope, while the second is a transformed square root function. Using the endpoints of each domain specified and a few additional points, we obtain the following:

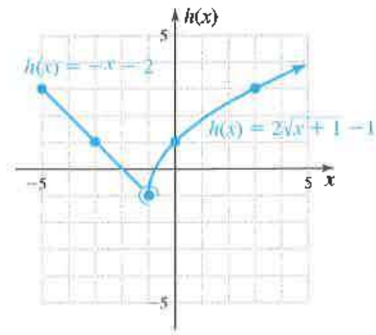
For $h(x) = -x - 2$, $-5 \le x < -1$,

x	$h(x)$
−5	3
−3	1
−1	(−1)

For $h(x) = 2\sqrt{x+1} - 1$, $x \ge -1$,

x	$h(x)$
−1	−1
0	1
3	3

After plotting the points from the first piece, we connect them with a line segment noting the left endpoint is included, while the right endpoint is not (indicated using a semicircle around the point). Then we plot the points from the second piece and draw a square root graph, noting the left endpoint here *is* included, and the graph rises to the right. From the graph we note the complete domain of h is $x \in [-5, \infty)$, and the range is $y \in [-1, \infty)$.



Now try Exercises 9 through 12 ►

Most graphing calculators are able to graph piecewise-defined functions. Consider Example 3.

EXAMPLE 3 ► **Graphing a Piecewise-Defined Function Using Technology**

Graph the function $f(x) = \begin{cases} x + 5, & -5 \le x < 2 \\ (x-4)^2 + 3, & x \ge 2 \end{cases}$ on a graphing calculator and evaluate $f(2)$.

Solution ► Both "pieces" are well known—the first is a line with slope $m = 1$ and y-intercept (0, 5). The second is a parabola that opens upward, shifted 4 units to the right and 3 units up. If we attempt to graph $f(x)$ using $Y_1 = X + 5$ and $Y_2 = (X - 4)^2 + 3$ as they stand, the resulting graph may be difficult to analyze because the pieces overlap and intersect (Figure 2.77). To graph the functions we must indicate the domain for each piece, separated by a slash and enclosed in parentheses. For instance, for the first piece we enter $Y_1 = X + 5/(X \ge -5$ and $X < 2)$, and for the second, $Y_2 = (X - 4)^2 + 3/(X \ge 2)$ (Figure 2.78). The slash looks like (is) the division symbol, but in this context, the calculator interprets it as a means of separating the function from the domain. The inequality symbols are accessed using the 2nd MATH **(TEST)** keys. As shown for Y_1, compound inequalities must be entered in two parts, using the logical connector "and": 2nd MATH ▷ **(LOGIC) 1:and.** The graph is shown in Figure 2.79, where we see the function is linear for $x \in [-5, 2)$ and quadratic for $x \in [2, \infty)$. Using the 2nd GRAPH **(TABLE)** feature reveals the calculator will give an **ERR:** (ERROR) message for inputs outside the domains of Y_1 and Y_2, and we see that f is defined for $x = 2$ only for Y_2: $f(2) = 7$ (Figure 2.80).

Figure 2.77

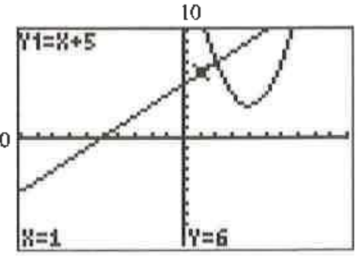


Figure 2.78

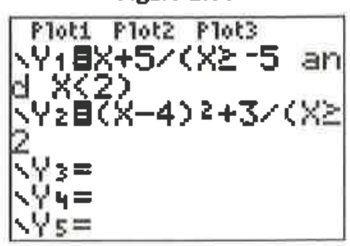


Figure 2.79

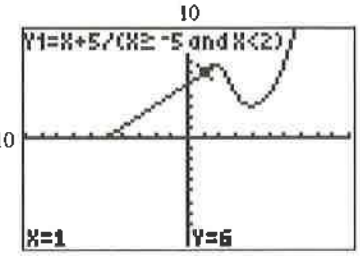
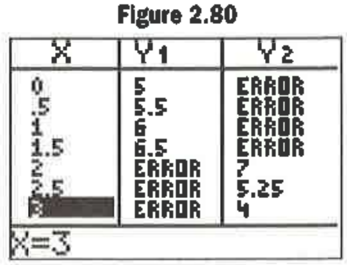


Figure 2.80



X	Y1	Y2
0	5	ERROR
.5	5.5	ERROR
1	6	ERROR
1.5	6.5	ERROR
2	ERROR	7
2.5	ERROR	5.25
3	ERROR	4

X=3

Now try Exercises 13 and 14 ►

As an alternative to plotting points, we can graph each piece of the function using transformations of a basic graph, then erase those parts that are outside of the corresponding domain. Repeat this procedure for each piece of the function. One interesting and highly instructive aspect of these functions is the opportunity to investigate restrictions on their domain and the ranges that result.

Piecewise and Continuous Functions

EXAMPLE 4 ▶ **Graphing a Piecewise-Defined Function**

Graph the function and state its domain and range:

$$f(x) = \begin{cases} -(x-3)^2 + 12, & 0 < x \leq 6 \\ 3, & x > 6 \end{cases}$$

Solution ▶ The first piece of f is a basic parabola, shifted three units right, reflected across the x-axis (opening downward), and shifted 12 units up. The vertex is at (3, 12) and the axis of symmetry is $x = 3$, producing the following graphs.

1. Graph first piece of f (Figure 2.81)
2. Erase portion outside domain. of $0 < x \leq 6$ (Figure 2.82).

Figure 2.81

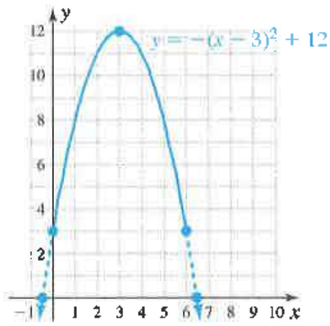
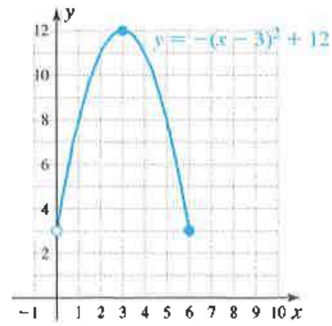


Figure 2.82



The second function is simply a horizontal line through (0, 3).

3. Graph second piece of f (Figure 2.83).
4. Erase portion outside domain of $x > 6$ (Figure 2.84).

Figure 2.83

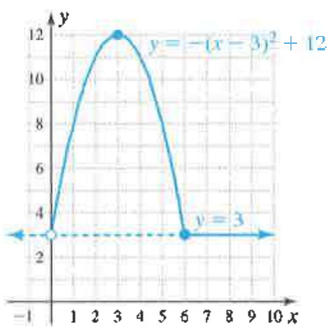
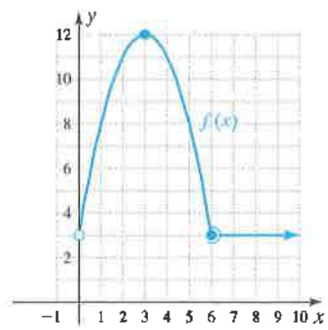


Figure 2.84



The domain of f is $x \in (0, \infty)$, and the corresponding range is $y \in [3, 12]$.

Now try Exercises 15 through 18 ▶

Piecewise and Discontinuous Functions

Notice that although the function in Example 4 was piecewise-defined, the graph was actually continuous—we could draw the entire graph without lifting our pencil. Piecewise graphs also come in the *discontinuous* variety, which makes the domain and range issues all the more important.

EXAMPLE 5 ▶ **Graphing a Discontinuous Piecewise-Defined Function**

Graph $g(x)$ and state the domain and range:

$$g(x) = \begin{cases} -\frac{1}{2}x + 6, & 0 \le x \le 4 \\ -|x - 6| + 10, & 4 < x \le 9 \end{cases}$$

Solution ▶ The first piece of g is a line, with y-intercept (0, 6) and slope $\frac{\Delta y}{\Delta x} = -\frac{1}{2}$.

1. Graph first piece of g (Figure 2.85)
2. Erase portion outside domain. of $0 \le x \le 4$ (Figure 2.86).

Figure 2.85

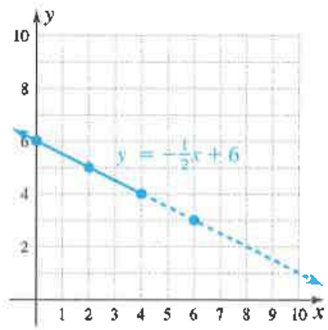
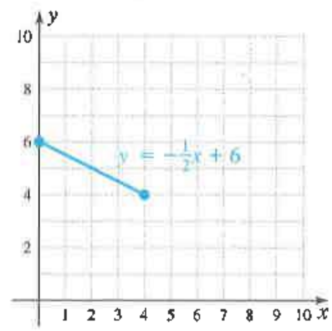


Figure 2.86



The second is an absolute value function, shifted right 6 units, reflected across the x-axis, then shifted up 10 units.

3. Graph second piece of g (Figure 2.87).
4. Erase portion outside domain of $4 < x \le 9$ (Figure 2.88).

Figure 2.87

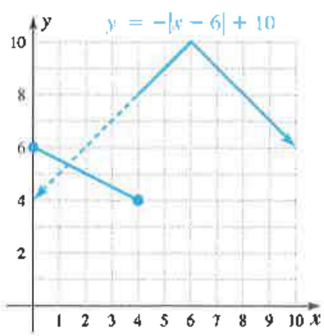
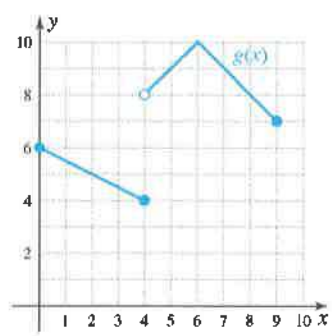


Figure 2.88



Note that the left endpoint of the absolute value portion is not included (this piece is not defined at $x = 4$), signified by the open dot. The result is a discontinuous graph, as there is no way to draw the graph other than by "jumping" the pencil from where one piece ends to where the next begins. Using a vertical boundary line, we note the domain of g includes all values between 0 and 9 inclusive: $x \in [0, 9]$. Using a horizontal boundary line shows the smallest y-value is 4 and the largest is 10, but no range values exist between 6 and 7. The range is $y \in [4, 6] \cup [7, 10]$.

WORTHY OF NOTE

As you graph piecewise-defined functions, keep in mind that they *are* functions and the end result must pass the vertical line test. This is especially important when we are drawing each piece as a complete graph, then erasing portions outside the effective domain.

Now try Exercises 19 and 20 ▶

EXAMPLE 6 ► **Graphing a Discontinuous Function**

The given piecewise-defined function is not continuous. Graph $h(x)$ to see why, then comment on what could be done to make it continuous.

$$h(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

Solution ► The first piece of h is unfamiliar to us, so we elect to graph it by plotting points, noting $x = 2$ is outside the domain. This produces the table shown. After connecting the points, the graph turns out to be a straight line, but with no corresponding y-value for $x = 2$. This leaves a "hole" in the graph at (2, 4), as designated by the open dot (see Figure 2.89).

x	$h(x)$
−4	−2
−2	0
0	2
2	—
4	6

Figure 2.89

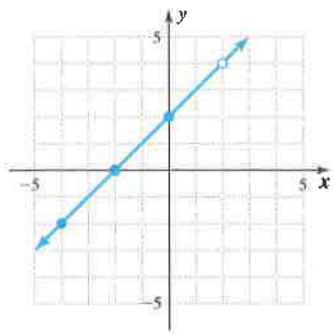
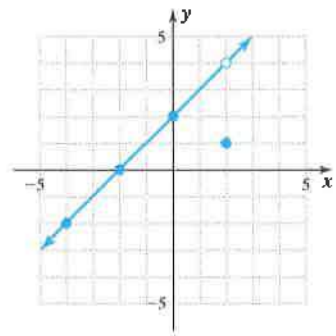


Figure 2.90



WORTHY OF NOTE

The discontinuity illustrated here is called a **removable discontinuity,** as the discontinuity can be removed by redefining a single point on the function. Note that after factoring the first piece, the denominator is a factor of the numerator, and writing the result in lowest terms gives $h(x) = \frac{(x+2)(x-2)}{x-2} = x + 2, x \neq 2$. This is precisely the equation of the line in Figure 2.89 [$y = x + 2$].

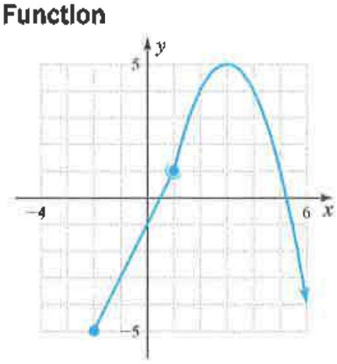
The second piece is pointwise-defined, and its graph is simply the point (2, 1) shown in Figure 2.90. It's interesting to note that while the domain of h is all real numbers (h *is* defined at all points), the range is $y \in (-\infty, 4) \cup (4, \infty)$ as the function never takes on the value $y = 4$. In order for h to be continuous, we would need to redefine the second piece as $y = 4$ when $x = 2$.

Now try Exercises 21 through 26 ►

To develop these concepts more fully, it will help to practice finding the equation of a piecewise-defined function *given its graph,* a process similar to that of Example 10 in Section 2.2.

EXAMPLE 7 ► **Determining the Equation of a Piecewise-Defined Function**

Determine the equation of the piecewise-defined function shown, including the domain for each piece.



Solution ► By counting $\frac{\Delta y}{\Delta x}$ from (−2, −5) to (1, 1), we find the linear portion has slope $m = 2$, and the y-intercept must be (0, −1). The equation of the line is $y = 2x - 1$. The second piece appears to be a parabola with vertex (h, k) at (3, 5). Using this vertex with the point (1, 1) in the general form $y = a(x - h)^2 + k$ gives

$$y = a(x - h)^2 + k \quad \text{general form, parabola is shifted right and up}$$
$$1 = a(1 - 3)^2 + 5 \quad \text{substitute 1 for } x \text{, 1 for } y \text{, 3 for } h \text{, 5 for } k$$
$$-4 = a(-2)^2 \quad \text{simplify; subtract 5}$$
$$-4 = 4a \quad (-2)^2 = 4$$
$$-1 = a \quad \text{divide by 4}$$

The equation of the parabola is $y = -(x - 3)^2 + 5$. Considering the domains shown in the figure, the equation of this piecewise-defined function must be

$$p(x) = \begin{cases} 2x - 1, & -2 \le x < 1 \\ -(x - 3)^2 + 5, & x \ge 1 \end{cases}$$

B. You've just seen how we can graph functions that are piecewise-defined

Now try Exercises 27 through 30 ►

C. Applications of Piecewise-Defined Functions

The number of applications for piecewise-defined functions is practically limitless. It is actually fairly rare for a single function to accurately model a situation over a long period of time. Laws change, spending habits change, and technology can bring abrupt alterations in many areas of our lives. To accurately model these changes often requires a piecewise-defined function.

EXAMPLE 8 ► **Modeling with a Piecewise-Defined Function**

For the first half of the twentieth century, per capita spending on police protection can be modeled by $S(t) = 0.54t + 12$, where $S(t)$ represents per capita spending on police protection in year t (1900 corresponds to year 0). After 1950, perhaps due to the growth of American cities, this spending greatly increased: $S(t) = 3.65t - 144$. Write these as a piecewise-defined function $S(t)$, state the domain for each piece, then graph the function. According to this model, how much was spent (per capita) on police protection in 2000 and 2010? How much will be spent in 2014?

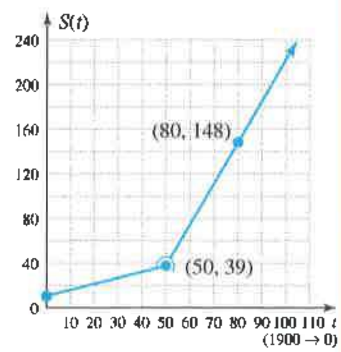
Source: Data taken from the *Statistical Abstract of the United States* for various years.

Solution ►

function name — function pieces — effective domain

$$S(t) = \begin{cases} 0.54t + 12, & 0 \le t \le 50 \\ 3.65t - 144, & t > 50 \end{cases}$$

Since both pieces are linear, we can graph each part using two points. For the first function, $S(0) = 12$ and $S(50) = 39$. For the second function $S(50) \approx 39$ and $S(80) = 148$. The graph for each piece is shown in the figure. Evaluating S at $t = 100$:



$$\begin{aligned} S(t) &= 3.65t - 144 \\ S(100) &= 3.65(100) - 144 \\ &= 365 - 144 \\ &= 221 \end{aligned}$$

About \$221 per capita was spent on police protection in the year 2000. For 2010, the model indicates that \$257.50 per capita was spent: $S(110) = 257.5$. By 2014, this function projects the amount spent will grow to $S(114) = 272.1$ or \$272.10 per capita.

Now try Exercises 33 through 44 ►

Step Functions

The last group of piecewise-defined functions we'll explore are the **step functions,** so called because the pieces of the function form a series of horizontal steps. These functions find frequent application in the way consumers are charged for services, and have several applications in number theory. Perhaps the most common is called the **greatest integer function,** though recently its alternative name, **floor function,** has gained popularity (see Figure 2.91). This is in large part due to an improvement in notation and as a better contrast to **ceiling functions.** The floor function of a real number x, denoted $f(x) = \lfloor x \rfloor$ or $[\![x]\!]$ (we will use the first), is the largest integer less than or equal to x. For instance, $\lfloor 5.9 \rfloor = 5$, $\lfloor 7 \rfloor = 7$, and $\lfloor -3.4 \rfloor = -4$.

In contrast, the ceiling function $C(x) = \lceil x \rceil$ is the smallest integer greater than or equal to x, meaning $\lceil 5.9 \rceil = 6$, $\lceil 7 \rceil = 7$, and $\lceil -3.4 \rceil = -3$ (see Figure 2.92). In simple terms, for any noninteger value on the number line, the floor function returns the integer to the left, while the ceiling function returns the integer to the right. A graph of each function is shown.

Figure 2.91

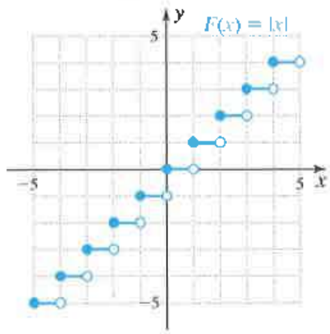
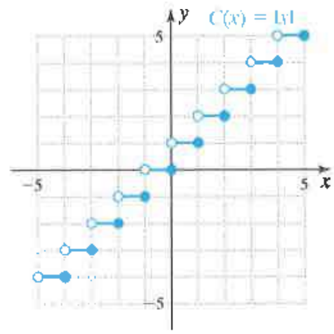


Figure 2.92



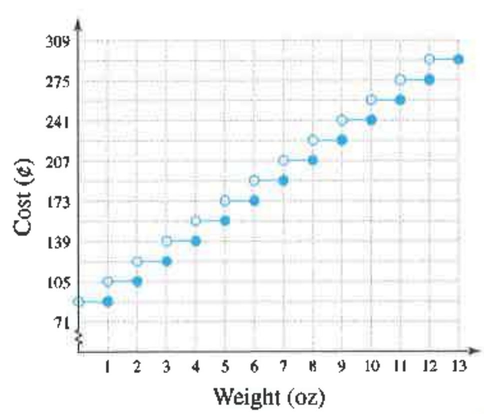
One common application of floor functions is the price of theater admission, where children 12 and under receive a discounted price. Right up until the day they're 13, they qualify for the lower price: $\lfloor 12\frac{364}{365} \rfloor = 12$. Applications of ceiling functions would include how phone companies charge for the minutes used (charging the 12-min rate for a phone call that only lasted 11.3 min: $\lceil 11.3 \rceil = 12$), and postage rates, as in Example 9.

EXAMPLE 9 ▶ Modeling Using a Step Function

In 2009 the first-class postage rate for large envelopes sent through the U.S. mail was 88¢ for the first ounce, then an additional 17¢ per ounce thereafter, up to 13 ounces. Graph the function and state its domain and range. Use the graph to state the cost of mailing a report weighing (a) 7.5 oz, (b) 8 oz, and (c) 8.1 oz in a large envelope.

Solution ▶ The 88¢ charge applies to letters weighing between 0 oz and 1 oz. Zero is not included since we have to mail *something*, but 1 is included since a large envelope and its contents weighing exactly one ounce still costs 88¢. The graph will be a horizontal line segment.

The function is defined for all weights between 0 and 13 oz, excluding zero and including 13: $x \in (0, 13]$. The range consists of single outputs corresponding to the step intervals: $R \in \{88, 105, 122, \ldots, 275, 292\}$.



a. The cost of mailing a 7.5-oz report is 207¢.

b. The cost of mailing an 8.0-oz report is still 207¢.

c. The cost of mailing an 8.1-oz report is 207 + 17 = 224¢, since this brings you up to the next step.

C. You've just seen how we can solve applications involving piecewise-defined functions

Now try Exercises 45 through 48 ▶

2.5 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. A function whose entire graph can be drawn without lifting your pencil is called a ________ function.

2. The input values for which each part of a piecewise function is defined is the ________ of the function.

3. A graph is called ________ if it has no sharp turns or jagged edges.

4. When graphing $2x + 3$ over a domain of $x > 0$, we leave an ________ dot at (0, 3).

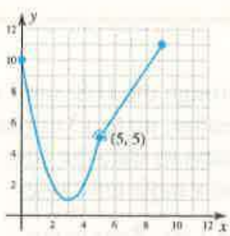
5. Discuss/Explain how to determine if a piecewise-defined function is continuous, without having to graph the function. Illustrate with an example.

6. Discuss/Explain how it is possible for the domain of a function to be defined for all real numbers, but have a range that is defined on more than one interval. Construct an illustrative example.

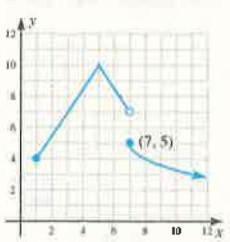
▶ DEVELOPING YOUR SKILLS

For Exercises 7 and 8, (a) use the correct notation to write them as a single piecewise-defined function and state the domain for each piece by inspecting the graph, then (b) state the range of the function.

7. $Y_1 = X^2 - 6x + 10$; $Y_2 = \frac{3}{2}X - \frac{5}{2}$



8. $Y_1 = -1.5|X - 5| + 10$; $Y_2 = -\sqrt{X - 7} + 5$



Evaluate each piecewise-defined function as indicated (if possible).

9. $h(x) = \begin{cases} -2 & x < -2 \\ |x| & -2 \le x < 3 \\ 5 & x \ge 3 \end{cases}$

$h(-5)$, $h(-2)$, $h(-\frac{1}{2})$, $h(0)$, $h(2.999)$, and $h(3)$

10. $H(x) = \begin{cases} 2x + 3 & x < 0 \\ x^2 + 1 & 0 \le x < 2 \\ 5 & x > 2 \end{cases}$

$H(-3)$, $H(-\frac{3}{2})$, $H(-0.001)$, $H(1)$, $H(2)$, and $H(3)$

11. $p(x) = \begin{cases} 5 & x < -3 \\ x^2 - 4 & -3 \le x \le 3 \\ 2x + 1 & x > 3 \end{cases}$

$p(-5)$, $p(-3)$, $p(-2)$, $p(0)$, $p(3)$, and $p(5)$

12. $q(x) = \begin{cases} -x - 3 & x < -1 \\ 2 & -1 \le x < 2 \\ -\frac{1}{2}x^2 + 3x - 2 & x \ge 2 \end{cases}$

$q(-3)$, $q(-1)$, $q(0)$, $q(1.999)$, $q(2)$, and $q(4)$

Graph each piecewise-defined function using a graphing calculator. Then evaluate each at $x = 2$ and $x = 0$.

13. $p(x) = \begin{cases} x + 2 & -6 \le x \le 2 \\ 2|x - 4| & x > 2 \end{cases}$

14. $q(x) = \begin{cases} \sqrt{x + 4} & -4 \le x \le 0 \\ |x - 2| & 0 < x \le 7 \end{cases}$

Graph each piecewise-defined function and state its domain and range. Use transformations of the toolbox functions where possible.

15. $g(x) = \begin{cases} -(x - 1)^2 + 5 & -2 \le x \le 4 \\ 2x - 12 & x > 4 \end{cases}$

16. $h(x) = \begin{cases} \frac{1}{2}x + 1 & x \le 0 \\ (x - 2)^2 - 3 & 0 < x \le 5 \end{cases}$

17. $H(x) = \begin{cases} -x + 3 & x < 1 \\ -|x - 5| + 6 & 1 \le x < 9 \end{cases}$

18. $w(x) = \begin{cases} \sqrt[3]{x - 1} & x < 2 \\ (x - 3)^2 & 2 \le x \le 6 \end{cases}$

19. $f(x) = \begin{cases} -x - 3 & x < -3 \\ 9 - x^2 & -3 \le x < 2 \\ 4 & x \ge 2 \end{cases}$

20. $h(x) = \begin{cases} -\frac{1}{2}x - 1 & x < -3 \\ -|x| + 5 & -3 \le x \le 5 \\ 3\sqrt{x - 5} & x > 5 \end{cases}$

21. $p(x) = \begin{cases} \frac{1}{2}x + 1 & x \ne 4 \\ 2 & x = 4 \end{cases}$

22. $q(x) = \begin{cases} \frac{1}{2}(x - 1)^3 - 1 & x \ne 3 \\ -2 & x = 3 \end{cases}$

Each of the following functions has a removable discontinuity. Graph the first piece of each function, then find the value of *c* so that a continuous function results.

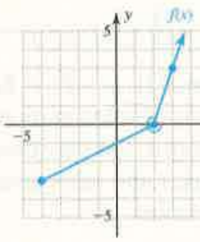
23. $f(x) = \begin{cases} \dfrac{x^2 - 9}{x + 3} & x \ne -3 \\ c & x = -3 \end{cases}$

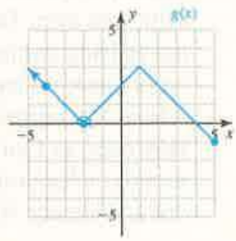
24. $f(x) = \begin{cases} \dfrac{x^2 - 3x - 10}{x - 5} & x \ne 5 \\ c & x = 5 \end{cases}$

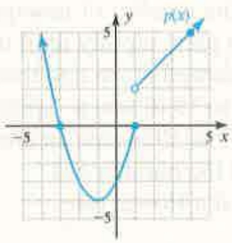
25. $f(x) = \begin{cases} \dfrac{x^3 - 1}{x - 1} & x \ne 1 \\ c & x = 1 \end{cases}$

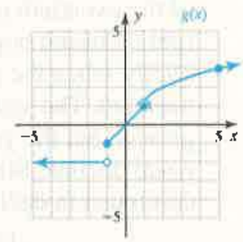
26. $f(x) = \begin{cases} \dfrac{4x - x^3}{x + 2} & x \ne -2 \\ c & x = -2 \end{cases}$

Determine the equation of each piecewise-defined function shown, including the domain for each piece. Assume all pieces are toolbox functions.

27. 

28. 

29. 

30. 

▶ WORKING WITH FORMULAS

31. **Definition of absolute value:** $|x| = \begin{cases} -x & x < 0 \\ x & x \ge 0 \end{cases}$

The absolute value function can be stated as a piecewise-defined function, a technique that is sometimes useful in graphing variations of the function or solving absolute value equations and inequalities. How does this definition ensure that the absolute value of a number is always positive? Use this definition to help sketch the graph of $f(x) = \frac{|x|}{x}$. Discuss what you notice.

32. **Sand dune function:**

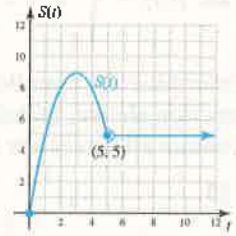
$$f(x) = \begin{cases} -|x - 2| + 1 & 1 \le x < 3 \\ -|x - 4| + 1 & 3 \le x < 5 \\ -|x - 2k| + 1 & 2k - 1 \le x < 2k + 1, \text{ for } k \in N \end{cases}$$

There are a number of interesting graphs that can be created using piecewise-defined functions, and these functions have been the basis for more than one piece of modern art. (a) Use the descriptive name and the pieces given to graph the function *f*. Is the function accurately named? (b) Use any combination of the toolbox functions to explore your own creativity by creating a piecewise-defined function with some interesting or appealing characteristics. (c) For $y = -|x - 2| + 1$, solve for *x* in terms of *y*.

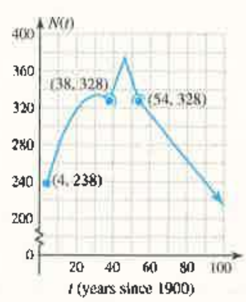
▶ APPLICATIONS

For Exercises 33 and 34, (a) write the information given as a piecewise-defined function, and state the domain for each piece by inspecting the graph. (b) Give the range of each.

33. **Results from advertising:** Due to heavy advertising, initial sales of the Lynx Digital Camera grew very rapidly, but started to decline once the advertising blitz was over. During the advertising campaign, sales were modeled by the function $S(t) = -t^2 + 6t$, where $S(t)$ represents hundreds of sales in month t. However, as Lynx Inc. had hoped, the new product secured a foothold in the market and sales leveled out at a steady 500 sales per month.



34. **Decline of newspaper publishing:** From the turn of the twentieth century, the number of newspapers (per thousand population) grew rapidly until the 1930s, when the growth slowed down and then declined. The years 1940 to 1946 saw a "spike" in growth, but the years 1947 to 1954 saw an almost equal decline. Since 1954 the number has continued to decline, but at a slower rate.



The number of papers N per thousand population for each period, respectively, can be approximated by

$N_1(t) = -0.13t^2 + 8.1t + 208,$

$N_2(t) = -5.75|t - 46| + 374,$ and

$N_3(t) = -2.45t + 460.$

Source: Data from the *Statistical Abstract of the United States*, various years; data from *The First Measured Century, The AEI Press*, Caplow, Hicks, and Wattenberg, 2001.

35. **Families that own stocks:** The percentage of American households that own publicly traded stocks began rising in the early 1950s, peaked in 1970, then began to decline until 1980 when there was a dramatic increase due to easy access over the Internet, an improved economy, and other factors. This phenomenon is modeled by the function $P(t)$, where $P(t)$ represents the percentage of households owning stock in year t, with 1950 corresponding to year 0.

$$P(t) = \begin{cases} -0.03t^2 + 1.28t + 1.68 & 0 \le t \le 30 \\ 1.89t - 43.5 & t > 30 \end{cases}$$

a. According to this model, what percentage of American households held stock in the years 1955, 1965, 1975, 1985, and 1995? If this pattern continues, what percentage held stock in 2005? What percent will hold stock in 2015?

b. Why is there a discrepancy in the outputs of each piece of the function for the year 1980 ($t = 30$)? According to how the function is defined, which output should be used?

Source: 2004 *Statistical Abstract of the United States*, Table 1204; various other years.

36. **Dependence on foreign oil:** America's dependency on foreign oil has always been a "hot" political topic, with the amount of imported oil fluctuating over the years due to political climate, public awareness, the economy, and other factors. The amount of crude oil imported can be approximated by the function given, where $A(t)$ represents the number of barrels imported in year t (in billions), with 1980 corresponding to year 0.

$$A(t) = \begin{cases} 0.047t^2 - 0.38t + 1.9 & 0 \le t < 8 \\ -0.075t^2 + 1.495t - 5.265 & 8 \le t \le 11 \\ 0.133t + 0.685 & t > 11 \end{cases}$$

a. Use $A(t)$ to estimate the number of barrels imported in the years 1983, 1989, 1995, and 2005. If this trend continues, how many barrels will be imported in 2015?

b. What was the minimum number of barrels imported between 1980 and 1988?

Source: 2004 *Statistical Abstract of the United States*, Table 897; various other years.

37. **Energy rationing:** In certain areas of the United States, power blackouts have forced some counties to ration electricity. Suppose the cost is \$0.09 per kilowatt (kW) for the first 1000 kW a household uses. After 1000 kW, the cost increases to 0.18 per kW. (a) Write these charges for electricity in the form of a piecewise-defined function $C(h)$, where $C(h)$ is the cost for h kilowatt hours. Include the domain for each piece. Then (b) sketch the graph and determine the cost for 1200 kW.

38. Water rationing: Many southwestern states have a limited water supply, and some state governments try to control consumption by manipulating the cost of water usage. Suppose for the first 5000 gal a household uses per month, the charge is \$0.05 per gallon. Once 5000 gal is used the charge doubles to \$0.10 per gallon. (a) Write these charges for water usage in the form of a piecewise-defined function $C(w)$, where $C(w)$ is the cost for w gallons of water. Include the domain for each piece. Then (b) sketch the graph and determine the cost to a household that used 9500 gal of water during a very hot summer month.

39. Pricing for natural gas: A local gas company charges \$0.75 per therm for natural gas, up to 25 therms. Once the 25 therms has been exceeded, the charge doubles to \$1.50 per therm due to limited supply and great demand. (a) Write these charges for natural gas consumption in the form of a piecewise-defined function $C(t)$, where $C(t)$ is the charge for t therms. Include the domain for each piece. Then (b) sketch the graph and determine the cost to a household that used 45 therms during a very cold winter month.

40. Multiple births: The number of multiple births has steadily increased in the United States during the twentieth century and beyond. Between 1985 and 1995 the number of twin births could be modeled by the function $T(x) = -0.21x^2 + 6.1x + 52$, where x is the number of years since 1980 and T is in thousands. After 1995, the incidence of twins becomes more linear, with $T(x) = 4.53x + 28.3$ serving as a better model. (a) Write the piecewise-defined function modeling the incidence of twins for these years. Include the domain of each piece. Then (b) sketch the graph and use the function to estimate the incidence of twins in 1990, 2000, and 2005. If this trend continued, how many sets of twins were born in 2010?

Source: National Vital Statistics Report, Vol. 50, No. 5, February 12, 2002

41. U.S. military expenditures: Except for the year 1991 when military spending was cut drastically, the amount spent by the U.S. government on national defense and veterans' benefits rose steadily from 1980 to 1992. These expenditures can be modeled by the function $S(t) = -1.35t^2 + 31.9t + 152$, where $S(t)$ is in billions of dollars and 1980 corresponds to $t = 0$. From 1992 to 1996 this spending declined, then began to rise in the following years. From 1992 to 2002, military-related spending can be modeled by $S(t) = 2.5t^2 - 80.6t + 950$.

Source: 2004 *Statistical Abstract of the United States*, Table 492

(a) Write $S(t)$ as a single piecewise-defined function. Include stating the domain for each piece. Then (b) sketch the graph and use the function to estimate the amount spent by the United States in 2005, 2008, and 2012 if this trend continues.

42. Amusement arcades: At a local amusement center, the owner has the SkeeBall machines programmed to reward very high scores. For scores of 200 or less, the function $T(x) = \frac{x}{10}$ models the number of tickets awarded (rounded to the nearest whole). For scores over 200, the number of tickets is modeled by $T(x) = 0.001x^2 - 0.3x + 40$. (a) Write these equation models of the number of tickets awarded in the form of a piecewise-defined function. Include the domain for each piece. Then (b) sketch the graph and find the number of tickets awarded to a person who scores 390 points.

43. Phone service charges: When it comes to phone service, a large number of calling plans are available. Under one plan, the first 30 min of any phone call costs only 3.3¢ per minute. The charge increases to 7¢ per minute thereafter. (a) Write this information in the form of a piecewise-defined function. Include the domain for each piece. Then (b) sketch the graph and find the cost of a 46-min phone call.

44. Overtime wages: Tara works on an assembly line, putting together computer monitors. She is paid \$9.50 per hour for regular time (0, 40 hr], \$14.25 for overtime (40, 48 hr], and when demand for computers is high, \$19.00 for double-overtime (48, 84 hr]. (a) Write this information in the form of a simplified piecewise-defined function. Include the domain for each piece. (b) Then sketch the graph and find the gross amount of Tara's check for the week she put in 54 hr.

45. Admission prices: At Wet Willy's Water World, infants under 2 are free, then admission is charged according to age. Children 2 and older but less than 13 pay \$2, teenagers 13 and older but less than 20 pay \$5, adults 20 and older but less than 65 pay \$7, and senior citizens 65 and older get in at the teenage rate. (a) Write this information in the form of a piecewise-defined function. Include the domain for each piece. Then (b) sketch the graph and find the cost of admission for a family of nine which includes: one grandparent (70), two adults (44/45), 3 teenagers, 2 children, and one infant.

46. Demographics: One common use of the floor function $y = \lfloor x \rfloor$ is the reporting of ages. As of 2007, the record for longest living human is 122 yr, 164 days for the life of Jeanne Calment, formerly of France. While she actually lived $x = 122\frac{164}{365}$ years, ages are normally reported using the floor function, or the greatest integer number of years less than or equal to the actual age: $\lfloor 122\frac{164}{365} \rfloor = 122$ years. (a) Write a function $A(t)$ that gives a person's age, where $A(t)$ is the reported age at time t. (b) State the domain of the function (be sure to consider Madame Calment's record). Report the age of a person who has been living for (c) 36 years; (d) 36 years, 364 days; (e) 37 years; and (f) 37 years, 1 day.

47. Postage rates: The postal charge function from Example 9 is simply a transformation of the basic ceiling function $y = \lceil x \rceil$. Using the ideas from Section 2.2, (a) write the postal charges as a step function $C(w)$, where $C(w)$ is the cost of mailing a large envelope weighing w ounces, and (b) state the domain of the function. Then use the function to find the cost of mailing reports weighing: (c) 0.7 oz, (d) 5.1 oz, (e) 5.9 oz; (f) 6 oz, and (g) 6.1 oz.

48. Cell phone charges: A national cell phone company advertises that calls of 1 min or less do not count toward monthly usage. Calls lasting longer than 1 min are calculated normally using a ceiling function, meaning a call of 1 min, 1 sec will be counted as a 2-min call. Using the ideas from Section 2.2, (a) write the cell phone charges as a piecewise-defined function $C(m)$, where $C(m)$ is the cost of a call lasting m minutes, and include the domain of the function. Then (b) graph the function, and (c) use the graph or function to determine if a cell phone subscriber has exceeded the 30 free minutes granted by her calling plan for calls lasting 2 min 3 sec, 13 min 46 sec, 1 min 5 sec, 3 min 59 sec, and 8 min 2 sec. (d) What was the actual usage in minutes and seconds?

49. Combined absolute value graphs: Carefully graph the function $h(x) = |x - 2| - |x + 3|$ using a table of values over the interval $x \in [-5, 5]$. Is the function continuous? Write this function in piecewise-defined form and state the domain for each piece.

50. Combined absolute value graphs: Carefully graph the function $H(x) = |x - 2| + |x + 3|$ using a table of values over the interval $x \in [-5, 5]$. Is the function continuous? Write this function in piecewise-defined form and state the domain for each piece.

▶ EXTENDING THE CONCEPT

51. You've heard it said, *"any number divided by itself is one."* Consider the functions $f(x) = \frac{x+2}{x+2}$, and $g(x) = \frac{|x+2|}{x+2}$. Are these functions continuous?

52. Find a linear function $h(x)$ that will make the function shown a *continuous* function. Be sure to include its domain.

$$f(x) = \begin{cases} x^2 & x < 1 \\ h(x) & \\ 2x + 3 & x > 3 \end{cases}$$

▶ MAINTAINING YOUR SKILLS

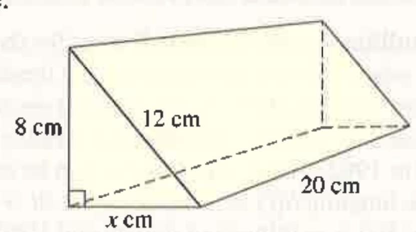
53. (Appendix A.5) Solve: $\frac{3}{x-2} + 1 = \frac{30}{x^2 - 4}$.

54. (Appendix A.5) Compute the following and write the result in lowest terms:

$$\frac{x^3 + 3x^2 - 4x - 12}{x - 3} \cdot \frac{2x - 6}{x^2 + 5x + 6} \div (3x - 6)$$

55. (1.4) Find an equation of the line perpendicular to $3x + 4y = 8$, and through the point $(0, -2)$. Write the result in slope-intercept form.

56. (Appendix A.6/1.1) For the figure shown, (a) use the Pythagorean Theorem to find the length of the missing side and (b) state the area of the triangular side.



2.6 Variation: The Toolbox Functions in Action

LEARNING OBJECTIVES

In Section 2.6 you will see how we can:

- ❑ **A.** Solve direct variations
- ❑ **B.** Solve inverse variations
- ❑ **C.** Solve joint variations

A study of direct and inverse variation offers perhaps our clearest view of how mathematics is used to model real-world phenomena. While the basis of our study is elementary, involving only the toolbox functions, the applications are at the same time elegant, powerful, and far reaching. In addition, these applications unite some of the most important ideas in algebra, including functions, transformations, rates of change, and graphical analysis, to name a few.

A. Toolbox Functions and Direct Variation

If a car gets 24 miles per gallon (mpg) of gas, we could express the distance d it could travel as $d = 24g$. Table 2.4 verifies the distance traveled by the car changes in *direct* or *constant proportion* to the number of gallons used, and here we say, "distance traveled *varies directly* with gallons used." The equation $d = 24g$ is called a **direct variation,** and the coefficient 24 is called the **constant of variation.** Using the rate of change notation, $\frac{\Delta \text{distance}}{\Delta \text{gallons}} = \frac{\Delta d}{\Delta g} = \frac{24}{1}$, and we note this is actually a *linear equation* with slope $m = 24$. When working with variations, the constant k is preferred over m, and in general we have the following:

Table 2.4

g	d
1	24
2	48
3	72
4	96

Direct Variation

y varies directly with x, or *y is directly proportional to x*, if there is a nonzero constant k such that

$$y = kx.$$

k is called the *constant of variation*

EXAMPLE 1 ▸ **Writing a Variation Equation**

Write the variation equation for these statements:

a. Wages earned varies directly with the number of hours worked.

b. The value of an office machine varies directly with time.

c. The circumference of a circle varies directly with the length of the diameter.

Solution ▸ **a.** Wages varies directly with hours worked: $W = kh$

b. The Value of an office machine varies directly with time: $V = kt$

c. The Circumference varies directly with the diameter: $C = kd$

Now try Exercises 7 through 10 ▸

Once we determine the relationship between two variables is a direct variation, we try to find the value of k and develop an equation model that can more generally be applied. Note that "varies directly" indicates that one value is a constant multiple of the other. In Example 1, you may have realized that if any one relationship between the variables is known, we can solve for k by substitution. For instance, if the circumference of a circle is 314 cm when the diameter is 100 cm, $C = kd$ becomes $314 = k(100)$ and division shows $k = 3.14$ (our estimate for π). The result is a formula for the circumference of *any* circle. This suggests the following procedure:

Solving Applications of Variation

1. Translate the information given into an equation model, using k as the constant of variation.
2. Substitute the first relationship (pair of values) given and solve for k.
3. Substitute this value for k in the original model to obtain the variation equation.
4. Use the variation equation to complete the application.

EXAMPLE 2 ▶ Solving an Application of Direct Variation

The weight of an astronaut on the surface of another planet **varies directly** with their weight on Earth. An astronaut weighing 140 lb on Earth weighs only 53.2 lb on Mars. How much would a 170-lb astronaut weigh on Mars?

Solution ▶

1. $M = kE$ "Mars weight varies directly with Earth weight"
2. $53.2 = k(140)$ substitute 53.2 for M and 140 for E
   $k = 0.38$ solve for k (constant of variation)

Substitute this value of k in the original equation to obtain the variation equation, then find the weight of a 170-lb astronaut that landed on Mars.

3. $M = 0.38E$ variation equation
4. $= 0.38(170)$ substitute 170 for E
   $= 64.6$ result

An astronaut weighing 170 lb on Earth weighs only 64.6 lb on Mars.

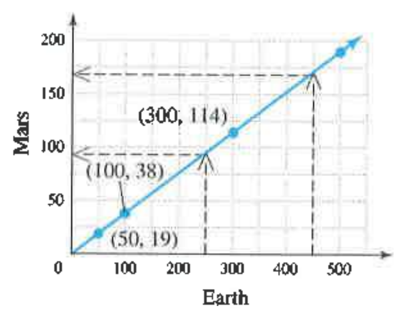
Now try Exercises 11 through 14 ▶

The toolbox function from Example 2 was a line with slope $k = 0.38$, or $k = \frac{19}{50}$ as a fraction in simplest form. As a rate of change, $k = \frac{\Delta M}{\Delta E} = \frac{19}{50}$, and we see that for every 50 additional pounds on Earth, the weight of an astronaut would increase by only 19 lb on Mars.

EXAMPLE 3 ▶ Making Estimates from the Graph of a Variation

The scientists at NASA are planning to send additional probes to the red planet (Mars), that will weigh from 250 to 450 lb. Graph the variation equation from Example 2, then *use the graph* to estimate the corresponding range of weights on Mars. Check your estimate using the variation equation.

Solution ▶ After selecting an appropriate scale, begin at (0, 0) and count off the slope $k = \frac{\Delta M}{\Delta E} = \frac{19}{50}$. This gives the points (50, 19), (100, 38), (200, 76), and so on. From the graph (see dashed arrows), it appears the weights corresponding to 250 lb and 450 lb on Earth are near 95 lb and 170 lb on Mars. Using the equation gives



$M = 0.38E$ variation equation
$= 0.38(250)$ substitute 250 for E
$= 95,$
$M = 0.38E$ variation equation
$= 0.38(450)$ substitute 450 for E
$= 171,$ very close to our estimate from the graph.

Now try Exercises 15 and 16 ▶

When toolbox functions are used to model variations, our knowledge of their graphs and defining characteristics strengthens a contextual understanding of the application. Consider Examples 4 and 5, where the squaring function is used.

EXAMPLE 4 ▶ Writing Variation Equations

Write the variation equation for these statements:

a. In free fall, the distance traveled by an object varies directly with the square of the time.

b. The area of a circle varies directly with the square of its radius.

Solution ▶

a. Distance varies directly with the square of the time: $D = kt^2$.

b. Area varies directly with the square of the radius: $A = kr^2$.

Now try Exercises 17 through 20 ▶

Both variations in Example 4 use the squaring function, where k represents the amount of stretch or compression applied, and whether the graph will open upward or downward. However, regardless of the function used, the four-step solution process remains the same.

EXAMPLE 5 ▶ Solving an Application of Direct Variation

The range of a projectile varies directly with the square of its initial velocity. As part of a circus act, Bailey the Human Bullet is shot out of a cannon with an initial velocity of 80 feet per second (ft/sec), into a net 200 ft away.

a. Find the constant of variation and write the variation equation.

b. Graph the equation and *use the graph* to estimate how far away the net should be placed if initial velocity is increased to 95 ft/sec.

c. Determine the accuracy of the estimate from (b) using the variation equation.

Solution ▶

a. 1. $R = kv^2$ "Range varies directly with the square of the velocity"

2. $200 = k(80)^2$ substitute 200 for R and 80 for v

$k = 0.03125$ solve for k (constant of variation)

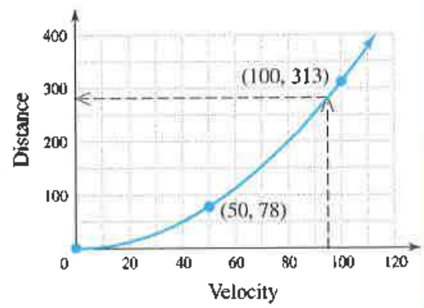
3. $R = 0.03125v^2$ variation equation (substitute 0.03125 for k)

b. Since velocity and distance are positive, we again use only QI. The graph is a parabola that opens upward, with the vertex at (0, 0). Selecting velocities from 50 to 100 ft/sec, we have:

$R = 0.03125v^2$ variation equation

$= 0.03125(50)^2$ substitute 50 for v

$= 78.125$ result



Likewise substituting 100 for v gives $R = 312.5$ ft. Scaling the axes and using (0, 0), (50, 78), and (100, 313) produces the graph shown. At 95 ft/sec (dashed lines), it appears the net should be placed about 280 ft away.

c. Using the variation equation gives:

4. $R = 0.03125v^2$ variation equation

$= 0.03125(95)^2$ substitute 95 for v

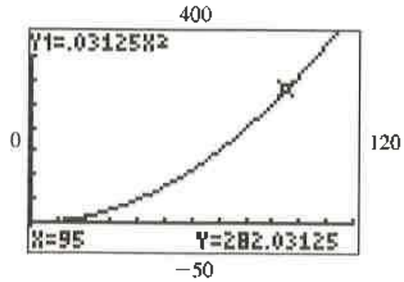
$R = 282.03125$ result

Our estimate was off by about 2 ft. The net should be placed about 282 ft away.

Now try Exercises 21 through 26 ▶

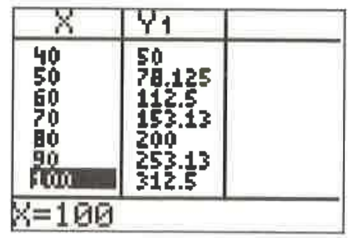
We now have a complete picture of this relationship, in which the required information can be presented graphically (Figure 2.93), numerically (Figure 2.94), verbally, and in equation form. This enables the people requiring the information, i.e., Bailey himself (for obvious reasons) and the Circus Master who is responsible, to make more informed (and safe) decisions.

Figure 2.93



Range R varies as the square of the velocity

Figure 2.94



$R = 0.03125v^2$

A. You've just seen how we can solve direct variations

Note: For Examples 7 and 8, the four steps of the solution process were used in sequence, but not numbered.

B. Inverse Variation

Table 2.5

Price (dollars)	Demand (1000s)
8	288
9	144
10	96
11	72
12	57.6

Numerous studies have been done that relate the price of a commodity to the demand—the willingness of a consumer to pay that price. For instance, if there is a sudden increase in the price of a popular tool, hardware stores know there will be a corresponding decrease in the demand for that tool. The question remains, "What is this rate of decrease?" Can it be modeled by a linear function with a negative slope? A parabola that opens downward? Some other function? Table 2.5 shows some (simulated) data regarding price versus demand. It appears that a linear function is not appropriate because the rate of change in the number of tools sold is not constant. Likewise a quadratic model seems inappropriate, since we don't expect demand to suddenly start rising again as the price continues to increase. This phenomenon is actually an example of **inverse variation,** modeled by a transformation of the reciprocal function $y = \frac{k}{x}$. We will often rewrite the equation as $y = k(\frac{1}{x})$ to clearly see the inverse relationship. In the case at hand, we might write $D = k(\frac{1}{P})$, where k is the constant of variation, D represents the demand for the product, and P the price of the product. In words, we say that "demand *varies inversely* as the price." In other applications of inverse variation, one quantity may vary inversely as the *square* of another [Example 6(b)], and in general we have

Inverse Variation

y varies inversely with x, or *y is inversely proportional to x,* if there is a nonzero constant k such that

$$y = k\left(\frac{1}{x}\right).$$

k is called the *constant of variation*

EXAMPLE 6 ▶ Writing Inverse Variation Equations

Write the variation equation for these statements:

a. In a closed container, pressure varies inversely with the volume of gas.

b. The intensity of light varies inversely with the square of the distance from the source.

Solution ▶ **a.** Pressure varies inversely with the Volume of gas: $P = k(\frac{1}{V})$.

b. Intensity of light varies inversely with the square of the distance: $I = k\left(\frac{1}{d^2}\right)$.

Now try Exercises 27 through 30 ▶

EXAMPLE 7 ▶ Solving an Application of Inverse Variation

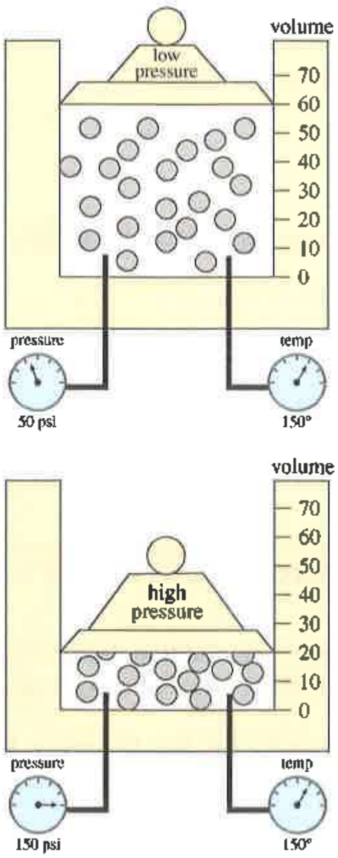
Boyle's law tells us that in a closed container with constant temperature, the volume of a gas varies inversely with the pressure applied (see illustration). Suppose the air pressure in a closed cylinder is 50 pounds per square inch (psi) when the volume of the cylinder is 60 in^3.

a. Find the constant of variation and write the variation equation.

b. Use the equation to find the volume, if the pressure is increased to 150 psi.

Solution ▶

Illustration of Boyle's Law



a. $V = k\left(\frac{1}{P}\right)$ "volume varies inversely with the pressure"

$60 = k\left(\frac{1}{50}\right)$ substitute 60 for V and 50 for P.

$k = 3000$ constant of variation

$V = 3000\left(\frac{1}{P}\right)$ variation equation (substitute 3000 for k)

b. Using the variation equation we have:

$V = 3000\left(\frac{1}{P}\right)$ variation equation

$= 3000\left(\frac{1}{150}\right)$ substitute 150 for P

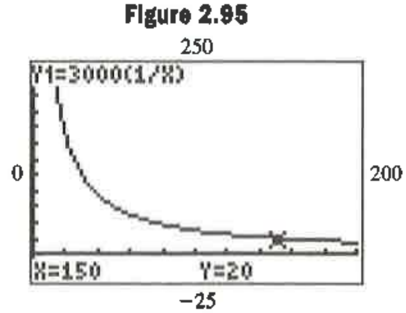
$= 20$ result

When the pressure is increased to 150 psi, the volume decreases to 20 in^3.

Now try Exercises 31 through 34 ▶

As an application of the reciprocal function, the relationship in Example 7 is easily graphed as a transformation of $y = \frac{1}{x}$. Using an appropriate scale and values in QI, only a vertical stretch of 3000 is required and the result is shown in Figure 2.95. As noted, when the pressure increases the volume decreases, or in notation: as $P \to \infty$, $V \to 0$. Applications of this sort can be as sophisticated as the manufacturing of industrial pumps and synthetic materials, or as simple as cooking a homemade dinner. Simply based on the equation, how much pressure is required to reduce the volume of gas to 1 in^3?

Figure 2.95



B. You've just seen how we can solve inverse variations

C. Joint or Combined Variations

Figure 2.96

Just as some decisions might be based on many considerations, often the relationship between two variables depends on a combination of factors. Imagine a wooden plank laid across the banks of a stream for hikers to cross the streambed (see Figure 2.96). The amount of weight the plank will support depends on the type of wood, the width and height of the plank's cross section, and the distance between the supported ends **(see Exercises 59 and 60).** This is an example of a **joint variation,** which can combine any number of variables in different ways. Two general possibilities are: (1) *y varies jointly with the product of x and p:* $y = kxp$; and (2) *y varies jointly with the product of x and p, and inversely with the square of q:* $y = kxp(\frac{1}{q^2})$. For practice writing joint variations as an equation model, **see Exercises 35 through 40.**

EXAMPLE 8 ▶ Solving an Application of Joint Variation

The amount of fuel used by a certain ship traveling at a uniform speed varies jointly with the distance it travels and the square of the velocity. If 200 barrels of fuel are used to travel 10 mi at 20 nautical miles per hour, how far does the ship travel on 500 barrels of fuel at 30 nautical miles per hour?

Solution ▶

$F = kdv^2$ "Fuel use *varies jointly* with distance and velocity squared"

$200 = k(10)(20)^2$ substitute 200 for F, 10 for d, and 20 for v

$200 = 4000k$ simplify and solve for k

$0.05 = k$ constant of variation

$F = 0.05dv^2$ equation of variation

To find the distance traveled at 30 nautical miles per hour using 500 barrels of fuel, substitute 500 for F and 30 for v:

$F = 0.05dv^2$ equation of variation

$500 = 0.05d(30)^2$ substitute 500 for F and 30 for v

$500 = 45d$ simplify

$11.\overline{1} = d$ result

If 500 barrels of fuel are consumed while traveling 30 nautical miles per hour, the ship covers a distance of just over 11 mi.

Now try Exercises 41 through 44 ▶

It's interesting to note that the ship covers just over one additional mile, but consumes 2.5 times the amount of fuel. The additional speed requires a great deal more fuel.

There is a variety of additional applications in the Exercise Set. **See Exercises 47 through 55.**

☑ **C.** You've just seen how we can solve joint variations

2.6 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. The phrase "y varies directly with x" is written $y = kx$, where k is called the ________ of variation.

2. If more than two quantities are related in a variation equation, the result is called a ________ variation.

3. For a right circular cylinder, $V = \pi r^2 h$ and we say, the volume varies ________ with the ________ and the ________ of the radius.

4. The statement "y varies inversely with the square of x" is written ________.

5. Discuss/Explain the general procedure for solving applications of variation. Include references to keywords, and illustrate using an example.

6. The basic percent formula is *amount equals percent times base*, or $A = PB$. In words, write this out as a direct variation with B as the constant of variation, then as an inverse variation with the amount A as the constant of variation.

▶ DEVELOPING YOUR SKILLS

Write the variation equation for each statement.

7. distance traveled varies directly with rate of speed
8. cost varies directly with the quantity purchased
9. force varies directly with acceleration
10. length of a spring varies directly with attached weight

For Exercises 11 and 12, find the constant of variation and write the variation equation. Then use the equation to complete the table.

11. y varies directly with x; $y = 0.6$ when $x = 24$.

x	y
500	
	16.25
750	

12. w varies directly with v; $w = \frac{1}{3}$ when $v = 5$.

v	w
291	
	21.8
339	

13. **Wages and hours worked:** Wages earned varies directly with the number of hours worked. Last week I worked 37.5 hr and my gross pay was \$344.25. Write the variation equation and determine how much I will gross this week if I work 35 hr. What does the value of k represent in this case?

14. **Pagecount and thickness of books:** The thickness of a paperback book varies directly as the number of pages. A book 3.2 cm thick has 750 pages. Write the variation equation and approximate the thickness of *Roget's 21st Century Thesaurus* (paperback—2nd edition), which has 957 pages.

15. **Building height and number of stairs:** The number of stairs in the stairwells of tall buildings and other structures varies directly as the height of the structure. The base and pedestal for the Statue of Liberty are 47 m tall, with 192 stairs from ground level to the observation deck at the top of the pedestal (at the statue's feet). (a) Find the constant of variation and write the variation equation, (b) graph the variation equation, (c) use the graph to estimate the number of stairs from ground level to the observation deck in the statue's crown 81 m above ground level, and (d) use the equation to check this estimate. Was it close?

16. **Projected images:** The height of a projected image varies directly as the distance of the projector from the screen. At a distance of 48 in., the image on the screen is 16 in. high. (a) Find the constant of variation and write the variation equation, (b) graph the variation equation, (c) use the graph to estimate the height of the image if the projector is placed at a distance of 5 ft 3 in., and (d) use the equation to check this estimate. Was it close?

Write the variation equation for each statement.

17. Surface area of a cube varies directly with the square of a side.
18. Potential energy in a spring varies directly with the square of the distance the spring is compressed.
19. Electric power varies directly with the square of the current (amperes).
20. Manufacturing cost varies directly as the square of the number of items made.

For Exercises 21 and 22, find the constant of variation and write the variation equation. Then use the equation to complete the table.

21. p varies directly with the square of q; $p = 280$ when $q = 50$

q	p
45	
	338.8
70	

22. n varies directly with m squared; $n = 24.75$ when $m = 30$

m	n
40	
	99
88	

For Exercises 23 to 26, supply the relationship indicated (a) in words, (b) in equation form, (c) graphically, and (d) in table form, then (e) solve the application.

23. **The Borg Collective:** The surface area of a cube varies directly as the square of one side. A cube with sides of $14\sqrt{3}$ cm has a surface area of 3528 cm^2. Find the surface area in square meters of the spaceships used by the Borg Collective in *Star Trek—The Next Generation,* cubical spacecraft with sides of 3036 m.

24. **Geometry and geography:** The area of an equilateral triangle varies directly as the square of one side. A triangle with sides of 50 yd has an area of 1082.5 yd^2. Find the area in mi^2 of the region bounded by straight lines connecting the cities of Cincinnati, Ohio, Washington, D.C., and Columbia, South Carolina, which are each approximately 400 mi apart.

25. **Galileo and gravity:** The distance an object falls varies directly as the square of the time it has been falling. The cannonballs dropped by Galileo from the Leaning Tower of Pisa fell about 169 ft in 3.25 sec. How long would it take a hammer, accidentally dropped from a height of 196 ft by a bridge repair crew, to splash into the water below? According to the equation, if a camera accidentally fell out of the *News 4 Eye-in-the-Sky* helicopter and hit the ground in 2.75 sec, how high was the helicopter?

26. **Soap bubble surface area:** When a child blows small soap bubbles, they come out in the form of a sphere because the surface tension in the soap seeks to minimize the surface area. The surface area of any sphere varies directly with the square of its radius. A soap bubble with a $\frac{3}{4}$ in. radius has a surface area of approximately 7.07 in^2. What is the radius of a seventeenth-century cannonball that has a surface area of 113.1 in^2? What is the surface area of an orange with a radius of $1\frac{1}{2}$ in.?

Write the variation equation for each statement.

27. The force of gravity varies inversely as the square of the distance between objects.
28. Pressure varies inversely as the area over which it is applied.
29. The safe load of a beam supported at both ends varies inversely as its length.
30. The intensity of sound varies inversely as the square of its distance from the source.

For Exercises 31 through 34, find the constant of variation and write the variation equation. Then use the equation to complete the table or solve the application.

31. Y varies inversely as the square of Z; $Y = 1369$ when $Z = 3$

Z	Y
37	
	2.25
111	

32. A varies inversely with B; $A = 2450$ when $B = 0.8$

B	A
140	
	6.125
560	

33. Gravitational force: The effect of Earth's gravity on an object (its weight) varies inversely as the square of its distance from the center of the planet (assume the Earth's radius is 6400 km). If the weight of an astronaut is 75 kg on Earth (when $r = 6400$), what would this weight be at an altitude of 1600 km *above the surface* of the Earth?

34. Popular running shoes: The demand for a popular new running shoe varies inversely with the price of the shoes. When the wholesale price is set at $45, the manufacturer ships 5500 orders per week to retail outlets. Based on this information, how many orders would be shipped per week if the wholesale price rose to $55?

Write the variation equation for each statement.

35. Interest earned varies jointly with the rate of interest and the length of time on deposit.

36. Horsepower varies jointly as the number of cylinders in the engine and the square of the cylinder's diameter.

37. The area of a trapezoid varies jointly with its height and the sum of the bases.

38. The area of a triangle varies jointly with its base and its height.

39. The volume of metal in a circular coin varies directly with the thickness of the coin and the square of its radius.

40. The electrical resistance in a wire varies directly with its length and inversely as the cross-sectional area of the wire.

For Exercises 41–44, find the constant of variation and write the related variation equation. Then use the equation to complete the table or solve the application.

41. C varies jointly with R and inversely with S squared, and $C = 21$ when $R = 7$ and $S = 1.5$.

R	S	C
120		22.5
200	12.5	
	15	10.5

42. J varies jointly with P and inversely with the square root of Q, and $J = 19$ when $P = 4$ and $Q = 25$.

P	Q	J
47.5		118.75
112	31.36	
	44.89	66.5

43. Kinetic energy: Kinetic energy (energy attributed to motion) varies jointly with the mass of the object and the square of its velocity. Assuming a unit mass of $m = 1$, an object with a velocity of 20 m per sec (m/s) has kinetic energy of 200 J. How much energy is produced if the velocity is increased to 35 m/s?

44. Safe load: The load that a horizontal beam can support varies jointly as the width of the beam, the square of its height, and inversely as the length of the beam. A beam 4 in. wide and 8 in. tall can safely support a load of 1 ton when the beam has a length of 12 ft. How much could a similar beam 10 in. tall safely support?

▶ WORKING WITH FORMULAS

45. Required interest rate: $R(A) = \sqrt[3]{A} - 1$

To determine the simple interest rate R that would be required for each dollar ($1) left on deposit for 3 yr to grow to an amount A, the formula $R(A) = \sqrt[3]{A} - 1$ can be applied. (a) To what function family does this formula belong? (b) Complete the table using a calculator, then use the table to estimate the interest rate required for each $1 to grow to $1.17. (c) Compare your estimate to the value you get by evaluating $R(1.17)$. (d) For $R = \sqrt[3]{A} - 1$, solve for A in terms of R.

Amount A	Rate R
1.0	
1.05	
1.10	
1.15	
1.20	
1.25	

46. **Force between charged particles:** $F = k\frac{Q_1Q_2}{d^2}$
The force between two charged particles is given by the formula shown, where F is the force (in joules—J), Q_1 and Q_2 represent the electrical charge on each particle (in coulombs—C), and d is the distance between them (in meters). If the particles have a like charge, the force is repulsive; if the charges are unlike, the force is attractive. (a) Write the variation equation in words. (b) Solve for k and use the formula to find the electrical constant k, given $F = 0.36\text{J}$, $Q_1 = 2 \times 10^{-6}\,\text{C}$, $Q_2 = 4 \times 10^{-6}\,\text{C}$, and $d = 0.2$ m. Express the result in scientific notation.

▶ APPLICATIONS

Find the constant of variation "*k*" and write the variation equation, then use the equation to solve.

47. **Cleanup time:** The time required to pick up the trash along a stretch of highway varies inversely as the number of volunteers who are working. If 12 volunteers can do the cleanup in 4 hr, how many volunteers are needed to complete the cleanup in just 1.5 hr?

48. **Wind power:** The wind farms in southern California contain wind generators whose power production varies directly with the cube of the wind's speed. If one such generator produces 1000 W of power in a 25 mph wind, find the power it generates in a 35 mph wind.

49. **Pull of gravity:** The weight of an object on the moon varies directly with the weight of the object on Earth. A 96-kg object on Earth would weigh only 16 kg on the moon. How much would a fully suited 250-kg astronaut weigh on the moon?

50. **Period of a pendulum:** The time that it takes for a simple pendulum to complete one period (swing over and back) varies directly as the square root of its length. If a pendulum 20 ft long has a period of 5 sec, find the period of a pendulum 30 ft long.

51. **Stopping distance:** The stopping distance of an automobile varies directly as the square root of its speed when the brakes are applied. If a car requires 108 ft to stop from a speed of 25 mph, estimate the stopping distance if the brakes were applied when the car was traveling 45 mph.

52. **Supply and demand:** A chain of hardware stores finds that the demand for a special power tool varies inversely with the advertised price of the tool. If the price is advertised at \$85, there is a monthly demand for 10,000 units at all participating stores. Find the projected demand if the price were lowered to \$70.83.

53. **Cost of copper tubing:** The cost of copper tubing varies jointly with the length and the diameter of the tube. If a 36-ft spool of $\frac{1}{4}$-in.-diameter tubing costs \$76.50, how much does a 24-ft spool of $\frac{3}{8}$-in.-diameter tubing cost?

54. **Electrical resistance:** The electrical resistance of a copper wire varies directly with its length and inversely with the square of the diameter of the wire. If a wire 30 m long with a diameter of 3 mm has a resistance of 25 Ω, find the resistance of a wire 40 m long with a diameter of 3.5 mm.

55. **Volume of phone calls:** The number of phone calls per day between two cities varies directly as the product of their populations and inversely as the square of the distance between them. The city of Tampa, Florida (pop. 300,000), is 430 mi from the city of Atlanta, Georgia (pop. 420,000). Telecommunications experts estimate there are about 300 calls per day between the two cities. Use this information to estimate the number of daily phone calls between Amarillo, Texas (pop. 170,000), and Denver, Colorado (pop. 550,000), which are also separated by a distance of about 430 mi. Note: Population figures are for the year 2000 and rounded to the nearest ten-thousand.
Source: 2005 World Almanac, p. 626.

56. **Internet commerce:** The likelihood of an eBay® item being sold for its "Buy it Now®" price P, varies directly with the feedback rating of the seller, and inversely with the cube of $\frac{P}{MSRP}$, where MSRP represents the manufacturer's suggested retail price. A power eBay® seller with a feedback rating of 99.6%, knows she has a 60% likelihood of selling an item at 90% of the MSRP. What is the likelihood a seller with a 95.3% feedback rating can sell the same item at 95% of the MSRP?

57. **Volume of an egg:** The volume of an egg laid by an average chicken varies jointly with its length and the square of its width. An egg measuring 2.50 cm wide and 3.75 cm long has a volume of 12.27 cm^3. A Barret's Blue Ribbon hen can lay an egg measuring 3.10 cm wide and 4.65 cm long. (a) What is the volume of this egg? (b) As a percentage, how much greater is this volume than that of an average chicken's egg?

58. Athletic performance: Researchers have estimated that a sprinter's time in the 100-m dash varies directly as the square root of her age and inversely as the number of hours spent training each week. At 20 yr old, Gail trains 10 hr per week (hr/wk) and has an average time of 11 sec. Assuming she continues to train 10 hr/wk, (a) what will her average time be at 30 yr old? (b) If she wants to keep her average time at 11 sec, how many hours per week should she train?

59. Maximum safe load: The maximum safe load M that can be placed on a uniform horizontal beam supported at both ends varies directly as the width w and the square of the height h of the beam's cross section, and inversely as its length L (width and height are assumed to be in inches, and length in feet). (a) Write the variation equation. (b) If a beam 18 in. wide, 2 in. high, and 8 ft long can safely support 270 lb, what is the safe load for a beam of like dimensions with a length of 12 ft?

60. Maximum safe load: Suppose a 10-ft wooden beam with dimensions 4 in. by 6 in. is made from the same material as the beam in Exercise 59 (the same k value can be used). (a) What is the maximum safe load if the beam is placed so that width is 6 in. and height is 4 in.? (b) What is the maximum safe load if the beam is placed so that width is 4 in. and height is 6 in.?

▶ EXTENDING THE CONCEPT

61. The gravitational force F between two celestial bodies varies jointly as the product of their masses and inversely as the square of the distance d between them. The relationship is modeled by Newton's law of universal gravitation: $F = k\frac{m_1m_2}{d^2}$. Given that $k = 6.67 \times 10^{-11}$, what is the gravitational force exerted by a 1000-kg sphere on another identical sphere that is 10 m away?

62. The intensity of light and sound both vary inversely as the square of their distance from the source.

a. Suppose you're relaxing one evening with a copy of *Twelfth Night* (Shakespeare), and the reading light is placed 5 ft from the surface of the book. At what distance would the intensity of the light be twice as great?

b. *Tamino's Aria* (*The Magic Flute*—Mozart) is playing in the background, with the speakers 12 ft away. At what distance from the speakers would the intensity of sound be three times as great?

▶ MAINTAINING YOUR SKILLS

63. (Appendix A.2) Evaluate: $\left(\frac{2x^4}{3x^3y}\right)^{-2}$

64. (Appendix A.4) Solve: $x^3 + 6x^2 + 8x = 0$.

65. (2.4) State the domains of f and g given here:

a. $f(x) = \frac{x - 3}{x^2 - 16}$

b. $g(x) = \frac{x - 3}{\sqrt{x^2 - 16}}$

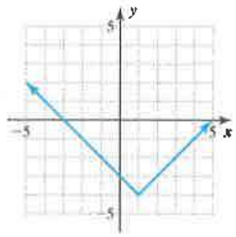
66. (2.3) Graph by using transformations of the parent function and plotting a minimum number of points: $f(x) = -2|x - 3| + 5$.

MAKING CONNECTIONS

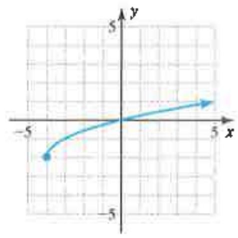
Making Connections: Graphically, Symbolically, Numerically, and Verbally

Eight graphs (a) through (h) are given. Match the characteristics shown in 1 through 16 to one of the eight graphs.

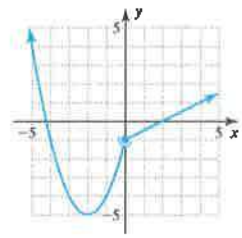
(a)



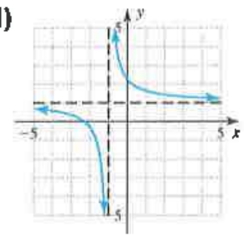
(b)



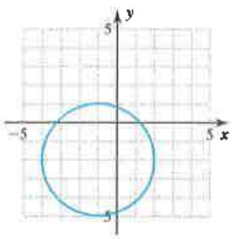
(c)



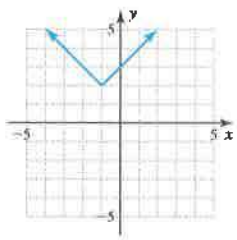
(d)



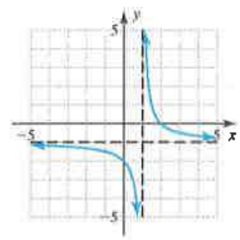
(e)



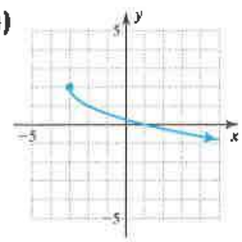
(f)



(g)



(h)



1. ____ domain: $x \in (-\infty, 1) \cup (1, \infty)$
2. ____ $y = \sqrt{x+4} - 2$
3. ____ $f(x)\uparrow$ for $x \in (1, \infty)$
4. ____ horizontal asymptote at $y = -1$
5. ____ $y = \dfrac{1}{x+1} + 1$
6. ____ domain: $x \in [-4, 2]$
7. ____ $y = |x - 1| - 4$
8. ____ $f(x) \leq 0$ for $x \in [1, \infty)$
9. ____ domain: $x \in [-4, \infty)$
10. ____ $f(-3) = -1, f(5) = 1$
11. ____ basic function is shifted 3 units left, reflected across x-axis, then shifted up 2 units
12. ____ basic function is shifted 1 unit left, 2 units up
13. ____ $f(-3) = -4, f(2) = 0$
14. ____ as $x \to \infty, y \to 1$
15. ____ $f(x) > 0$ for $x \in (-\infty, \infty)$
16. ____ $y = \begin{cases} (x+2)^2 - 5, & x < 0 \\ \frac{1}{2}x - 1, & x \geq 0 \end{cases}$

SUMMARY AND CONCEPT REVIEW

SECTION 2.1 Analyzing the Graph of a Function

KEY CONCEPTS

- A function f is even (symmetric to the y-axis), if and only if when a point (x, y) is on the graph, then $(-x, y)$ is also on the graph. In function notation: $f(-x) = f(x)$.
- A function f is odd (symmetric to the origin), if and only if when a point (x, y) is on the graph, then $(-x, -y)$ is also on the graph. In function notation: $f(-x) = -f(x)$.

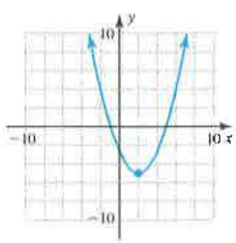
Intuitive descriptions of the characteristics of a graph are given here. The formal definitions can be found within Section 2.1.

- A function is *increasing* in an interval if the graph rises from left to right (larger inputs produce larger outputs).
- A function is *decreasing* in an interval if the graph falls from left to right (larger inputs produce smaller outputs).
- A function is *positive* in an interval if the graph is above the x-axis in that interval.
- A function is *negative* in an interval if the graph is below the x-axis in that interval.
- A function is *constant* in an interval if the graph is parallel to the x-axis in that interval.
- A maximum value can be a *local* maximum, or *global* maximum. An *endpoint* maximum can occur at the endpoints of the domain. Similar statements can be made for minimum values.

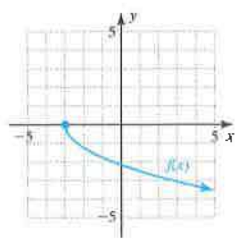
EXERCISES

State the domain and range for each function $f(x)$ given. Then state the intervals where f is increasing or decreasing and intervals where f is positive or negative. Assume all endpoints have integer values.

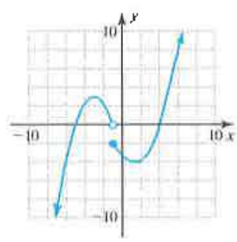
1.



2.



3.



4. Determine whether the following are even $[f(-k) = f(k)]$, odd $[f(-k) = -f(k)]$, or neither.

   a. $f(x) = 2x^5 - \sqrt[3]{x}$ b. $g(x) = x^4 - \dfrac{\sqrt[3]{x}}{x}$

   c. $p(x) = |3x| - x^3$ d. $q(x) = \dfrac{x^2 - |x|}{x}$

5. Draw the function f that has all of the following characteristics, then name the zeroes of the function and the location of all local maximum and minimum values. [*Hint:* Write them in the form $(c, f(c))$.]

   a. Domain: $x \in [-6, 10)$ b. Range: $y \in [-8, 6)$

   c. $f(0) = 0$ d. $f(x)\downarrow$ for $x \in (-6, -3) \cup (3, 7.5)$

   e. $f(x)\uparrow$ for $x \in (-3, 3) \cup (7.5, 10)$ f. $f(x) < 0$ for $x \in (-6, 0) \cup (6, 9)$

   g. $f(x) > 0$ for $x \in (0, 6) \cup (9, 10)$

6. Use a graphing calculator to find the maximum and minimum values of $f(x) = 2x^5 - \sqrt[3]{x}$. Round to the nearest hundredth.

SECTION 2.2 The Toolbox Functions and Transformations

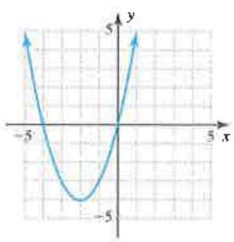
KEY CONCEPTS

- The *toolbox functions* and graphs commonly used in mathematics are
  - the identity function $f(x) = x$
  - squaring function: $f(x) = x^2$
  - square root function: $f(x) = \sqrt{x}$
  - absolute value function: $f(x) = |x|$
  - cubing function: $f(x) = x^3$
  - cube root function: $f(x) = \sqrt[3]{x}$
- For a basic or parent function $y = f(x)$, the general equation of the transformed function is $y = af(x \pm h) \pm k$. For any function $y = f(x)$ and $h, k > 0$,
  - the graph of $y = f(x) + k$ is the graph of $y = f(x)$ shifted upward k units
  - the graph of $y = f(x) - k$ is the graph of $y = f(x)$ shifted downward k units
  - the graph of $y = f(x + h)$ is the graph of $y = f(x)$ shifted left h units
  - the graph of $y = f(x - h)$ is the graph of $y = f(x)$ shifted right h units
  - the graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected across the x-axis
  - the graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected across the y-axis
  - $y = af(x)$ results in a vertical stretch when $a > 1$
  - $y = af(x)$ results in a vertical compression when $0 < a < 1$
- Transformations are applied in the following order: (1) horizontal shifts, (2) reflections, (3) stretches or compressions, and (4) vertical shifts.

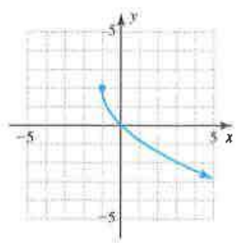
EXERCISES

Identify the function family for each graph given, then (a) describe the end-behavior; (b) name the x- and y-intercepts; (c) identify the vertex, initial point, or point of inflection (as applicable); and (d) state the domain and range.

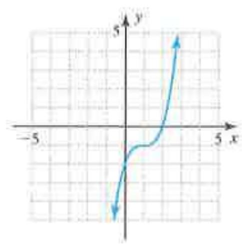
7.



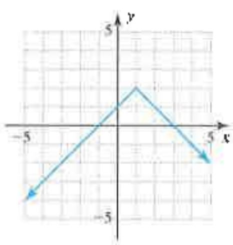
8.



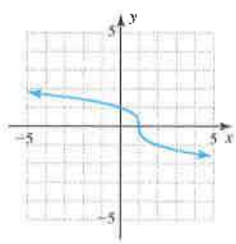
9.



10.



11.



Identify each function as belonging to the linear, quadratic, square root, cubic, cube root, or absolute value family. Then sketch the graph using shifts of a parent function and a few characteristic points.

12. $f(x) = -(x + 2)^2 - 5$

13. $f(x) = 2|x + 3|$

14. $f(x) = x^3 - 1$

15. $f(x) = \sqrt{x - 5} + 2$

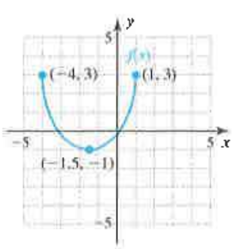
16. $f(x) = \sqrt[3]{x + 2}$

17. Apply the transformations indicated for the graph of $f(x)$ given.

a. $f(x - 2)$

b. $-f(x) + 4$

c. $\frac{1}{2}f(x)$



SECTION 2.3 Absolute Value Functions, Equations, and Inequalities

KEY CONCEPTS

- To solve absolute value equations and inequalities, begin by writing the equation in simplified form, with the absolute value isolated on one side.
- If X and Y represent algebraic expressions and k is a nonnegative constant:
  - Absolute value equations: $|X| = k$ is equivalent to $X = -k$ or $X = k$; $|X| = |Y|$ is equivalent to $X = Y$ or $X = -Y$
  - "Less than" inequalities: $|X| < k$ is equivalent to $-k < X < k$
  - "Greater than" inequalities: $|X| > k$ is equivalent to $X < -k$ or $X > k$
- These properties also apply when the symbols "≤" or "≥"are used.
- If the absolute value quantity has been isolated on the left, the solution to a less-than inequality will be a single interval, while the solution to a greater-than inequality will consist of two disjoint intervals.
- The multiplicative property states that for algebraic expressions A and B, $|AB| = |A||B|$.
- Absolute value equations and inequalities can be solved graphically using the intersect method *or* the zeroes/x-intercept method.

EXERCISES

Solve each equation or inequality. Write solutions to inequalities in interval notation.

18. $7 = |x - 3|$

19. $-2|x + 2| = -10$

20. $|-2x + 3| = 13$

21. $\dfrac{|2x + 5|}{3} + 8 = 9$

22. $-3|x + 2| - 2 < -14$

23. $\left|\dfrac{x}{2} - 9\right| \leq 7$

24. $|3x + 5| = -4$

25. $3|x + 1| < -9$

26. $2|x + 1| > -4$

27. $5|m - 2| - 12 \leq 8$

28. $\dfrac{|3x - 2|}{2} + 6 \geq 10$

29. Monthly rainfall received in Omaha, Nebraska, rarely varies by more than 1.7 in. from an average of 2.5 in. per month. (a) Use this information to write an absolute value inequality model, then (b) solve the inequality to find the highest and lowest amounts of monthly rainfall for this city.

SECTION 2.4 Basic Rational Functions and Power Functions; More on the Domain

KEY CONCEPTS

- A rational function is one of the form $V(x) = \dfrac{p(x)}{d(x)}$, where p and d are polynomials and $d(x) \neq 0$.
- The most basic rational functions are the reciprocal function $f(x) = \dfrac{1}{x}$ and the reciprocal square function $g(x) = \dfrac{1}{x^2}$.
- The line $y = k$ is a horizontal asymptote of V if as $|x|$ increases without bound, $V(x)$ approaches k: as $|x| \to \infty$, $V(x) \to k$.
- The line $x = h$ is a vertical asymptote of V if as x approaches h, $V(x)$ increases/decreases without bound: as $x \to h$, $|V(x)| \to \infty$.
- The reciprocal and reciprocal square functions can be transformed using the same shifts, stretches, and reflections as applied to other basic functions, with the asymptotes also shifted.
- A power function can be written in the form $f(x) = x^p$ where p is a constant real number and x is a variable. If $p = \dfrac{1}{n}$, where n is a natural number, $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ is called a root function in x.
- Given the rational exponent $\frac{m}{n}$ is in simplest form, the domain of $f(x) = x^{\frac{m}{n}}$ is $(-\infty, \infty)$ if n is odd, and $[0, \infty)$ if n is even.

EXERCISES

Sketch the graph of each function using shifts of the parent function (not by using a table of values). Find and label the x- and y-intercepts (if they exist) and redraw the asymptotes.

30. $f(x) = \dfrac{1}{x+2} - 1$

31. $h(x) = \dfrac{-1}{(x-2)^2} - 3$

32. In a certain county, the cost to keep public roads free of trash is given by $C(p) = \dfrac{-7500}{p - 100} - 75$, where $C(p)$ represents the cost (thousands of dollars) to keep p percent of the trash picked up. (a) Find the cost to pick up 30%, 50%, 70%, and 90% of the trash, and comment on the results. (b) Sketch the graph using the transformation of a toolbox function. (c) Use mathematical notation to describe what happens if the county tries to keep 100% of the trash picked up.

33. Use a graphing calculator to graph the functions $f(x) = x^1$, $g(x) = x^{\frac{1}{2}}$, and $h(x) = x^{\pi}$ in the same viewing window. What is the domain of each function?

34. The expression $T = \dfrac{2\pi}{37{,}840} r^{\frac{3}{2}}$ models the time T (in hr) it takes for a satellite to complete one revolution around the Earth, where r represents the radius (in km) of the orbit measured from the center of the Earth. If the Earth has a radius of 6370 km, (a) how long does it takes for a satellite at a height of 200 km to complete one orbit? (b) What is the orbital height of a satellite that completes one revolution in 4 days (96 hr)?

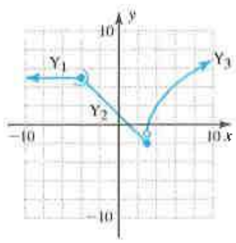
SECTION 2.5 Piecewise-Defined Functions

KEY CONCEPTS

- Each piece of a piecewise-defined function has a domain over which that piece is defined.
- To evaluate a piecewise-defined function, identify the domain interval containing the input value, then use the piece of the function corresponding to this interval.
- To graph a piecewise-defined function you can plot points, or graph each piece in its entirety, then erase portions of the graph outside the domain indicated for each piece.
- If the graph of a function can be drawn without lifting your pencil from the paper, the function is continuous.
- A discontinuity is said to be removable if we can redefine the function to "fill the hole."
- Step functions are discontinuous and formed by a series of horizontal steps.
- The floor function $\lfloor x \rfloor$ gives the largest integer less than or equal to x.
- The ceiling function $\lceil x \rceil$ is the smallest integer greater than or equal to x.

EXERCISES

35. For the graph and functions given, (a) use the correct notation to write the relation as a single piecewise-defined function, stating the effective domain for each piece by inspecting the graph; and (b) state the range of the function: $Y_1 = 5$, $Y_2 = -X + 1$, $Y_3 = 3\sqrt{X - 3} - 1$.



36. Use a table of values as needed to graph $h(x)$, then state its domain and range. If the function has a pointwise (removable) **discontinuity,** state how the second piece could be redefined so that a continuous function results.

$$h(x) = \begin{cases} \dfrac{x^2 - 2x - 15}{x + 3}, & x \neq -3 \\ -6, & x = -3 \end{cases}$$

37. Evaluate the piecewise-defined function $p(x)$: $p(-4)$, $p(-2)$, $p(2.5)$, $p(2.99)$, $p(3)$, and $p(3.5)$

$$p(x) = \begin{cases} -4, & x < -2 \\ -\lfloor x \rfloor - 2, & -2 \leq x < 3 \\ 3\sqrt{x} - 9, & x \geq 3 \end{cases}$$

38. Sketch the graph of the function and state its domain and range. Use transformations of the toolbox functions where possible.

$$q(x) = \begin{cases} 2\sqrt{-x-3} - 4, & x \leq -3 \\ -2|x| + 2, & -3 < x < 3 \\ 2\sqrt{x-3} - 4, & x \geq 3 \end{cases}$$

39. Many home improvement outlets now rent flatbed trucks in support of customers that purchase large items. The cost is \$20 per hour for the first 2 hr, \$30 for the next 2 hr, then \$40 for each hour afterward. Write this information as a piecewise-defined function, then sketch its graph. What is the total cost to rent this truck for 5 hr?

SECTION 2.6 Variation: The Toolbox Functions in Action

KEY CONCEPTS

- *Direct variation:* If there is a nonzero constant k such that $y = kx$, we say, "y varies directly with x" or "y is directly proportional to x" (k is called the constant of variation).
- *Inverse variation:* If there is a nonzero constant k such that $y = k\left(\frac{1}{x}\right)$ we say, "y varies inversely with x" or y is inversely proportional to x.
- In some cases, direct and inverse variations work simultaneously to form a *joint variation.*
- The process for solving variation applications can be found on page 178.

EXERCISES

Find the constant of variation and write the equation model, then use this model to complete the table.

40. y varies directly as the cube root of x; $y = 52.5$ when $x = 27$.

x	y
216	
	12.25
729	

41. z varies directly as v and inversely as the square of w; $z = 1.62$ when $w = 8$ and $v = 144$.

v	w	z
196	7	
	1.25	17.856
24		48

42. Given t varies jointly with u and v, and inversely as w, if $t = 30$ when $u = 2$, $v = 3$, and $w = 5$, find t when $u = 8$, $v = 12$, and $w = 15$.

43. The time that it takes for a simple pendulum to complete one period (swing over and back) is directly proportional to the square root of its length. If a pendulum 16 ft long has a period of 3 sec, find the time it takes for a 36-ft pendulum to complete one period.

PRACTICE TEST

1. Determine the following from the graph shown.

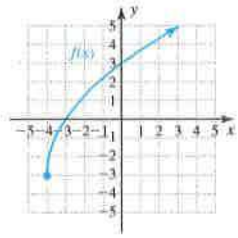
a. the domain and range

b. estimate the value of $f(-1)$

c. interval(s) where $f(x)$ is negative or positive

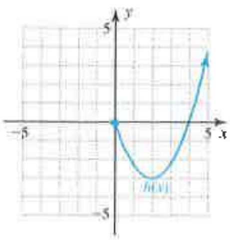
d. interval(s) where $f(x)$ is increasing, decreasing, or constant.

e. an equation for $f(x)$

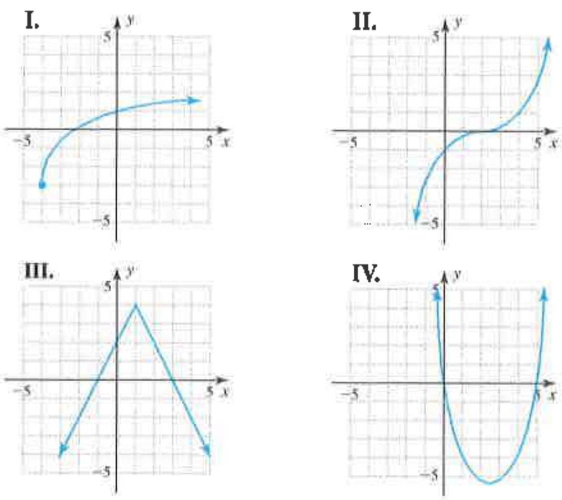


For the function $h(x)$ whose partial graph is given.

2. Complete the graph if h is known to be even.
3. Complete the graph if h is known to be odd.



4. Use a graphing calculator to find the maximum and minimum values of $f(x) = |x^2 + 4x - 11| - 7$. Round answers to nearest hundredth when necessary.
5. Each function graphed here is from a toolbox function family. For each graph, (a) identify the function family, (b) state the domain and range, (c) identify x- and y-intercepts, (d) discuss the end-behavior, and (e) solve the inequalities $f(x) > 0$ and $f(x) < 0$.



Sketch each graph using a transformation.

6. $f(x) = |x - 2| + 3$
7. $g(x) = -(x + 3)^2 - 2$

Solve each inequality. Write the solutions in interval notation.

8. $\frac{2}{3}|3x - 1| > 14$ 9. $5 - 2|x + 2| \geq 1$

10. Use a graphing calculator to solve the equation.

$$1.7|x - 0.75| + 3 = 3 - \frac{3}{5}\left|x - \frac{3}{4}\right|$$

11. Sketch the graph of $f(x) = \frac{2}{x + 3}$. Find and label the x- and y-intercepts, if they exist, along with all asymptotes.

12. After the engine is cut at $t = 0$, a boat coasts for a while before stopping. The distance D it travels over t sec can be modeled by $D(t) = 60 - \frac{240}{(t + 2)^2}$. If the boat comes to a complete stop after traveling 59 ft, use a graphing calculator to determine the time required to stop. Round your answer to the nearest tenth.

13. Find the domains of the following functions.
    a. $f(x) = 2.09x^{\frac{2}{5}}$ b. $g(x) = -4.22x^{\frac{5}{2}}$
    c. $h(t) = 4.5t^{\pi}$

14. Identify the vertical and horizontal asymptotes of $g(x) = \frac{3}{(x + 2)^2} - 1$.

15. Using time-lapse photography, the spread of a liquid is tracked in one-fifth of a second intervals, as a small amount of liquid is dropped on a piece of fabric. A power function provides a reasonable model for the first second. Use a graphing calculator to (a) graph a scatterplot of the data and (b) find an equation model using a power regression (round to two decimal places). Use the equation to estimate (c) the size of the stain after 0.5 sec and (d) how long it will take the stain to reach a size of 15 mm.

Time (sec)	Size (mm)
0.2	0.39
0.4	1.27
0.6	3.90
0.8	10.60
1	21.50

16. The following function has two removable discontinuities. Find the values of a and b so that a continuous function results.

$$g(x) = \begin{cases} \frac{x^3 + x^2 - 4x - 4}{x^2 - x - 2}, & x \neq -1, 2 \\ a, & x = -1 \\ b, & x = 2 \end{cases}$$

17. The annual output of a wind turbine varies jointly with the square of the blade diameter and the cube of the average wind speed. If a 10-ft-diameter turbine in 12 mph average winds produces 2300 KWH/year, how much will a 6 ft-diameter turbine produce in 15 mph average winds? Round to the nearest KWH/year.

18. Given $h(x) = \begin{cases} 4, & x < -2 \\ 2x, & -2 \leq x \leq 2 \\ x^2, & x > 2 \end{cases}$
    a. Find $h(-3)$, $h(-2)$, and $h(\frac{5}{2})$
    b. Sketch the graph of h. Label important points.

19. By observing a significantly smaller object orbiting a large celestial body, astronomers can easily determine the mass of the larger. Appealing to Kepler's third law of planetary motion, we know the mass of the large body varies directly with the cube of the mean distance to the smaller and inversely with the square of its orbital period. Write the variation equation. Using the mean Earth/Sun distance of 1.496×10^8 km and the Earth's orbital period of 1 yr, the mass of the Sun has been calculated to be 1.98892×10^{30} kg. Given the orbital period of Mars is 1.88 yr, find its mean distance from the Sun.

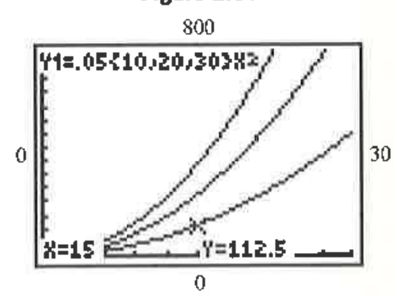
20. The maximum load that can be supported by a rectangular beam varies jointly with its width and its height squared and inversely with its length. If a beam 10 ft long, 3 in. wide, and 4 in. high can support 624 lb, how many pounds could a 12-ft-long beam with the same dimensions support?

CALCULATOR EXPLORATION AND DISCOVERY

Studying Joint Variations

Although a graphing calculator is limited to displaying the relationship between only two variables (for the most part), it has a feature that enables us to see how these two are related with respect to a third. Consider the variation equation from Example 8 in Section 2.6: $F = 0.05dv^2$. If we want to investigate the relationship between fuel consumption and velocity, we can have the calculator display multiple versions of the relationship *simultaneously for different values of d*. This is accomplished using the "{" and "}" symbols, which are 2nd functions to the parentheses. When the calculator sees values between these grouping symbols and separated by commas, it is programmed to use each value independently of the others, graphing or evaluating the relation for each value in the set. We illustrate by graphing the relationship $f = 0.05dv^2$ for three different values of d. Enter the equation on the Y= screen as $Y_1 = 0.05\{10, 20, 30\}X^2$, which tells the calculator to graph the equations $Y_1 = 0.05(10)X^2$, $Y_1 = 0.05(20)X^2$, and $Y_1 = 0.05(30)X^2$ on the same grid. Note that since d is constant, each graph is a parabola. Set the viewing window using the values given in Example 8 as a guide. The result is the graph shown in Figure 2.97, where we can study the relationship between these three variables using the up ▲ and down ▼ arrows. From our work with the toolbox functions and transformations, we know the widest parabola used the coefficient "10," while the narrowest parabola used the coefficient "30." As shown, the graph tells us that at a speed of 15 nautical miles per hour ($X = 15$), it will take 112.5 barrels of fuel to travel 10 mi (the first number in the list). After pressing the ▼ key, the cursor jumps to the second curve, which shows values of $X = 15$ and $Y = 225$. This means at 15 nautical miles per hour, it would take 225 barrels of fuel to travel 20 mi. Use these ideas to complete the following exercises:

Figure 2.97



Exercise 1: The comparison of distance covered versus fuel consumption at different speeds also makes an interesting study. This time velocities are constant values and the distance varies. On the Y= screen, enter $Y_1 = 0.05x\{10, 20, 30\}^2$. What family of equations results? Use the up/down arrow keys for $x = 15$ (a distance of 15 mi) to find how many barrels of fuel it takes to travel 15 mi at 10 mph, 15 mi at 20 mph, and 15 mi at 30 mph. Comment on what you notice.

Exercise 2: The maximum safe load S for a wooden horizontal plank supported at both ends varies jointly with the width W of the beam, the square of its thickness T, and inversely with its length L. A plank 10 ft long, 12 in. wide, and 1 in. thick will safely support 450 lb. Find the value of k and write the variation equation, then use the equation to explore:

a. Safe load versus thickness for a constant width and given lengths (quadratic function). Use $w = 8$ in. and $\{8, 12, 16\}$ for L.

b. Safe load versus length for a constant width and given thickness (reciprocal functional). Use $w = 8$ in. and $\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ for thickness.

STRENGTHENING CORE SKILLS

Variation and Power Functions: $y = kx^p$

You may have noticed that applications of power functions (Section 2.4) can also be stated as variations (Section 2.6): From the general equation shown in the title of this feature, "... y varies directly as x to the p power." Due to the nature of real data and data collection, applications of power functions based on regression yield values of k (the constant of variation) and p (the power) that cannot be written in exact form. However, "fixed" relationships modeled by power functions produce values of k and p that can be written in exact form. For instance, the power function that models planetary orbits states: The time T it takes a planet to complete one orbit varies directly with its orbital radius to the three-halves power: $T = kR^{\frac{3}{2}}$. Here, the power is exactly $\frac{3}{2}$ and the constant of variation turns out to be exactly 1 (also **see Section 2.4, Exercise 78**). Many times, finding this constant takes more effort, and utilizes the skills developed in this and previous chapters. Consider the following.

Illustration 1 ▸ The volume V of a sphere varies directly with its surface area S to the three-halves power. If the volume is approximately 33.51 cm^3 when the surface area is 50.30 cm^2, (a) find the constant of variation yielded by these values (round to two decimal places), and (b) find the exact constant of variation dictated by the geometry of the sphere and write the variation equation.

Solution ▸ a.

$$V = kS^{\frac{3}{2}} \qquad \text{variation equation}$$

$$33.51 = k(50.30)^{\frac{3}{2}} \qquad \text{substitute 33.51 for } V\text{, 50.30 for } S$$

$$\frac{33.51}{(50.30)^{\frac{3}{2}}} = k \qquad \text{solve for } k$$

$$0.09 \approx k \qquad \text{approximate value for } k$$

This gives $V \approx 0.09S^{\frac{3}{2}}$ as an approximate relationship.

b. To find the true constant of variation fixed by the nature of spheres, we begin with the same set up, but *substitute the actual formulas* for volume and surface area, then simplify.

$$V = kS^{\frac{3}{2}} \qquad \text{variation equation}$$

$$\frac{4}{3}\pi r^3 = k(4\pi r^2)^{\frac{3}{2}} \qquad \text{substitute } \frac{4}{3}\pi r^3 \text{ for } V\text{, } 4\pi r^2 \text{ for } S$$

$$\frac{4}{3}\pi r^3 = 8\pi^{\frac{3}{2}}r^3k \qquad \text{properties of exponents: } 4^{\frac{3}{2}} = 8$$

$$\pi = 6\pi^{\frac{3}{2}}k \qquad \text{multiply by } \frac{3}{4}\text{, divide by } r^3$$

$$\frac{\pi}{6\pi^{\frac{3}{2}}} = k \qquad \text{solve for } k$$

$$\frac{1}{6\pi^{\frac{1}{2}}} = k \qquad \text{simplify (exact form)}$$

The constant of variation for this relationship is $\frac{1}{6\sqrt{\pi}}$, giving a variation equation of $V = \frac{1}{6\sqrt{\pi}}S^{\frac{3}{2}}$. Note that $\frac{1}{6\sqrt{\pi}} \approx 0.09$.

Studies of this type are important, because as the radius of the sphere gets larger, so does the error generated by using an approximate value. Using a radius of $r = 18$ cm with the approximate relationship $[V \approx 0.09\,(4\pi 18^2)^{3/2}]$, gives a volume of near 23,381.6 cm^3, while the volume found using the exact value for k is about 24,429.0 cm^3.

Exercise 1: Use this *Strengthening Core Skills* feature to find the exact constant of variation for the following relationship: The volume of a cube varies directly with its surface area to the three-halves power.

CUMULATIVE REVIEW CHAPTERS 1–2

1. Given $f(x) = 2x^3 + 4x^2 + 8x - 7$, find $f(-2)$ and $f\left(\frac{1}{2}\right)$.

2. Find the solution set for: $2 - x < 5$ **and** $3x + 2 < 8$.

3. The area of a circle is 69 cm^2. Find the circumference of the same circle.

4. The surface area of a cylinder is $A = 2\pi r^2 + 2\pi rh$. Write r in terms of A and h (solve for r).

5. Solve for x: $-2(3 - x) + 5x = 4(x + 1) - 7$.

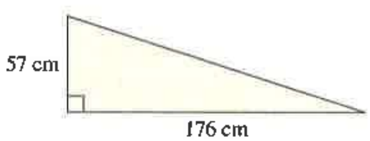
6. Evaluate without using a calculator: $\left(\frac{27}{8}\right)^{-\frac{2}{3}}$.

7. Find the slope of each line:
   a. through the points: $(-4, 7)$ and $(2, 5)$.
   b. the line with equation $3x - 5y = 20$.

8. Graph using transformations of a parent function.
   a. $f(x) = \sqrt{x - 2} + 3$.
   b. $f(x) = -|x + 2| - 3$.

9. Graph the line passing through $(-3, 2)$ with a slope of $m = \frac{1}{2}$, then state its equation.

10. Find (a) the length of the hypotenuse and (b) the perimeter of the triangle shown.

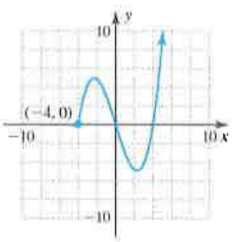


11. Sketch the graph of $h(x) = \dfrac{-1}{(x - 1)^2} + 3$ using a transformation of the parent function.

12. Graph by plotting the y-intercept, then counting $\dfrac{\Delta y}{\Delta x}$ to find additional points: $y = \frac{1}{3}x - 2$

13. Graph the piecewise-defined function
$$f(x) = \begin{cases} x^2 - 4, & x < 2 \\ x - 1, & 2 \le x \le 8 \end{cases}$$ and determine the following:
    a. the domain and range
    b. the value of $f(-3)$, $f(-1)$, $f(1)$, $f(2)$, and $f(3)$
    c. the zeroes of the function
    d. interval(s) where $f(x)$ is negative/positive
    e. location of any local max/min values
    f. interval(s) where $f(x)$ is increasing/decreasing

14. The graph of a function $h(x)$ is shown. (a) State the domain and estimate the range of h. (b) What are the zeroes of the function? (c) What is the value of $h(-1)$? (d) If $h(k) = 9$ what is the value of k?

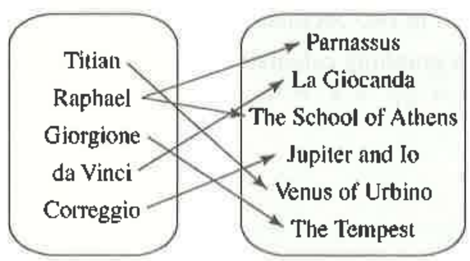


15. Add the rational expressions:
    a. $\dfrac{-2}{x^2 - 3x - 10} + \dfrac{1}{x + 2}$
    b. $\dfrac{b^2}{4a^2} - \dfrac{c}{a}$

16. Simplify the radical expressions:
    a. $\dfrac{-10 + \sqrt{72}}{4}$
    b. $\dfrac{1}{\sqrt{2}}$

17. Perform the division by factoring the numerator: $(x^3 - 5x^2 + 2x - 10) \div (x - 5)$.

18. Determine if the following relation is a function. If not, how is the definition of a function violated?



19. Find the center and radius of the circle defined by $x^2 + 6x + y^2 - 12y + 36 = 0$.

20. The amount of pressure (in pounds per square inch—psi) felt by a professional pearl diver as she dives to harvest oysters, is 14.7 more than 0.43 times the depth of the dive (in feet). Write the equation model for this situation. If the oyster bed is at a depth of 60 ft, how much pressure is felt?

21. The *National Geographic Atlas of the World* is a very large, rectangular book with an almost inexhaustible panoply of information about the world we live in. The length of the front cover is 16 cm more than its width, and the area of the cover is 1457 cm^2. Use this information to write an equation model, then use the quadratic formula to determine the length and width of the *Atlas*.

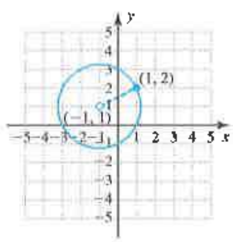
22. During a table tennis tournament, the championship game between J.W. and Mike took a dramatic and unexpected turn. At one point in the game, J.W. was losing 5–15. Facing a crushing loss, he summoned all his willpower and battled on to a 21–19 victory! Assuming the game score relationship is linear, find the slope of the line between these two scores and discuss its meaning in this context.

23. Solve by factoring:

a. $6x^2 - 7x = 20$

b. $x^3 + 5x^2 - 15 = 3x$

24. A theorem from elementary geometry states, "*A line tangent to a circle is perpendicular to the radius at the point of tangency.*" Find the equation of the tangent line for the circle and radius shown.



25. A triangle has its vertices at $(-4, 5)$, $(4, -1)$, and $(0, 8)$. Find the perimeter of the triangle and determine whether or not it is a *right* triangle.

Exercises 26 through 30 require the use of a graphing calculator.

26. Use the zeroes method to solve the equation $2.7(x - 3) + 0.3 = 1.8 - 1.2(x + 4)$. Round your answer to two decimal places.

27. Use a graphing calculator to graph the circle defined by $(x + 2)^2 + y^2 = 4$.

28. Use a graphing calculator and the intersection-of-graphs method to solve the inequality. $|1.2(x - 0.5)| < 0.4x + 1.4$

29. Graph the following piecewise-defined function using a graphing calculator. Then use the TRACE command to evaluate the function at $x = 1.2$.

$$f(x) = \begin{cases} -x + 2 & -5 \le x < -2 \\ (x - 2)^2 - 3 & x \ge 1 \end{cases}$$

30. The data given shows the growth of the total U.S. National Debt (in billions) for the years 1993 to 1999 (1993 → 1), and for the years 2001 to 2007 (2001 → 1), for each set of data, enter the data into a graphing calculator, then

Year (scaled)	1993 to 1999	2001 to 2007
1	4.5	5.9
2	4.8	6.4
3	5.0	7.0
4	5.3	7.6
5	5.5	8.2
6	5.6	8.7
7	5.8	9.2

a. Set an appropriate window for viewing the scatterplots, and determine if the associations are linear or nonlinear.

b. If linear, find the regression equation for each data set and graph both (as Y_1 and Y_2) in the same window (round to two decimal places).

c. During which 7-year period did the national debt increase faster? How much faster?

CONNECTIONS TO CALCULUS

Chapter 2 actually highlights numerous concepts and skills that transfer directly into a study of calculus. In the *Connections to Calculus* introduction (page 105), we noted that analyzing very small differences is one such skill, with this task carried out using the absolute value concept. The ability to solve a wide variety of equation types will also be a factor of your success in calculus. Here we'll explore how these concepts and skills are "connected."

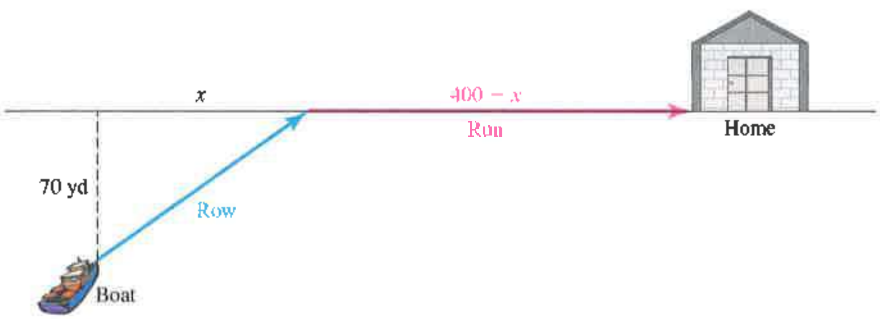
Solving Various Types of Equations

The need to solve equations of various types occurs frequently in both *differential* and *integral* calculus, and the required skills will span a broad range of your algebraic experience. Here we'll solve a type of radical equation that occurs frequently in a study of *optimization* [finding the maximum or minimum value(s) of a function].

EXAMPLE 1 ▶ Minimizing Response Time

A boater is 70 yd away from a straight shoreline when she gets an emergency call from her home, 400 yd downshore. Knowing she can row at 200 yd/min and run at 300 yd/min, how far downshore should she land the boat to make it home in the shortest time possible?

Solution ▶ As with other forms of problem solving, drawing an accurate sketch is an important first step.



From the diagram, we note the rowing distance will be $\sqrt{x^2 + 4900}$ (using the Pythagorean theorem), and the running distance will be $400 - x$ (total minus distance downshore).

From the relationship time $= \frac{\text{distance}}{\text{rate}}$, we find the total time required to reach home is $t(x) = \frac{\sqrt{x^2 + 4900}}{200} + \frac{400 - x}{300}$. Using the tools of calculus it can be shown that the distance x downshore that results in the shortest possible time, is a zero of $T(x) = \frac{x}{200\sqrt{x^2 + 4900}} - \frac{1}{300}$. Find the zero(es) of $T(x)$ and state the result in both exact and approximate form.

Solution ▶ Begin by isolating the radical on one side.

$$\frac{x}{200\sqrt{x^2+4900}} - \frac{1}{300} = 0 \qquad f'(x) = 0$$

$$\frac{x}{200\sqrt{x^2+4900}} = \frac{1}{300} \qquad \text{add } \frac{1}{300}$$

$$300x = 200\sqrt{x^2+4900} \qquad \text{clear denominators}$$

$$1.5x = \sqrt{x^2+4900} \qquad \text{divide by 200}$$

$$2.25x^2 = x^2 + 4900 \qquad \text{square both sides}$$

$$1.25x^2 = 4900 \qquad \text{subtract } x^2$$

$$x^2 = 3920 \qquad \text{divide by 1.25}$$

$$x = \sqrt{3920} \qquad \text{solve for } x;\ x > 0 \text{ (distance)}$$

$$x = 28\sqrt{5} \qquad \text{simplify radical (exact form)}$$

$$\approx 62.6 \qquad \text{approximate form}$$

The boater should row to a spot about 63 yd downshore, then run the remaining 337 yd.

Now try Exercises 1 and 2 ▶

In addition to radical equations, equations involving rational exponents are often seen in a study of calculus. Many times, solving these equations involves combining the basic properties of exponents with other familiar skills such as factoring, or in this case, *factoring least powers.*

EXAMPLE 2 ▶ Modeling the Motion of a Particle

Suppose the motion of an object floating in turbulent water is modeled by the function $d(t) = \sqrt{t}(t^2 - 9t + 22)$, where $d(t)$ represents the displacement (in meters) at t sec. Using the tools of calculus, it can be shown that the velocity v of the particle is given by $v(t) = \frac{5}{2}t^{\frac{3}{2}} - \frac{27}{2}t^{\frac{1}{2}} + 11t^{-\frac{1}{2}}$. Find any time(s) t when the particle is motionless ($v = 0$).

Solution ▶ Set the equation equal to zero and factor out the fraction and least power.

$$\frac{5}{2}t^{\frac{3}{2}} - \frac{27}{2}t^{\frac{1}{2}} + 11t^{-\frac{1}{2}} = 0 \qquad \text{original equation}$$

$$5t^{\frac{4}{2}}\left(\frac{1}{2}\right)t^{-\frac{1}{2}} - 27t^{\frac{2}{2}}\left(\frac{1}{2}\right)t^{-\frac{1}{2}} + 22\left(\frac{1}{2}\right)t^{-\frac{1}{2}} = 0 \qquad \text{rewrite to help factor } \frac{1}{2}t^{-\frac{1}{2}} \text{ (least power)}$$

$$\frac{1}{2}t^{-\frac{1}{2}}(5t^2 - 27t + 22) = 0 \qquad \text{common factor}$$

$$\frac{1}{2}t^{-\frac{1}{2}}(5t - 22)(t - 1) = 0 \qquad \text{factor the trinomial}$$

$$t^{-\frac{1}{2}} \neq 0;\ t = \frac{22}{5} \text{ or } t = 1 \qquad \text{result}$$

The particle is temporarily motionless at $t = 4.4$ sec and $t = 1$ sec.

Now try Exercises 3 and 4 ▶

Absolute Value Inequalities and Delta/Epsilon Form

While the terms may mean little to you now, the concept of absolute value plays an important role in the *precise definition of a limit, intervals of convergence,* and *derivatives.* In the case of *limits,* the study of calculus concerns itself with very small differences, as in the difference between the number 3 itself, and a number very close to 3. Consider the function $f(x) = \frac{x^2 - 9}{x - 3}$. From the implicit domain and the figures shown, we see that $f(x)$ (shown as Y_1) is not defined at 3, but is defined for any number near 3.

X	Y_1
2.6	5.6
2.7	5.7
2.8	5.8
2.9	5.9
2.99	5.99
2.999	5.999
2.9999	5.9999

Y_1=5.9999

X	Y_1
3.4	6.4
3.3	6.3
3.2	6.2
3.1	6.1
3.01	6.01
3.001	6.001
3.0001	6.0001

Y_1=6.0001

The figures also suggest that when x is a number very close to 3, $f(x)$ is a number very close to 6. Alternatively, we might say, "if the difference between x and 3 is very small, the difference between $f(x)$ and 6 is very small." The most convenient way to express this idea and make it practical is through the use of absolute value (which allows that the difference can be either positive or negative). Using the symbols δ (delta) and ϵ (epsilon) to represent very small (and possibly unequal) numbers, we can write this phrase in *delta/epsilon form* as

$$\textit{if } |x - 3| < \delta, \textit{ then } |f(x) - 6| < \epsilon$$

For now, we'll simply practice translating similar relationships from words into symbols, leaving any definitive conclusions for our study of limits in Chapter 12, or a future study of calculus.

EXAMPLE 3 ► **Using Delta/Epsilon Form**

Use a graphing calculator to explore the value of $g(x) = \frac{x^2 + 3x - 10}{x - 2}$ when x is near 2, then write the relationship in delta/epsilon form.

Solution ► Using a graphing calculator and the approach outlined above produces the tables in the figures.

X	Y_1
1.6	6.6
1.7	6.7
1.8	6.8
1.9	6.9
1.99	6.99
1.999	6.999
1.9999	6.9999

Y_1=6.9999

X	Y_1
2.4	7.4
2.3	7.3
2.2	7.2
2.1	7.1
2.01	7.01
2.001	7.001
2.0001	7.0001

Y_1=7.0001

From these, it appears that, "if the difference between x and 2 is very small, the difference between $f(x)$ and 7 is very small." In delta/epsilon form:

$$\text{if } |x - 2| < \delta, \text{ then } |f(x) - 7| < \epsilon.$$

Now try Exercises 5 through 8 ►

At first, modeling this relationship may seem like a minor accomplishment. But historically and in a practical sense, it is actually a major achievement as it enables us to "tame the infinite," since we can now verify that no matter how small ϵ is, there is a corresponding δ that *guarantees*

$$|f(x) - 7| < \epsilon \quad \text{whenever} \quad |x - 2| < \delta$$

$f(x)$ is *infinitely* close to 7 $\qquad$ x is *infinitely* close to 2

This observation leads directly to the precise definition of a limit, the type of "limit" referred to in our *Introduction to Calculus,* found in the Preface (page xxii). As noted there, such limits will enable us to find a precise formula for the instantaneous speed of a cue ball as it falls, and a precise formula for the volume of an irregular solid.

Connections to Calculus Exercises

Solve the following equations.

1. To find the length of a rectangle with maximum area that can be circumscribed by a circle of radius 3 in. requires that we solve $\sqrt{9 - x^2} - \dfrac{x^2}{\sqrt{9 - x^2}} = 0$, where the length of the rectangle is $2x$. To the nearest hundredth, what is the length of the rectangle?

2. To find the height of an isosceles triangle with maximum area that can be inscribed in a circle of radius $r = 5$ in. requires that we solve $\dfrac{-5x}{\sqrt{25 - x^2}} + \sqrt{25 - x^2} - \dfrac{x^2}{\sqrt{25 - x^2}} = 0$, where the height is $5 + x$. What is the height of the triangle?

3. If the motion of a particle in turbulent air is modeled by $d = \sqrt{t}(2t^2 - 9t + 18)$, the velocity of the particle is given by $v = 5t^{\frac{3}{2}} - \dfrac{27}{2}t^{\frac{1}{2}} + 9t^{-\frac{1}{2}}$ (d in meters, t in seconds). Find any time(s) t when velocity $v = 0$.

4. In order for a light source to provide maximum (circular) illumination to a workroom, the light must be hung at a certain height. While the complete development requires trigonometry, we find that maximum illumination is obtained at the solutions of the equation shown, where h is the height of the light, k is a constant, and the radius of illumination is 12 ft. Solve the equation for h by factoring the least power and simplifying the result: $k\dfrac{(h^2 + 12^2)^{\frac{3}{2}} - 3h^2(h^2 + 12^2)^{\frac{1}{2}}}{(h^2 + 12^2)^3} = 0$.

Use a graphing calculator to explore the value of the function given for values of x near the one indicated. Then write the relationship in words and in delta/epsilon form.

5. $h(x) = \dfrac{4x^2 - 9}{2x - 3}$; $x = \dfrac{3}{2}$

6. $v(x) = \dfrac{x^3 + 27}{x + 3}$; $x = -3$

7. $w(x) = \dfrac{7x^3 - 28x}{x^2 - 4}$; $x = 2$

8. $F(x) = \dfrac{x^2 + 7x}{x}$; $x = 0$