Further mathematics HL guide

First examinations 2014

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Diploma Programme

5041

IB mission statement

The International Baccalaureate aims to develop inquiring, knowledgeable and caring young people who help to create a better and more peaceful world through intercultural understanding and respect.

To this end the organization works with schools, governments and international organizations to develop challenging programmes of international education and rigorous assessment.

These programmes encourage students across the world to become active, compassionate and lifelong learners who understand that other people, with their differences, can also be right.

IB learner profile

The aim of all IB programmes is to develop internationally minded people who, recognizing their common humanity and shared guardianship of the planet, help to create a better and more peaceful world.

IB learners strive to be:

Inquirers	They develop their natural curiosity. They acquire the skills necessary to conduct inquiry and research and show independence in learning. They actively enjoy learning and this love of learning will be sustained throughout their lives.
Knowledgeable	They explore concepts, ideas and issues that have local and global significance. In so doing, they acquire in-depth knowledge and develop understanding across a broad and balanced range of disciplines.
Thinkers	They exercise initiative in applying thinking skills critically and creatively to recognize and approach complex problems, and make reasoned, ethical decisions.
Communicators	They understand and express ideas and information confidently and creatively in more than one language and in a variety of modes of communication. They work effectively and willingly in collaboration with others.
Principled	They act with integrity and honesty, with a strong sense of fairness, justice and respect for the dignity of the individual, groups and communities. They take responsibility for their own actions and the consequences that accompany them.
Open-minded	They understand and appreciate their own cultures and personal histories, and are open to the perspectives, values and traditions of other individuals and communities. They are accustomed to seeking and evaluating a range of points of view, and are willing to grow from the experience.
Caring	They show empathy, compassion and respect towards the needs and feelings of others. They have a personal commitment to service, and act to make a positive difference to the lives of others and to the environment.
Risk-takers	They approach unfamiliar situations and uncertainty with courage and forethought, and have the independence of spirit to explore new roles, ideas and strategies. They are brave and articulate in defending their beliefs.
Balanced	They understand the importance of intellectual, physical and emotional balance to achieve personal well-being for themselves and others.
Reflective	They give thoughtful consideration to their own learning and experience. They are able to assess and understand their strengths and limitations in order to support their learning and personal development.

Contents

Purpose of this document

This publication is intended to guide the planning, teaching and assessment of the subject in schools. Subject teachers are the primary audience, although it is expected that teachers will use the guide to inform students and parents about the subject.

This guide can be found on the subject page of the online curriculum centre (OCC) at http://occ.ibo.org, a password-protected IB website designed to support IB teachers. It can also be purchased from the IB store at http://store.ibo.org.

Additional resources

Additional publications such as teacher support materials, subject reports and grade descriptors can also be found on the OCC. Specimen and past examination papers as well as markschemes can be purchased from the IB store.

Teachers are encouraged to check the OCC for additional resources created or used by other teachers. Teachers can provide details of useful resources, for example: websites, books, videos, journals or teaching ideas.

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The Diploma Programme

The Diploma Programme is a rigorous pre-university course of study designed for students in the 16 to 19 age range. It is a broad-based two-year course that aims to encourage students to be knowledgeable and inquiring, but also caring and compassionate. There is a strong emphasis on encouraging students to develop intercultural understanding, open-mindedness, and the attitudes necessary for them to respect and evaluate a range of points of view.

The Diploma Programme hexagon

The course is presented as six academic areas enclosing a central core (see figure 1). It encourages the concurrent study of a broad range of academic areas. Students study: two modern languages (or a modern language and a classical language); a humanities or social science subject; an experimental science; mathematics; one of the creative arts. It is this comprehensive range of subjects that makes the Diploma Programme a demanding course of study designed to prepare students effectively for university entrance. In each of the academic areas students have flexibility in making their choices, which means they can choose subjects that particularly interest them and that they may wish to study further at university.
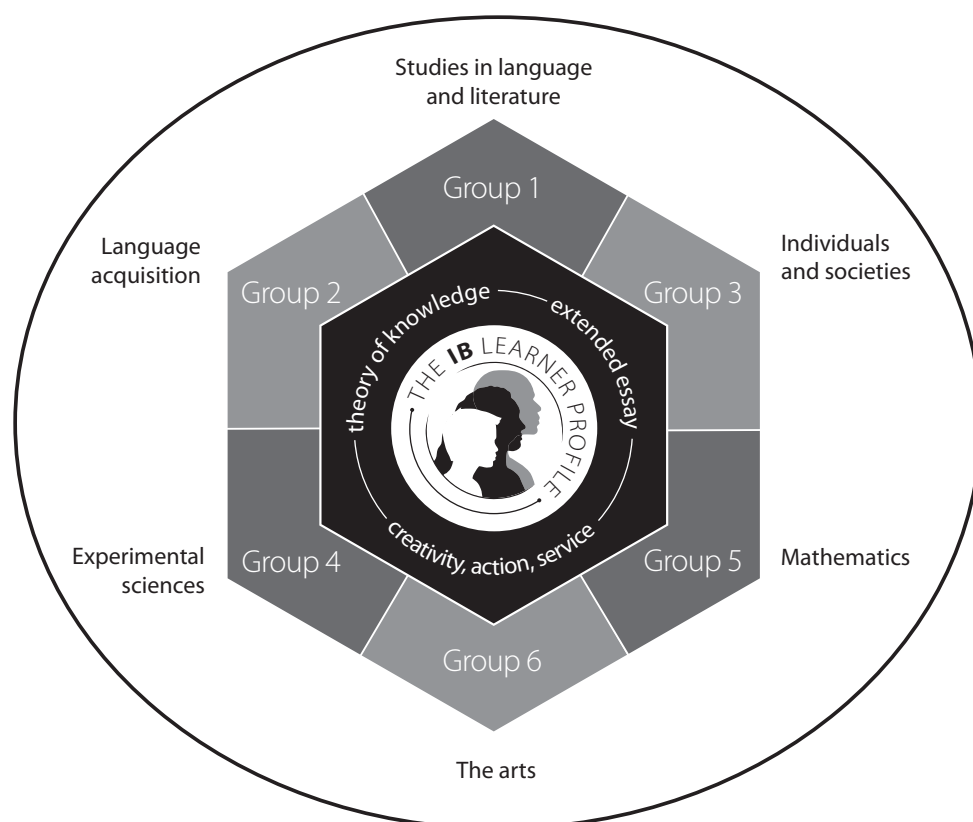

Figure 1
Diploma Programme model

Choosing the right combination

Students are required to choose one subject from each of the six academic areas, although they can choose a second subject from groups 1 to 5 instead of a group 6 subject. Normally, three subjects (and not more than four) are taken at higher level (HL), and the others are taken at standard level (SL). The IB recommends 240 teaching hours for HL subjects and 150 hours for SL. Subjects at HL are studied in greater depth and breadth than at SL.

At both levels, many skills are developed, especially those of critical thinking and analysis. At the end of the course, students' abilities are measured by means of external assessment. Many subjects contain some element of coursework assessed by teachers. The courses are available for examinations in English, French and Spanish, with the exception of groups 1 and 2 courses where examinations are in the language of study.

The core of the hexagon

All Diploma Programme students participate in the three course requirements that make up the core of the hexagon. Reflection on all these activities is a principle that lies at the heart of the thinking behind the Diploma Programme.

The theory of knowledge course encourages students to think about the nature of knowledge, to reflect on the process of learning in all the subjects they study as part of their Diploma Programme course, and to make connections across the academic areas. The extended essay, a substantial piece of writing of up to 4,000 words, enables students to investigate a topic of special interest that they have chosen themselves. It also encourages them to develop the skills of independent research that will be expected at university. Creativity, action, service involves students in experiential learning through a range of artistic, sporting, physical and service activities.

The IB mission statement and the IB learner profile

The Diploma Programme aims to develop in students the knowledge, skills and attitudes they will need to fulfill the aims of the IB, as expressed in the organization's mission statement and the learner profile. Teaching and learning in the Diploma Programme represent the reality in daily practice of the organization's educational philosophy.

Nature of the subject

Introduction

The nature of mathematics can be summarized in a number of ways: for example, it can be seen as a well-defined body of knowledge, as an abstract system of ideas, or as a useful tool. For many people it is probably a combination of these, but there is no doubt that mathematical knowledge provides an important key to understanding the world in which we live. Mathematics can enter our lives in a number of ways: we buy produce in the market, consult a timetable, read a newspaper, time a process or estimate a length. Mathematics, for most of us, also extends into our chosen profession: visual artists need to learn about perspective; musicians need to appreciate the mathematical relationships within and between different rhythms; economists need to recognize trends in financial dealings; and engineers need to take account of stress patterns in physical materials. Scientists view mathematics as a language that is central to our understanding of events that occur in the natural world. Some people enjoy the challenges offered by the logical methods of mathematics and the adventure in reason that mathematical proof has to offer. Others appreciate mathematics as an aesthetic experience or even as a cornerstone of philosophy. This prevalence of mathematics in our lives, with all its interdisciplinary connections, provides a clear and sufficient rationale for making the study of this subject compulsory for students studying the full diploma.

Summary of courses available

Because individual students have different needs, interests and abilities, there are four different courses in mathematics. These courses are designed for different types of students: those who wish to study mathematics in depth, either as a subject in its own right or to pursue their interests in areas related to mathematics; those who wish to gain a degree of understanding and competence to understand better their approach to other subjects; and those who may not as yet be aware how mathematics may be relevant to their studies and in their daily lives. Each course is designed to meet the needs of a particular group of students. Therefore, great care should be taken to select the course that is most appropriate for an individual student.

In making this selection, individual students should be advised to take account of the following factors:

- their own abilities in mathematics and the type of mathematics in which they can be successful

- their own interest in mathematics and those particular areas of the subject that may hold the most interest for them

- their other choices of subjects within the framework of the Diploma Programme

- their academic plans, in particular the subjects they wish to study in future

- their choice of career.

Teachers are expected to assist with the selection process and to offer advice to students.

Mathematical studies SL

This course is available only at standard level, and is equivalent in status to mathematics SL, but addresses different needs. It has an emphasis on applications of mathematics, and the largest section is on statistical techniques. It is designed for students with varied mathematical backgrounds and abilities. It offers students opportunities to learn important concepts and techniques and to gain an understanding of a wide variety of mathematical topics. It prepares students to be able to solve problems in a variety of settings, to develop more sophisticated mathematical reasoning and to enhance their critical thinking. The individual project is an extended piece of work based on personal research involving the collection, analysis and evaluation of data. Students taking this course are well prepared for a career in social sciences, humanities, languages or arts. These students may need to utilize the statistics and logical reasoning that they have learned as part of the mathematical studies SL course in their future studies.

Mathematics SL

This course caters for students who already possess knowledge of basic mathematical concepts, and who are equipped with the skills needed to apply simple mathematical techniques correctly. The majority of these students will expect to need a sound mathematical background as they prepare for future studies in subjects such as chemistry, economics, psychology and business administration.

Mathematics HL

This course caters for students with a good background in mathematics who are competent in a range of analytical and technical skills. The majority of these students will be expecting to include mathematics as a major component of their university studies, either as a subject in its own right or within courses such as physics, engineering and technology. Others may take this subject because they have a strong interest in mathematics and enjoy meeting its challenges and engaging with its problems.

Further mathematics HL

This course is available only at higher level. It caters for students with a very strong background in mathematics who have attained a high degree of competence in a range of analytical and technical skills, and who display considerable interest in the subject. Most of these students will expect to study mathematics at university, either as a subject in its own right or as a major component of a related subject. The course is designed specifically to allow students to learn about a variety of branches of mathematics in depth and also to appreciate practical applications. It is expected that students taking this course will also be taking mathematics HL.

Note: Mathematics HL is an ideal course for students expecting to include mathematics as a major component of their university studies, either as a subject in its own right or within courses such as physics, engineering or technology. It should not be regarded as necessary for such students to study further mathematics HL. Rather, further mathematics HL is an optional course for students with a particular aptitude and interest in mathematics, enabling them to study some wider and deeper aspects of mathematics, but is by no means a necessary qualification to study for a degree in mathematics.

Further mathematics HL—course details

The nature of the subject is such that it focuses on different branches of mathematics to encourage students to appreciate the diversity of the subject. Students should be equipped at this stage in their mathematical progress to begin to form an overview of the characteristics that are common to all mathematical thinking, independent of topic or branch.

All categories of student can register for mathematics HL only **or** for further mathematics HL only **or** for both. However, students registering for further mathematics HL will be presumed to know the topics in the core syllabus of mathematics HL and to have studied one of the options, irrespective of whether they have also registered for mathematics HL.

Examination questions are intended to be comparable in difficulty with those set on the four options in the mathematics HL course. The challenge for students will be to reach an equivalent level of understanding across all topics. There is no internal assessment component in this course. Although not a requirement, it is expected that students studying further mathematics HL will also be studying mathematics HL and therefore will be required to undertake a mathematical exploration for the internal assessment component of that course.

Prior learning

Mathematics is a linear subject, and it is expected that most students embarking on a Diploma Programme (DP) mathematics course will have studied mathematics for at least 10 years. There will be a great variety of topics studied, and differing approaches to teaching and learning. Thus students will have a wide variety of skills and knowledge when they start the further mathematics HL course. Most will have some background in arithmetic, algebra, geometry, trigonometry, probability and statistics. Some will be familiar with an inquiry approach, and may have had an opportunity to complete an extended piece of work in mathematics.

As previously stated, students registering for further mathematics HL will be presumed to know the topics in the core syllabus of mathematics HL and to have studied one of the options.

Links to the Middle Years Programme

The prior learning topics for the DP courses have been written in conjunction with the Middle Years Programme (MYP) mathematics guide. The approaches to teaching and learning for DP mathematics build on the approaches used in the MYP. These include investigations, exploration and a variety of different assessment tools.

A continuum document called *Mathematics: The MYP–DP continuum* (November 2010) is available on the DP mathematics home pages of the OCC. This extensive publication focuses on the alignment of mathematics across the MYP and the DP. It was developed in response to feedback provided by IB World Schools, which expressed the need to articulate the transition of mathematics from the MYP to the DP. The publication also highlights the similarities and differences between MYP and DP mathematics, and is a valuable resource for teachers.

Mathematics and theory of knowledge

The *Theory of knowledge guide* (March 2006) identifies four ways of knowing, and it could be claimed that these all have some role in the acquisition of mathematical knowledge. While perhaps initially inspired by data from sense perception, mathematics is dominated by reason, and some mathematicians argue that their subject is a language, that it is, in some sense, universal. However, there is also no doubt that mathematicians perceive beauty in mathematics, and that emotion can be a strong driver in the search for mathematical knowledge.

As an area of knowledge, mathematics seems to supply a certainty perhaps missing in other disciplines. This may be related to the "purity" of the subject that makes it sometimes seem divorced from reality. However, mathematics has also provided important knowledge about the world, and the use of mathematics in science and technology has been one of the driving forces for scientific advances.

Despite all its undoubted power for understanding and change, mathematics is in the end a puzzling phenomenon. A fundamental question for all knowers is whether mathematical knowledge really exists independently of our thinking about it. Is it there "waiting to be discovered" or is it a human creation?

Students' attention should be drawn to questions relating theory of knowledge (TOK) and mathematics, and they should be encouraged to raise such questions themselves, in mathematics and TOK classes. This includes questioning all the claims made above. Examples of issues relating to TOK are given in the "Links" column of the syllabus. Teachers could also discuss questions such as those raised in the "Areas of knowledge" section of the TOK guide.

Mathematics and the international dimension

Mathematics is in a sense an international language, and, apart from slightly differing notation, mathematicians from around the world can communicate within their field. Mathematics transcends politics, religion and nationality, yet throughout history great civilizations owe their success in part to their mathematicians being able to create and maintain complex social and architectural structures.

Despite recent advances in the development of information and communication technologies, the global exchange of mathematical information and ideas is not a new phenomenon and has been essential to the progress of mathematics. Indeed, many of the foundations of modern mathematics were laid many centuries ago by Arabic, Greek, Indian and Chinese civilizations, among others. Teachers could use timeline websites to show the contributions of different civilizations to mathematics, but not just for their mathematical content. Illustrating the characters and personalities of the mathematicians concerned and the historical context in which they worked brings home the human and cultural dimension of mathematics.

The importance of science and technology in the everyday world is clear, but the vital role of mathematics is not so well recognized. It is the language of science, and underpins most developments in science and technology. A good example of this is the digital revolution, which is transforming the world, as it is all based on the binary number system in mathematics.

Many international bodies now exist to promote mathematics. Students are encouraged to access the extensive websites of international mathematical organizations to enhance their appreciation of the international dimension and to engage in the global issues surrounding the subject.

Examples of global issues relating to international-mindedness (**Int**) are given in the "Links" column of the syllabus.

Aims

Group 5 aims

The aims of all mathematics courses in group 5 are to enable students to:

1. enjoy mathematics, and develop an appreciation of the elegance and power of mathematics

2. develop an understanding of the principles and nature of mathematics

3. communicate clearly and confidently in a variety of contexts

4. develop logical, critical and creative thinking, and patience and persistence in problem-solving

5. employ and refine their powers of abstraction and generalization

6. apply and transfer skills to alternative situations, to other areas of knowledge and to future developments

7. appreciate how developments in technology and mathematics have influenced each other

8. appreciate the moral, social and ethical implications arising from the work of mathematicians and the applications of mathematics

9. appreciate the international dimension in mathematics through an awareness of the universality of mathematics and its multicultural and historical perspectives

10. appreciate the contribution of mathematics to other disciplines, and as a particular "area of knowledge" in the TOK course.

Assessment objectives

Problem-solving is central to learning mathematics and involves the acquisition of mathematical skills and concepts in a wide range of situations, including non-routine, open-ended and real-world problems. Having followed a DP mathematics HL course, students will be expected to demonstrate the following.

1. **Knowledge and understanding**: recall, select and use their knowledge of mathematical facts, concepts and techniques in a variety of familiar and unfamiliar contexts.

2. **Problem-solving**: recall, select and use their knowledge of mathematical skills, results and models in both real and abstract contexts to solve problems.

3. **Communication and interpretation**: transform common realistic contexts into mathematics; comment on the context; sketch or draw mathematical diagrams, graphs or constructions both on paper and using technology; record methods, solutions and conclusions using standardized notation.

4. **Technology**: use technology, accurately, appropriately and efficiently both to explore new ideas and to solve problems.

5. **Reasoning**: construct mathematical arguments through use of precise statements, logical deduction and inference, and by the manipulation of mathematical expressions.

6. **Inquiry approaches**: investigate unfamiliar situations, both abstract and real-world, involving organizing and analysing information, making conjectures, drawing conclusions and testing their validity.

Syllabus outline

Syllabus component	Teaching hours
	HL
All topics are compulsory. Students must study all the sub-topics in each of the topics in the syllabus as listed in this guide. Students are also required to be familiar with all of the core topics in mathematics HL.	
Topic 1 Linear algebra	48
Topic 2 Geometry	48
Topic 3 Statistics and probability	48
Topic 4 Sets, relations and groups	48
Topic 5 Calculus	48
Topic 6 Discrete mathematics	48
Note: One of topics 3–6 will be assumed to have been taught as part of the mathematics HL course and therefore the total teaching hours will be 240 not 288.	
Total teaching hours	**240**

Approaches to the teaching and learning of further mathematics HL

Throughout the DP further mathematics HL course, students should be encouraged to develop their understanding of the methodology and practice of the discipline of mathematics. The processes of **mathematical inquiry**, **mathematical modelling and applications** and the **use of technology** should be introduced appropriately. These processes should be used throughout the course, and not treated in isolation.

Mathematical inquiry

The IB learner profile encourages learning by experimentation, questioning and discovery. In the IB classroom, students should generally learn mathematics by being active participants in learning activities rather than recipients of instruction. Teachers should therefore provide students with opportunities to learn through mathematical inquiry. This approach is illustrated in figure 2.
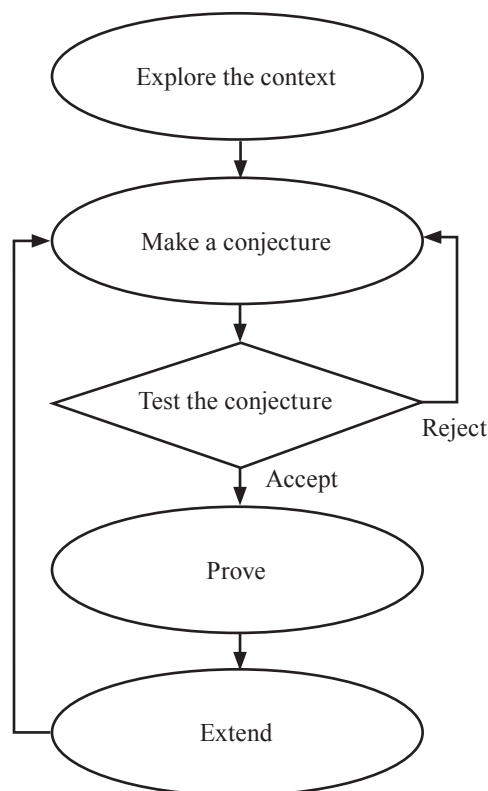
Figure 2

Mathematical modelling and applications

Students should be able to use mathematics to solve problems in the real world. Engaging students in the mathematical modelling process provides such opportunities. Students should develop, apply and critically analyse models. This approach is illustrated in figure 3.
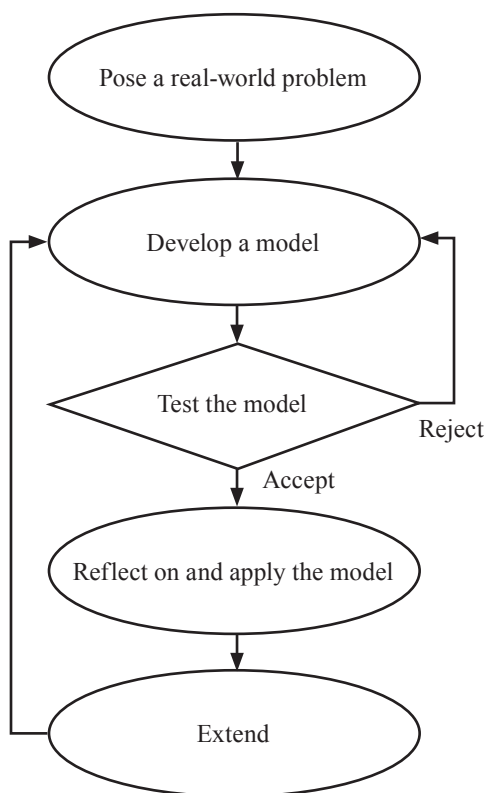
Figure 3

Technology

Technology is a powerful tool in the teaching and learning of mathematics. Technology can be used to enhance visualization and support student understanding of mathematical concepts. It can assist in the collection, recording, organization and analysis of data. Technology can increase the scope of the problem situations that are accessible to students. The use of technology increases the feasibility of students working with interesting problem contexts where students reflect, reason, solve problems and make decisions.

As teachers tie together the unifying themes of **mathematical inquiry**, **mathematical modelling and applications** and the **use of technology**, they should begin by providing substantial guidance, and then gradually encourage students to become more independent as inquirers and thinkers. IB students should learn to become strong communicators through the language of mathematics. Teachers should create a safe learning environment in which students are comfortable as risk-takers.

Teachers are encouraged to relate the mathematics being studied to other subjects and to the real world, especially topics that have particular relevance or are of interest to their students. Everyday problems and questions should be drawn into the lessons to motivate students and keep the material relevant; suggestions are provided in the "Links" column of the syllabus.

For further information on "Approaches to teaching a DP course", please refer to the publication *The Diploma Programme: From principles into practice* (April 2009). To support teachers, a variety of resources can be found on the OCC and details of workshops for professional development are available on the public website.

Format of the syllabus

- **Content**: this column lists, under each topic, the sub-topics to be covered.

- **Further guidance**: this column contains more detailed information on specific sub-topics listed in the content column. This clarifies the content for examinations.

- **Links**: this column provides useful links to the aims of the further mathematics HL course, with suggestions for discussion, real-life examples and ideas for further investigation. **These suggestions are only a guide for introducing and illustrating the sub-topic and are not exhaustive.** Links are labelled as follows.

  Appl   real-life examples and links to other DP subjects

  Aim 8   moral, social and ethical implications of the sub-topic

  Int   international-mindedness

  TOK   suggestions for discussion

Note that any syllabus references to other subject guides given in the "Links" column are correct for the current (2012) published versions of the guides.

Notes on the syllabus

- Formulae are only included in this document where there may be some ambiguity. All formulae required for the course are in the mathematics HL and further mathematics HL formula booklet.

- The term "technology" is used for any form of calculator or computer that may be available. However, there will be restrictions on which technology may be used in examinations, which will be noted in relevant documents.

- The terms "analysis" and "analytic approach" are generally used when referring to an approach that does not use technology.

Course of study

The content of topics 1 and 2 as well as three out of the four remaining topics in the syllabus must be taught, although not necessarily in the order in which they appear in this guide. Teachers are expected to construct a course of study that addresses the needs of their students and includes, where necessary, the topics noted in prior learning.

Time allocation

The recommended teaching time for higher level courses is 240 hours. The time allocations given in this guide are approximate, and are intended to suggest how the 240 hours allowed for the teaching of the syllabus might be allocated. However, the exact time spent on each topic depends on a number of factors, including the background knowledge and level of preparedness of each student. Teachers should therefore adjust these timings to correspond to the needs of their students.

Use of calculators

Students are expected to have access to a graphic display calculator (GDC) at all times during the course. The minimum requirements are reviewed as technology advances, and updated information will be provided to schools. It is expected that teachers and schools monitor calculator use with reference to the calculator policy. Regulations covering the types of calculators allowed in examinations are provided in the *Handbook of procedures for the Diploma Programme*. Further information and advice is provided in the *Mathematics HL/ SL: Graphic display calculators teacher support material* (May 2005) and on the OCC.

Mathematics HL and further mathematics HL formula booklet

Each student is required to have access to a clean copy of this booklet during the examination. It is recommended that teachers ensure students are familiar with the contents of this document from the beginning of the course. It is the responsibility of the school to download a copy from IBIS or the OCC, check that there are no printing errors, and ensure that there are sufficient copies available for all students.

Command terms and notation list

Teachers and students need to be familiar with the IB notation and the command terms, as these will be used without explanation in the examination papers. The "Glossary of command terms" and "Notation list" appear as appendices in this guide.

Syllabus content

Topic 1—Linear algebra

48 hours

The aim of this section is to introduce students to the principles of matrices, vector spaces and linear algebra, including eigenvalues and geometrical interpretations.

	Content	Further guidance	Links
1.1	Definition of a matrix: the terms element, row, column and order for $m \times n$ matrices.		Data storage and manipulation, eg stock control.
	Algebra of matrices: equality; addition; subtraction; multiplication by a scalar for $m \times n$ matrices.	Including use of the GDC.	Matrix operations to handle or process information.
	Multiplication of matrices.	Students should be familiar with the notation $\boldsymbol{I}$ and $\boldsymbol{0}$.	**TOK:** Given the many applications of matrices as seen in this course, consider the fact that mathematicians marvel at some of the deep connections between disparate parts of their subject. Is this evidence for a simple underlying mathematical reality?
	Properties of matrix multiplication: associativity, distributivity.		
	Identity and zero matrices.		
	Transpose of a matrix including $\boldsymbol{A}^{\mathrm{T}}$ notation: $(\boldsymbol{AB})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}} \boldsymbol{A}^{\mathrm{T}}$.	Students should be familiar with the definition of symmetric and skew-symmetric matrices.	

	Content	Further guidance	Links
1.2	Definition and properties of the inverse of a square matrix: $(AB)^{-1} = B^{-1} A^{-1}$, $(A^{T})^{-1} = (A^{-1})^{T}$, $(A^{n})^{-1} = (A^{-1})^{n}$.	The terms singular and non-singular matrices.	
	Calculation of A^{-1} . Use of elementary row operations to find A^{-1} .	Using a GDC.	
	Formulae for the inverse and determinant of a 2×2 matrix and the determinant of a 3×3 matrix.	The result $\det(AB) = \det A \det B$.	
1.3	Elementary row and column operations for matrices.		
	Scaling, swapping and pivoting.	Pivoting is the $R_1 + aR_2$ method.	
	Corresponding elementary matrices.		
	Row reduced echelon form.		
	Row space, column space and null space.		
	Row rank and column rank and their equality.		
1.4	Solutions of m linear equations in n unknowns: both augmented matrix method, leading to reduced row echelon form method, and inverse matrix method, when applicable.	Include the non-existent, unique and infinitely many cases.	**Int:** Also known as Gaussian elimination after the German mathematician CF Gauss (1777–1855). **Aim 8:** Linear optimization was developed by George Dantzig in the 1940s as a way of allocating scarce resources.

	Content	Further guidance	Links
1.5	The vector space $\mathbb{R}^n$. Linear combinations of vectors. Spanning set. Linear independence of vectors. Basis and dimension for a vector space. Subspaces.	Include linear dependence and use of determinants. Students should be familiar with the term orthogonal.	**Aim 8:** Link to Fourier analysis for waves, which has many uses in physics (Physics 4.5) and engineering.
1.6	Linear transformations: $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$, $T(k\boldsymbol{u}) = kT(\boldsymbol{u})$. Composition of linear transformations. Domain, range, codomain and kernel. Result and proof that the kernel is a subspace of the domain. Result and proof that the range is a subspace of the codomain. Rank-nullity theorem (proof not required).	The kernel of a linear transformation is the null space of its matrix representation. Dim(domain) = Dim(range) + Dim(kernel) where Dim(range) = rank and Dim(kernel) = nullity.	**Aim 8:** Lorenz transformations and their use in relativity and quantum mechanics (Physics 13.1).

Content		Further guidance	Links
1.7	Result that any linear transformation can be represented by a matrix, and the converse of this result.		
	Result that the numbers of linearly independent rows and columns are equal, and this is the dimension of the range of the transformation (proof not required).		
	Application of linear transformations to solutions of system of equations.	Using	
	Solution of $A\mathbf{x} = \mathbf{b}$.	(particular solution) + (any member of the null space).	
1.8	Geometric transformations represented by 2×2 matrices include general rotation, general reflection in $y = (\tan \alpha)x$, stretches parallel to axes, shears parallel to axes, and projection onto $y = (\tan \alpha)x$.		**Aim 8:** Computer graphics in three-dimensional modelling.
	Compositions of the above transformations.		
	Geometric interpretation of determinant.	New area $= |\det A| \times$ old area.	

	Content	Further guidance	Links
1.9	Eigenvalues and eigenvectors of 2×2 matrices.	Geometric interpretation.	Stochastic processes.
	Characteristic polynomial of 2×2 matrices.		Stock market values and trends.
	Diagonalization of 2×2 matrices (restricted to the case where there are distinct real eigenvalues).		Invariant states.
	Applications to powers of 2×2 matrices.		Representation of conics.
			TOK: "We can use mathematics successfully to model real-world processes. Is this because we create mathematics to mirror the world or because the world is intrinsically mathematical?"
			For example, the natural frequency of an object can be characterized by the eigenvalue of smallest magnitude (1940 Tacoma Narrows Bridge disaster).
			Aim 8: Damping noise in car design, test for cracks in solid objects, oil exploration, the Google PageRank formula, and "the $25 billion dollar eigenvector".
			Markov chains, link with genetics (Biology 4).

Topic 2—Geometry
48 hours

The aim of this section is to develop students' geometric intuition, visualization and deductive reasoning.

|	Content	Further guidance	Links
2.1	Similar and congruent triangles. Euclid's theorem for proportional segments in a right-angled triangle.		**TOK, Int:** The influence of Euclid's axiomatic approach on philosophy (Descartes) and politics (Jefferson: American Declaration of Independence). **TOK:** Hippasus' existence proof for irrational numbers and impact on separate development of number and geometry. **TOK:** Crisis over non-Euclidean geometry parallels with that of Cantor's set theory.
2.2	Centres of a triangle: orthocentre, incentre, circumcentre and centroid.	The terms altitude, angle bisector, perpendicular bisector, median. Proof of concurrency theorems. The terms inscribed and circumscribed.	**Appl:** centre of mass, triangulation.
2.3	Circle geometry. Tangents; arcs, chords and secants. In a cyclic quadrilateral, opposite angles are supplementary, and the converse.	Angle at centre theorem and corollaries. The tangent–secant and secant–secant theorems and intersecting chords theorems.	
2.4	Angle bisector theorem; Apollonius' circle theorem, Menelaus' theorem; Ceva's theorem; Ptolemy's theorem for cyclic quadrilaterals. Proofs of these theorems and converses.	The use of these theorems to prove further results.	

	Content	Further guidance	Links
2.5	Finding equations of loci. Coordinate geometry of the circle. Tangents to a circle.	The equations $(x-h)^2 + (y-k)^2 = r^2$ and $x^2 + y^2 + dx + ey + f = 0$.	**TOK:** Consequences of Descartes' unification of algebra and geometry.
2.6	Conic sections. The parabola, ellipse and hyperbola, including rectangular hyperbola. Focus–directrix definitions. Tangents and normals.	The standard forms $y^2 = 4ax$, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and their translations.	**Appl:** Satellite dish, headlight, orbits, projectiles (Physics 9.1). **TOK:** Kepler's difficulties accepting that an orbit was not a "perfect" circle.
2.7	Parametric equations. Parametric differentiation. Tangents and normals. The standard parametric equations of the circle, parabola, ellipse, rectangular hyperbola, hyperbola.	$x = a\cos\theta$, $y = a\sin\theta$, $x = at^2$, $y = 2at$, $x = a\cos\theta$, $y = b\sin\theta$, $x = ct$, $y = \dfrac{c}{t}$, $x = a\sec\theta$, $y = b\tan\theta$.	
2.8	The general conic $ax^2 + 2bxy + cy^2 + dx + ey + f = 0$, and the quadratic form $\mathbf{x}^\mathrm{T} \mathbf{A}\mathbf{x} = ax^2 + 2bxy + cy^2$. Diagonalizing the matrix $\mathbf{A}$ with the rotation matrix $\mathbf{P}$ and reducing the general conic to standard form.	The general conic can be rotated to give the form $\lambda_1 x_1^2 + \lambda_2 y_2^2 + dx + ey + f = 0$, where λ_1 and λ_2 are the eigenvalues of the matrix $\mathbf{A}$ in the quadratic form $\mathbf{x}^\mathrm{T} \mathbf{A}\mathbf{x}$.	

Geometry theorems—clarification of theorems used in topic 2

Teachers and students should be aware that some of the theorems mentioned in this section may be known by other names, or some names of theorems may be associated with different statements in some textbooks. To avoid confusion, on examination papers, theorems that may be misinterpreted are defined below.

Euclid's theorem for proportional segments in a right-angled triangle

The proportional segments p and q satisfy the following:
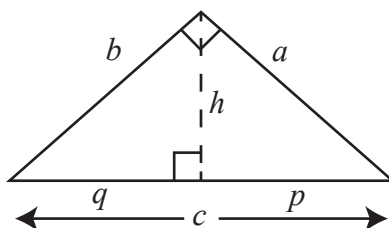
$h^2 = pq,$

$a^2 = pc,$

$b^2 = qc.$

Angle at centre theorem

The angle subtended by an arc at the circumference is half that subtended by the same arc at the centre.
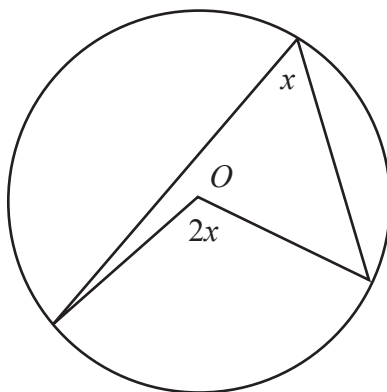
Corollaries

Angles subtended at the circumference by the same arc are equal.
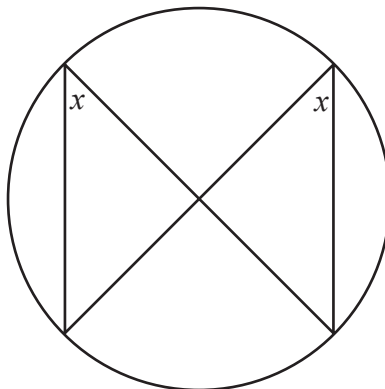
The angle in a semicircle is a right angle.
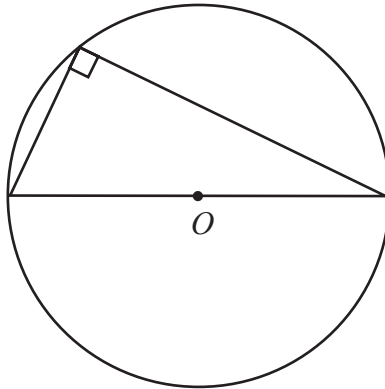
The alternate segment theorem: The angle between a tangent and a chord is equal to the angle subtended by the chord in the alternate segment.
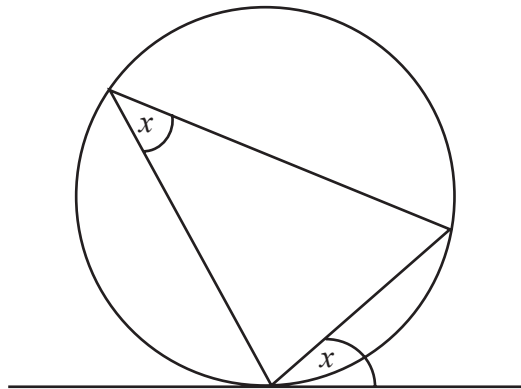
The tangent is perpendicular to the radius at the point of tangency.
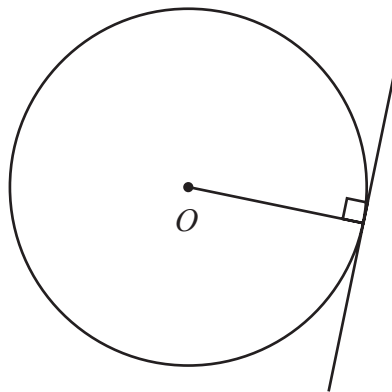
The intersecting chords theorem

$ab = cd$
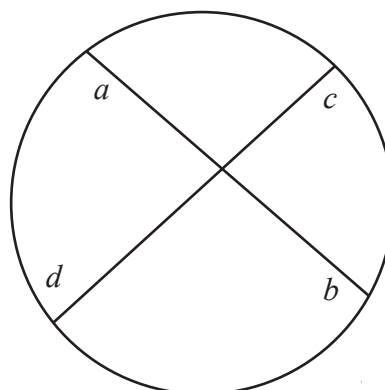
The tangent–secant and secant–secant theorems

$PT^2 = PA \times PB = PC \times PD$
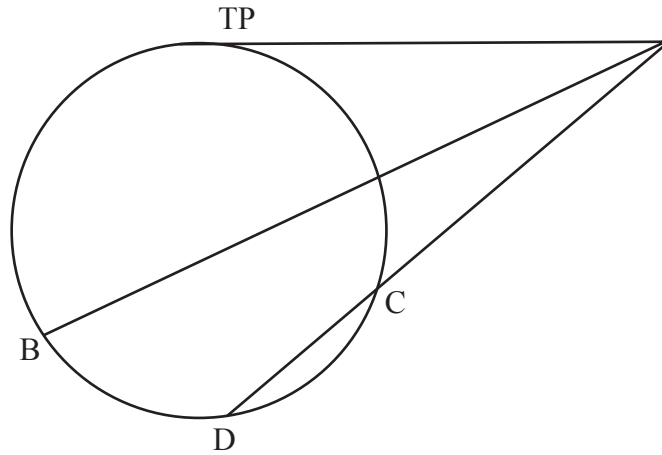
Apollonius' circle theorem (circle of Apollonius)

If A and B are two fixed points such that $\dfrac{PA}{PB}$ is a constant not equal to one, then the locus of P is a circle. This is called the circle of Apollonius.

Included: the converse of this theorem.

Menelaus' theorem

If a transversal meets the sides [BC], [CA] and [AB] of a triangle at D, E and F respectively, then
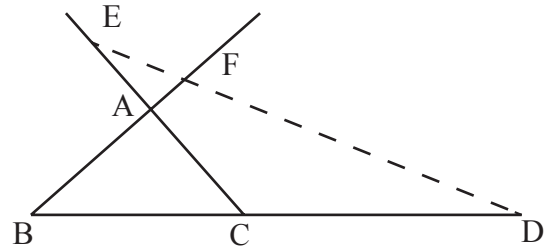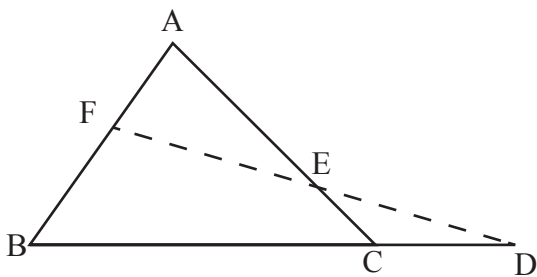
$$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1.$$

Converse: if D, E and F are points on the sides [BC], [CA] and [AB], respectively, of a triangle such that $\dfrac{BD}{DC} \times \dfrac{CE}{EA} \times \dfrac{AF}{FB} = -1$, then D, E and F are collinear.

Ceva's theorem

If three concurrent lines are drawn through the vertices A, B and C of a triangle ABC to meet the opposite sides at D, E and F respectively, then
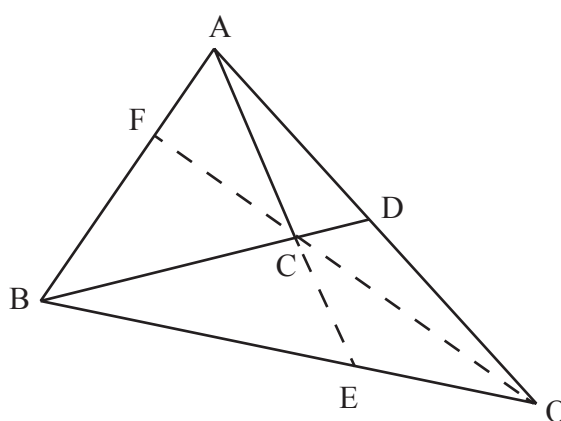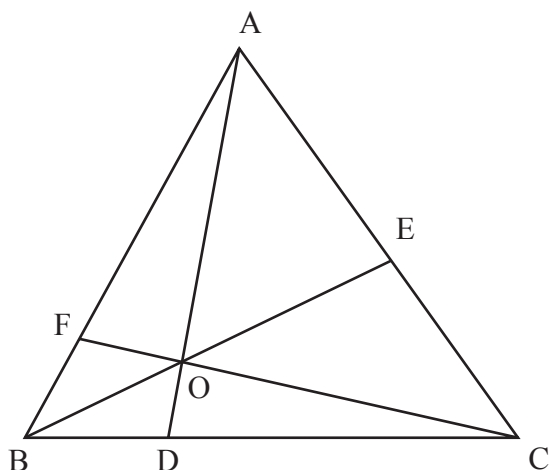
$$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = +1.$$

Converse: if D, E and F are points on [BC], [CA] and [AB], respectively, such that $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = +1$, then [AD], [BE] and [CF] are concurrent.

Note on Ceva's theorem and Menelaus' theorem

The statements and proofs of these theorems presuppose the idea of sensed magnitudes. Two segments [AB] and [PQ] of the same or parallel lines are said to have the same sense or opposite senses (or are sometimes called like or unlike) according to whether the displacements A → B and P → Q are in the same or opposite directions. The idea of sensed magnitudes may be used to prove the following theorem:

If A, B and C are any three collinear points, then AB + BC + CA = 0, where AB, BC and CA denote sensed magnitudes.

Ptolemy's theorem

If a quadrilateral is cyclic, the sum of the products of the two pairs of opposite sides equals the products of the diagonals. That is, for a cyclic quadrilateral ABCD, AB × CD + BC × DA = AC × BD .
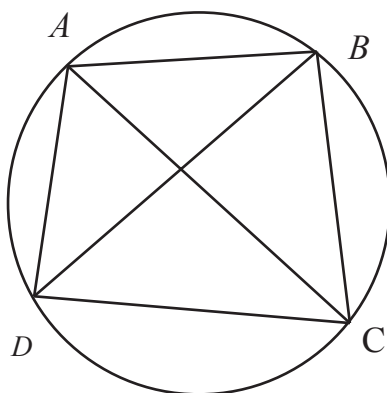
Angle bisector theorem

The angle bisector of an angle of a triangle divides the side of the triangle opposite the angle into segments proportional to the sides adjacent to the angle.

If ABC is the given triangle with (AD) as the bisector of angle BAC intersecting (BC) at point D, then

$$\frac{BD}{DC} = \frac{AB}{AC}$$ for internal bisectors

and

$$\frac{BD}{DC} = -\frac{AB}{AC}$$ for external bisectors.
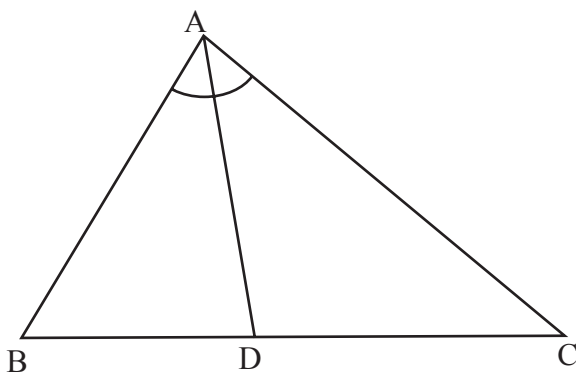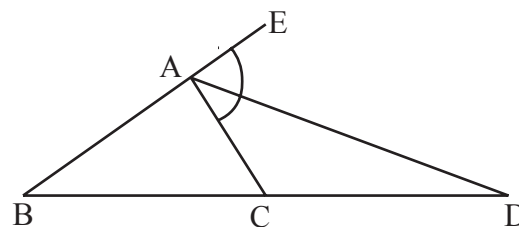
$$B\hat{A}D = C\hat{A}D \qquad\qquad C\hat{A}D = E\hat{A}D$$

Included: the converse of this theorem.

Topic 3—Statistics and probability

48 hours

The aims of this topic are to allow students the opportunity to approach statistics in a practical way; to demonstrate a good level of statistical understanding; and to understand which situations apply and to interpret the given results. It is expected that GDCs will be used throughout this option and that the minimum requirement of a GDC will be to find the probability distribution function (pdf), cumulative distribution function (cdf), inverse cumulative distribution function, p-values and test statistics, including calculations for the following distributions: binomial, Poisson, normal and t. Students are expected to set up the problem mathematically and then read the answers from the GDC, indicating this within their written answers. Calculator-specific or brand-specific language should not be used within these explanations.

	Content	Further guidance	Links
3.1	Cumulative distribution functions for both discrete and continuous distributions.		
	Geometric distribution.		
	Negative binomial distribution.		**Int:** Also known as Pascal's distribution.
	Probability generating functions for discrete random variables.	$G(t) = E(t^X) = \sum_x P(X = x)t^x$.	
	Using probability generating functions to find mean, variance and the distribution of the sum of n independent random variables.		**Aim 8:** Statistical compression of data files.
3.2	Linear transformation of a single random variable.	$E(aX + b) = aE(X) + b$,	
	Mean of linear combinations of n random variables.	$\text{Var}(aX + b) = a^2\text{Var}(X)$.	
	Variance of linear combinations of n independent random variables.		
	Expectation of the product of independent random variables.	$E(XY) = E(X)E(Y)$.	

	Content	Further guidance	Links
3.3	Unbiased estimators and estimates. Comparison of unbiased estimators based on variances.	T is an unbiased estimator for the parameter θ if $E(T) = \theta$. T_1 is a more efficient estimator than T_2 if $\mathrm{Var}(T_1) < \mathrm{Var}(T_2)$.	**TOK:** Mathematics and the world. In the absence of knowing the value of a parameter, will an unbiased estimator always be better than a biased one?
	$\bar{X}$ as an unbiased estimator for μ.	$\bar{X} = \sum_{i=1}^{n} \dfrac{X_i}{n}$.	
	S^2 as an unbiased estimator for σ^2.	$S^2 = \sum_{i=1}^{n} \dfrac{(X_i - \bar{X})^2}{n-1}$.	
3.4	A linear combination of independent normal random variables is normally distributed. In particular, $X \sim \mathrm{N}(\mu, \sigma^2) \ \Rightarrow \ \bar{X} \sim \mathrm{N}\!\left(\mu, \dfrac{\sigma^2}{n}\right)$. The central limit theorem.		**Aim 8/TOK:** Mathematics and the world. "Without the central limit theorem, there could be no statistics of any value within the human sciences." **TOK:** Nature of mathematics. The central limit theorem can be proved mathematically (formalism), but its truth can be confirmed by its applications (empiricism).

	Content	Further guidance	Links
3.5	Confidence intervals for the mean of a normal population.	Use of the normal distribution when σ is known and use of the t-distribution when σ is unknown, regardless of sample size. The case of matched pairs is to be treated as an example of a single sample technique.	**TOK:** Mathematics and the world. Claiming brand A is "better" on average than brand B can mean very little if there is a large overlap between the confidence intervals of the two means. **Appl:** Geography.
3.6	Null and alternative hypotheses, H_0 and H_1. Significance level. Critical regions, critical values, p-values, one-tailed and two-tailed tests. Type I and II errors, including calculations of their probabilities. Testing hypotheses for the mean of a normal population.	Use of the normal distribution when σ is known and use of the t-distribution when σ is unknown, regardless of sample size. The case of matched pairs is to be treated as an example of a single sample technique.	**TOK:** Mathematics and the world. In practical terms, is saying that a result is significant the same as saying that it is true? **TOK:** Mathematics and the world. Does the ability to test only certain parameters in a population affect the way knowledge claims in the human sciences are valued? **Appl:** When is it more important not to make a Type I error and when is it more important not to make a Type II error?

	Content	Further guidance	Links
3.7	Introduction to bivariate distributions.	Informal discussion of commonly occurring situations, eg marks in pure mathematics and statistics exams taken by a class of students, salary and age of teachers in a certain school. The need for a measure of association between the variables and the possibility of predicting the value of one of the variables given the value of the other variable.	**Appl:** Geographic skills. **Aim 8:** The correlation between smoking and lung cancer was "discovered" using mathematics. Science had to justify the cause.
	Covariance and (population) product moment correlation coefficient ρ.	$Cov(X, Y) = E[(X - \mu_x)(Y - \mu_y)]$ $= E(XY) - \mu_x \mu_y$, where $\mu_x = E(X)$, $\mu_y = E(Y)$. $\rho = \dfrac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$.	**Appl:** Using technology to fit a range of curves to a set of data.
	Proof that $\rho = 0$ in the case of independence and ± 1 in the case of a linear relationship between X and Y.	The use of ρ as a measure of association between X and Y, with values near 0 indicating a weak association and values near +1 or near -1 indicating a strong association.	**TOK:** Mathematics and the world. Given that a set of data may be approximately fitted by a range of curves, where would we seek for knowledge of which equation is the "true" model?
	Definition of the (sample) product moment correlation coefficient R in terms of n paired observations on X and Y. Its application to the estimation of ρ.	$R = \dfrac{\displaystyle\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\displaystyle\sum_{i=1}^{n}(X_i - \bar{X})^2 \sum_{i=1}^{n}(Y_i - \bar{Y})^2}}$ $= \dfrac{\displaystyle\sum_{i=1}^{n}X_iY_i - n\bar{X}\bar{Y}}{\sqrt{\left(\displaystyle\sum_{i=1}^{n}X_i^2 - n\bar{X}^2\right)\left(\sum Y_i^2 - n\bar{Y}^2\right)}}$.	**Aim 8:** The physicist Frank Oppenheimer wrote: "Prediction is dependent only on the assumption that observed patterns will be repeated." This is the danger of extrapolation. There are many examples of its failure in the past, eg share prices, the spread of disease, climate change.

(continued)

Content	Further guidance	Links
Informal interpretation of r, the observed value of R. Scatter diagrams.	Values of r near 0 indicate a weak association between X and Y, and values near ± 1 indicate a strong association.	(see notes above)
The following topics are based on the assumption of bivariate normality.	It is expected that the GDC will be used wherever possible in the following work.	
Use of the t-statistic to test the null hypothesis $\rho = 0$.	$R\sqrt{\dfrac{n-2}{1-R^2}}$ has the Student's t-distribution with $(n-2)$ degrees of freedom.	
Knowledge of the facts that the regression of X on Y ($\mathrm{E}(X)\,|\,Y=y$) and Y on X ($\mathrm{E}(Y)\,|\,X=x$) are linear.	$x - \overline{x} = \left(\dfrac{\displaystyle\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\displaystyle\sum_{i=1}^{n}(y_i - \overline{y})^2} \right)(y - \overline{y})$ $= \left(\dfrac{\displaystyle\sum_{i=1}^{n}x_i y_i - n\overline{x}\,\overline{y}}{\displaystyle\sum_{i=1}^{n}y_i^2 - n\overline{y}^2} \right)(y - \overline{y}),$	
Least-squares estimates of these regression lines (proof not required).	$y - \overline{y} = \left(\dfrac{\displaystyle\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\displaystyle\sum_{i=1}^{n}(x_i - \overline{x})^2} \right)(x - \overline{x})$	
The use of these regression lines to predict the value of one of the variables given the value of the other.	$= \left(\dfrac{\displaystyle\sum_{i=1}^{n}x_i y_i - n\overline{x}\,\overline{y}}{\displaystyle\sum_{i=1}^{n}x_i^2 - n\overline{x}^2} \right)(x - \overline{x}).$	

Topic 4—Sets, relations and groups
48 hours

The aims of this topic are to provide the opportunity to study some important mathematical concepts, and introduce the principles of proof through abstract algebra.

	Content	Further guidance	Links
4.1	Finite and infinite sets. Subsets.		**TOK:** Cantor theory of transfinite numbers, Russell's paradox, Godel's incompleteness theorems.
	Operations on sets: union; intersection; complement; set difference; symmetric difference.		
	De Morgan's laws: distributive, associative and commutative laws (for union and intersection).	Illustration of these laws using Venn diagrams. Students may be asked to prove that two sets are the same by establishing that $A \subseteq B$ and $B \subseteq A$.	**Appl:** Logic, Boolean algebra, computer circuits.
4.2	Ordered pairs: the Cartesian product of two sets.		
	Relations: equivalence relations; equivalence classes.	An equivalence relation on a set forms a partition of the set.	**Appl, Int:** Scottish clans.
4.3	Functions: injections; surjections; bijections.	The term codomain.	
	Composition of functions and inverse functions.	Knowledge that the function composition is not a commutative operation and that if f is a bijection from set A to set B then f^{-1} exists and is a bijection from set B to set A.	
4.4	Binary operations.	A binary operation $*$ on a non-empty set S is a rule for combining any two elements $a, b \in S$ to give a unique element c. That is, in this definition, a binary operation on a set is not necessarily closed.	
	Operation tables (Cayley tables).		

	Content	Further guidance	Links
4.5	Binary operations: associative, distributive and commutative properties.	The arithmetic operations on $\mathbb{R}$ and $\mathbb{C}$. Examples of distributivity could include the fact that, on $\mathbb{R}$, multiplication is distributive over addition but addition is not distributive over multiplication.	**TOK:** Which are more fundamental, the general models or the familiar examples?
4.6	The identity element e. The inverse a^{-1} of an element a. Proof that left-cancellation and right-cancellation by an element a hold, provided that a has an inverse. Proofs of the uniqueness of the identity and inverse elements.	Both the right-identity $a*e=a$ and left-identity $e*a=a$ must hold if e is an identity element. Both $a*a^{-1}=e$ and $a^{-1}*a=e$ must hold.	
4.7	The definition of a group $\{G,*\}$. The operation table of a group is a Latin square, but the converse is false. Abelian groups.	For the set G under a given operation $*$: • G is closed under $*$; • $*$ is associative; • G contains an identity element; • each element in G has an inverse in G. $a*b=b*a$, for all $a,b \in G$.	**Appl:** Existence of formula for roots of polynomials. **Appl:** Galois theory for the impossibility of such formulae for polynomials of degree 5 or higher.

	Content	Further guidance	Links
4.8	Examples of groups:	Cross-topic questions may be set in further mathematics examinations, so there may be questions on groups of matrices.	**Appl:** Rubik's cube, time measures, crystal structure, symmetries of molecules, strut and cable constructions, Physics H2.2 (special relativity), the 8-fold way, supersymmetry.
	• $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{C}$ under addition;		
	• integers under addition modulo n;		
	• non-zero integers under multiplication, modulo p, where p is prime;		
	• symmetries of plane figures, including equilateral triangles and rectangles;	The composition $T_2 \circ T_1$ denotes T_1 followed by T_2.	
	• invertible functions under composition of functions.		
4.9	The order of a group.		
	The order of a group element.		
	Cyclic groups.		**Appl:** Music circle of fifths, prime numbers.
	Generators.		
	Proof that all cyclic groups are Abelian.		

	Content	Further guidance	Links
4.10	Permutations under composition of permutations.	On examination papers: the form $p = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ or in cycle notation (132) will be used to represent the permutation $1 \to 3$, $2 \to 1$, $3 \to 2$.	**Appl:** Cryptography, campanology.
	Cycle notation for permutations.		
	Result that every permutation can be written as a composition of disjoint cycles.		
	The order of a combination of cycles.		
4.11	Subgroups, proper subgroups.	A proper subgroup is neither the group itself nor the subgroup containing only the identity element.	
	Use and proof of subgroup tests.	Suppose that $\{G, *\}$ is a group and H is a non-empty subset of G. Then $\{H, *\}$ is a subgroup of $\{G, *\}$ if $a * b^{-1} \in H$ whenever $a, b \in H$.	
		Suppose that $\{G, *\}$ is a finite group and H is a non-empty subset of G. Then $\{H, *\}$ is a subgroup of $\{G, *\}$ if H is closed under $*$.	
	Definition and examples of left and right cosets of a subgroup of a group.		
	Lagrange's theorem.		
	Use and proof of the result that the order of a finite group is divisible by the order of any element. (Corollary to Lagrange's theorem.)		**Appl:** Prime factorization, symmetry breaking.

	Content	Further guidance	Links
4.12	Definition of a group homomorphism.	Infinite groups as well as finite groups. Let $\{G, *\}$ and $\{H, \circ\}$ be groups, then the function $f : G \to H$ is a homomorphism if $f(a*b) = f(a) \circ f(b)$ for all $a, b \in G$.	
	Definition of the kernel of a homomorphism. Proof that the kernel and range of a homomorphism are subgroups.	If $f : G \to H$ is a group homomorphism, then $\mathrm{Ker}(f)$ is the set of $a \in G$ such that $f(a) = e_H$.	
	Proof of homomorphism properties for identities and inverses.	Identity: let e_G and e_H be the identity elements of $(G, *)$ and $(H, \circ)$, respectively, then $f(e_G) = e_H$. Inverse: $f(a^{-1}) = (f(a))^{-1}$ for all $a \in G$.	
	Isomorphism of groups.	Infinite groups as well as finite groups.	
	The order of an element is unchanged by an isomorphism.	The homomorphism $f : G \to H$ is an isomorphism if f is bijective.	

Topic 5—Calculus   48 hours

The aims of this topic are to introduce limit theorems and convergence of series, and to use calculus results to solve differential equations.

	Content	Further guidance	Links
5.1	Infinite sequences of real numbers and their convergence or divergence.	Informal treatment of limit of sum, difference, product, quotient; squeeze theorem. Divergent is taken to mean not convergent.	**TOK:** Zeno's paradox, impact of infinite sequences and limits on our understanding of the physical world.
5.2	Convergence of infinite series. Tests for convergence: comparison test; limit comparison test; ratio test; integral test. The p-series, $\sum \dfrac{1}{n^p}$. Series that converge absolutely. Series that converge conditionally. Alternating series. Power series: radius of convergence and interval of convergence. Determination of the radius of convergence by the ratio test.	The sum of a series is the limit of the sequence of its partial sums. Students should be aware that if $\lim\limits_{x\to\infty} x_n = 0$ then the series is not necessarily convergent, but if $\lim\limits_{x\to\infty} x_n \neq 0$, the series diverges. $\sum \dfrac{1}{n^p}$ is convergent for $p>1$ and divergent otherwise. When $p=1$, this is the harmonic series. Conditions for convergence. The absolute value of the truncation error is less than the next term in the series.	**TOK:** Euler's idea that $1-1+1-1+\ldots = \frac{1}{2}$. Was it a mistake or just an alternative view?

Content	Further guidance	Links
5.3 Continuity and differentiability of a function at a point.	Test for continuity: $\lim\limits_{x \to a-} f(x) = f(a) = \lim\limits_{x \to a+} f(x)$.	
Continuous functions and differentiable functions.	Test for differentiability: f is continuous at a and $\lim\limits_{h \to 0-} \dfrac{f(a+h) - f(a)}{h}$ and $\lim\limits_{h \to 0+} \dfrac{f(a+h) - f(a)}{h}$ exist and are equal. Students should be aware that a function may be continuous but not differentiable at a point, eg $f(x) = |x|$ and simple piecewise functions.	
5.4 The integral as a limit of a sum; lower and upper Riemann sums.		**Int:** How close was Archimedes to integral calculus? **Int:** Contribution of Arab, Chinese and Indian mathematicians to the development of calculus.
Fundamental theorem of calculus.	$\dfrac{d}{dx}\left[\int\limits_a^x f(y)\, dy \right] = f(x)$.	**Aim 8:** Leibniz versus Newton versus the "giants" on whose shoulders they stood—who deserves credit for mathematical progress?
Improper integrals of the type $\int\limits_a^\infty f(x)\, dx$.		**TOK:** Consider $f(x) = \dfrac{1}{x}$, $1 \le x \le \infty$. An infinite area sweeps out a finite volume. Can this be reconciled with our intuition? What does this tell us about mathematical knowledge?

	Content	Further guidance	Links
5.5	First-order differential equations.		**Appl:** Real-life differential equations, eg
	Geometric interpretation using slope fields, including identification of isoclines.		Newton's law of cooling,
			population growth,
	Numerical solution of	$y_{n+1} = y_n + hf(x_n, y_n)$,	carbon dating.
	$\dfrac{dy}{dx} = f(x, y)$	$x_{n+1} = x_n + h$,	
		where h is a constant.	
	using Euler's method.		
	Variables separable.		
	Homogeneous differential equation		
	$\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$		
	using the substitution $y = vx$.		
	Solution of		
	$y' + P(x)y = Q(x)$,		
	using the integrating factor.		

	Content	Further guidance	Links
5.6	Rolle's theorem. Mean value theorem. Taylor polynomials; the Lagrange form of the error term.		**Int, TOK:** Influence of Bourbaki on understanding and teaching of mathematics. **Int:** Compare with work of the Kerala school.
		Applications to the approximation of functions; formula for the error term, in terms of the value of the $(n+1)^{th}$ derivative at an intermediate point.	
	Maclaurin series for e^x, $\sin x$, $\cos x$, $\ln(1+x)$, $(1+x)^p$, $p \in \mathbb{Q}$.	Students should be aware of the intervals of convergence.	
	Use of substitution, products, integration and differentiation to obtain other series. Taylor series developed from differential equations.		
5.7	The evaluation of limits of the form $$\lim_{x \to a} \frac{f(x)}{g(x)} \text{ and } \lim_{x \to \infty} \frac{f(x)}{g(x)}.$$	The indeterminate forms $\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$.	
	Using l'Hôpital's rule or the Taylor series.	Repeated use of l'Hôpital's rule.	

Topic 6—Discrete mathematics

48 hours

The aim of this topic is to provide the opportunity for students to engage in logical reasoning, algorithmic thinking and applications.

	Content	Further guidance	Links
6.1	Strong induction. Pigeon-hole principle.	For example, proofs of the fundamental theorem of arithmetic and the fact that a tree with n vertices has $n-1$ edges.	**TOK:** Mathematics and knowledge claims. The difference between proof and conjecture, eg Goldbach's conjecture. Can a mathematical statement be true before it is proven? **TOK:** Proof by contradiction.
6.2	$a \mid b \Rightarrow b = na$ for some $n \in \mathbb{Z}$. The theorem $a \mid b$ and $a \mid c \Rightarrow a \mid (bx \pm cy)$ where $x, y \in \mathbb{Z}$. Division and Euclidean algorithms. The greatest common divisor, $\gcd(a, b)$, and the least common multiple, $\text{lcm}(a, b)$, of integers a and b. Prime numbers; relatively prime numbers and the fundamental theorem of arithmetic.	The division algorithm $a = bq + r$, $0 \leq r < b$. The Euclidean algorithm for determining the greatest common divisor of two integers.	**Int:** Euclidean algorithm contained in Euclid's *Elements*, written in Alexandria about 300 BCE. **Aim 8:** Use of prime numbers in cryptography. The possible impact of the discovery of powerful factorization techniques on internet and bank security.
6.3	Linear Diophantine equations $ax + by = c$.	General solutions required and solutions subject to constraints. For example, all solutions must be positive.	**Int:** Described in Diophantus' *Arithmetica* written in Alexandria in the 3rd century CE. When studying *Arithmetica*, a French mathematician, Pierre de Fermat (1601–1665) wrote in the margin that he had discovered a simple proof regarding higher-order Diophantine equations—Fermat's last theorem.

	Content	Further guidance	Links
6.4	Modular arithmetic. The solution of linear congruences. Solution of simultaneous linear congruences (Chinese remainder theorem).		**Int:** Discussed by Chinese mathematician Sun Tzu in the 3rd century CE.
6.5	Representation of integers in different bases.	On examination papers, questions that go beyond base 16 will not be set.	**Int:** Babylonians developed a base 60 number system and the Mayans a base 20 number system.
6.6	Fermat's little theorem.	$a^p = a \pmod{p}$, where p is prime.	**TOK:** Nature of mathematics. An interest may be pursued for centuries before becoming "useful".
6.7	Graphs, vertices, edges, faces. Adjacent vertices, adjacent edges. Degree of a vertex, degree sequence. Handshaking lemma. Simple graphs; connected graphs; complete graphs; bipartite graphs; planar graphs; trees; weighted graphs, including tabular representation. Subgraphs; complements of graphs. Euler's relation: $v - e + f = 2$; theorems for planar graphs including $e \le 3v - 6$, $e \le 2v - 4$, leading to the results that κ_5 and $\kappa_{3,3}$ are not planar.	Two vertices are adjacent if they are joined by an edge. Two edges are adjacent if they have a common vertex. It should be stressed that a graph should not be assumed to be simple unless specifically stated. The term adjacency table may be used. If the graph is simple and planar and $v \ge 3$, then $e \le 3v - 6$. If the graph is simple, planar, has no cycles of length 3 and $v \ge 3$, then $e \le 2v - 4$.	**Aim 8:** Symbolic maps, eg Metro and Underground maps, structural formulae in chemistry, electrical circuits. **TOK:** Mathematics and knowledge claims. Proof of the four-colour theorem. If a theorem is proved by computer, how can we claim to know that it is true? **Aim 8:** Importance of planar graphs in constructing circuit boards. **TOK:** Mathematics and knowledge claims. Applications of the Euler characteristic $(v - e + f)$ to higher dimensions. Its use in understanding properties of shapes that cannot be visualized.

	Content	Further guidance	Links
6.8	Walks, trails, paths, circuits, cycles.		
	Eulerian trails and circuits.	A connected graph contains an Eulerian circuit if and only if every vertex of the graph is of even degree.	**Int:** The "Bridges of Königsberg" problem.
	Hamiltonian paths and cycles.	Simple treatment only.	
6.9	Graph algorithms: Kruskal's; Dijkstra's.		
6.10	Chinese postman problem.	To determine the shortest route around a weighted graph going along each edge at least once.	**Int:** Problem posed by the Chinese mathematician Kwan Mei-Ko in 1962.
	Not required: Graphs with more than four vertices of odd degree.		
	Travelling salesman problem.	To determine the Hamiltonian cycle of least weight in a weighted complete graph.	**TOK:** Mathematics and knowledge claims. How long would it take a computer to test all Hamiltonian cycles in a complete, weighted graph with just 30 vertices?
	Nearest-neighbour algorithm for determining an upper bound.		
	Deleted vertex algorithm for determining a lower bound.		
6.11	Recurrence relations. Initial conditions, recursive definition of a sequence.		**TOK:** Mathematics and the world. The connections of sequences such as the Fibonacci sequence with art and biology.
	Solution of first- and second-degree linear homogeneous recurrence relations with constant coefficients.	Includes the cases where auxiliary equation has equal roots or complex roots.	
	The first-degree linear recurrence relation $u_n = au_{n-1} + b$.		
	Modelling with recurrence relations.	Solving problems such as compound interest, debt repayment and counting problems.	

Glossary of terminology: Discrete mathematics

Introduction

Teachers and students should be aware that many different terminologies exist in graph theory, and that different textbooks may employ different combinations of these. Examples of these are: vertex/node/junction/point; edge/route/arc; degree/order of a vertex; multiple edges/parallel edges; loop/self-loop.

In IB examination questions, the terminology used will be as it appears in the syllabus. For clarity, these terms are defined below.

Terminology

Bipartite graph	A graph whose vertices can be divided into two sets such that no two vertices in the same set are adjacent.
Circuit	A walk that begins and ends at the same vertex, and has no repeated edges.
Complement of a graph G	A graph with the same vertices as G but which has an edge between any two vertices if and only if G does not.
Complete bipartite graph	A bipartite graph in which every vertex in one set is joined to every vertex in the other set.
Complete graph	A simple graph in which each pair of vertices is joined by an edge.
Connected graph	A graph in which each pair of vertices is joined by a path.
Cycle	A walk that begins and ends at the same vertex, and has no other repeated vertices.
Degree of a vertex	The number of edges joined to the vertex; a loop contributes two edges, one for each of its end points.
Disconnected graph	A graph that has at least one pair of vertices not joined by a path.
Eulerian circuit	A circuit that contains every edge of a graph.
Eulerian trail	A trail that contains every edge of a graph.
Graph	Consists of a set of vertices and a set of edges.
Graph isomorphism between two simple graphs G and H	A one-to-one correspondence between vertices of G and H such that a pair of vertices in G is adjacent if and only if the corresponding pair in H is adjacent.
Hamiltonian cycle	A cycle that contains all the vertices of the graph.
Hamiltonian path	A path that contains all the vertices of the graph.

Loop	An edge joining a vertex to itself.
Minimum spanning tree	A spanning tree of a weighted graph that has the minimum total weight.
Multiple edges	Occur if more than one edge joins the same pair of vertices.
Path	A walk with no repeated vertices.
Planar graph	A graph that can be drawn in the plane without any edge crossing another.
Simple graph	A graph without loops or multiple edges.
Spanning tree of a graph	A subgraph that is a tree, containing every vertex of the graph.
Subgraph	A graph within a graph.
Trail	A walk in which no edge appears more than once.
Tree	A connected graph that contains no cycles.
Walk	A sequence of linked edges.
Weighted graph	A graph in which each edge is allocated a number or weight.
Weighted tree	A tree in which each edge is allocated a number or weight.

Assessment in the Diploma Programme

General

Assessment is an integral part of teaching and learning. The most important aims of assessment in the Diploma Programme are that it should support curricular goals and encourage appropriate student learning. Both external and internal assessment are used in the Diploma Programme. IB examiners mark work produced for external assessment, while work produced for internal assessment is marked by teachers and externally moderated by the IB.

There are two types of assessment identified by the IB.

• Formative assessment informs both teaching and learning. It is concerned with providing accurate and helpful feedback to students and teachers on the kind of learning taking place and the nature of students' strengths and weaknesses in order to help develop students' understanding and capabilities. Formative assessment can also help to improve teaching quality, as it can provide information to monitor progress towards meeting the course aims and objectives.

• Summative assessment gives an overview of previous learning and is concerned with measuring student achievement.

The Diploma Programme primarily focuses on summative assessment designed to record student achievement at or towards the end of the course of study. However, many of the assessment instruments can also be used formatively during the course of teaching and learning, and teachers are encouraged to do this. A comprehensive assessment plan is viewed as being integral with teaching, learning and course organization. For further information, see the IB *Programme standards and practices* document.

The approach to assessment used by the IB is criterion-related, not norm-referenced. This approach to assessment judges students' work by their performance in relation to identified levels of attainment, and not in relation to the work of other students. For further information on assessment within the Diploma Programme, please refer to the publication *Diploma Programme assessment: Principles and practice*.

To support teachers in the planning, delivery and assessment of the Diploma Programme courses, a variety of resources can be found on the OCC or purchased from the IB store (http://store.ibo.org). Teacher support materials, subject reports, internal assessment guidance, grade descriptors, as well as resources from other teachers, can be found on the OCC. Specimen and past examination papers as well as markschemes can be purchased from the IB store.

Methods of assessment

The IB uses several methods to assess work produced by students.

Assessment criteria

Assessment criteria are used when the assessment task is open-ended. Each criterion concentrates on a particular skill that students are expected to demonstrate. An assessment objective describes what students should be able to do, and assessment criteria describe how well they should be able to do it. Using assessment criteria allows discrimination between different answers and encourages a variety of responses. Each criterion comprises a set of hierarchically ordered level descriptors. Each level descriptor is worth one or more marks. Each criterion is applied independently using a best-fit model. The maximum marks for each criterion may differ according to the criterion's importance. The marks awarded for each criterion are added together to give the total mark for the piece of work.

Markbands

Markbands are a comprehensive statement of expected performance against which responses are judged. They represent a single holistic criterion divided into level descriptors. Each level descriptor corresponds to a range of marks to differentiate student performance. A best-fit approach is used to ascertain which particular mark to use from the possible range for each level descriptor.

Markschemes

This generic term is used to describe analytic markschemes that are prepared for specific examination papers. Analytic markschemes are prepared for those examination questions that expect a particular kind of response and/or a given final answer from the students. They give detailed instructions to examiners on how to break down the total mark for each question for different parts of the response. A markscheme may include the content expected in the responses to questions or may be a series of marking notes giving guidance on how to apply criteria.

Assessment outline

First examinations 2014

Assessment component	Weighting
External assessment (5 hours)	
Paper 1 (2 hours 30 minutes) Graphic display calculator required. Compulsory short- to medium-response questions based on the whole syllabus.	**50%**
Paper 2 (2 hours 30 minutes) Graphic display calculator required. Compulsory medium- to extended-response questions based on the whole syllabus.	**50%**

Assessment details

External assessment

Papers 1 and 2

These papers are externally set and externally marked. The papers are designed to allow students to demonstrate what they know and what they can do.

Markschemes are used to assess students in both papers. The markschemes are specific to each examination.

Calculators

Papers 1 and 2

Students must have access to a GDC at all times. However, not all questions will necessarily require the use of the GDC. Regulations covering the types of GDC allowed are provided in the *Handbook of procedures for the Diploma Programme*.

Mathematics HL and further mathematics HL formula booklet

Each student must have access to a clean copy of the formula booklet during the examination. It is the responsibility of the school to download a copy from IBIS or the OCC and to ensure that there are sufficient copies available for all students.

Awarding of marks

Marks may be awarded for method, accuracy, answers and reasoning, including interpretation.

In paper 1 and paper 2, full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations (in the form of, for example, diagrams, graphs or calculations). Where an answer is incorrect, some marks may be given for correct method, provided this is shown by written working. All students should therefore be advised to show their working.

Paper 1

Duration: 2 hours 30 minutes
Weighting: 50%
- This paper consists of short- to medium-response questions. A GDC is required for this paper, but not every question will necessarily require its use.

Syllabus coverage
- Knowledge of **all** topics in the syllabus is required for this paper. However, not all topics are necessarily assessed in every examination session.

Mark allocation

- This paper is worth **150** marks, representing 50% of the final mark.

- Questions of varying levels of difficulty and length are set. Therefore, individual questions may not necessarily be worth the same number of marks. The exact number of marks allocated to each question is indicated at the start of the question.

- The intention of this paper is to test students' knowledge across the breadth of the syllabus.

Question type

- Questions may be presented in the form of words, symbols, diagrams or tables, or combinations of these.

Paper 2

Duration: 2 hours 30 minutes
Weighting: 50%

- This paper consists of medium- to extended-response questions. A GDC is required for this paper, but not every question will necessarily require its use.

Syllabus coverage

- Knowledge of **all** topics in the core of the syllabus is required for this paper. However, not all topics are necessarily assessed in every examination session.

Mark allocation

- This paper is worth **150** marks, representing **50%** of the final mark.

- Questions of varying levels of difficulty and length are set. Therefore, individual questions may not necessarily each be worth the same number of marks. The exact number of marks allocated to each question is indicated at the start of the question.

- The intention of this paper is to test students' knowledge and understanding across the breadth of the syllabus.

Question type

- Questions may be presented in the form of words, symbols, diagrams or tables, or combinations of these.

Internal assessment

There is no internal assessment component in this course.

Glossary of command terms

Command terms with definitions

Students should be familiar with the following key terms and phrases used in examination questions, which are to be understood as described below. Although these terms will be used in examination questions, other terms may be used to direct students to present an argument in a specific way.

Calculate	Obtain a numerical answer showing the relevant stages in the working.
Comment	Give a judgment based on a given statement or result of a calculation.
Compare	Give an account of the similarities between two (or more) items or situations, referring to both (all) of them throughout.
Compare and contrast	Give an account of the similarities and differences between two (or more) items or situations, referring to both (all) of them throughout.
Construct	Display information in a diagrammatic or logical form.
Contrast	Give an account of the differences between two (or more) items or situations, referring to both (all) of them throughout.
Deduce	Reach a conclusion from the information given.
Demonstrate	Make clear by reasoning or evidence, illustrating with examples or practical application.
Describe	Give a detailed account.
Determine	Obtain the only possible answer.
Differentiate	Obtain the derivative of a function.
Distinguish	Make clear the differences between two or more concepts or items.
Draw	Represent by means of a labelled, accurate diagram or graph, using a pencil. A ruler (straight edge) should be used for straight lines. Diagrams should be drawn to scale. Graphs should have points correctly plotted (if appropriate) and joined in a straight line or smooth curve.
Estimate	Obtain an approximate value.
Explain	Give a detailed account, including reasons or causes.
Find	Obtain an answer, showing relevant stages in the working.
Hence	Use the preceding work to obtain the required result.
Hence or otherwise	It is suggested that the preceding work is used, but other methods could also receive credit.

Identify	Provide an answer from a number of possibilities.
Integrate	Obtain the integral of a function.
Interpret	Use knowledge and understanding to recognize trends and draw conclusions from given information.
Investigate	Observe, study, or make a detailed and systematic examination, in order to establish facts and reach new conclusions.
Justify	Give valid reasons or evidence to support an answer or conclusion.
Label	Add labels to a diagram.
List	Give a sequence of brief answers with no explanation.
Plot	Mark the position of points on a diagram.
Predict	Give an expected result.
Prove	Use a sequence of logical steps to obtain the required result in a formal way.
Show	Give the steps in a calculation or derivation.
Show that	Obtain the required result (possibly using information given) without the formality of proof. "Show that" questions do not generally require the use of a calculator.
Sketch	Represent by means of a diagram or graph (labelled as appropriate). The sketch should give a general idea of the required shape or relationship, and should include relevant features.
Solve	Obtain the answer(s) using algebraic and/or numerical and/or graphical methods.
State	Give a specific name, value or other brief answer without explanation or calculation.
Suggest	Propose a solution, hypothesis or other possible answer.
Verify	Provide evidence that validates the result.
Write down	Obtain the answer(s), usually by extracting information. Little or no calculation is required. Working does not need to be shown.

Notation list

Of the various notations in use, the IB has chosen to adopt a system of notation based on the recommendations of the International Organization for Standardization (ISO). This notation is used in the examination papers for this course without explanation. If forms of notation other than those listed in this guide are used on a particular examination paper, they are defined within the question in which they appear.

Because students are required to recognize, though not necessarily use, IB notation in examinations, it is recommended that teachers introduce students to this notation at the earliest opportunity. Students are **not** allowed access to information about this notation in the examinations.

Students must always use correct mathematical notation, not calculator notation.

$\mathbb{N}$	the set of positive integers and zero, $\{0, 1, 2, 3, ...\}$
$\mathbb{Z}$	the set of integers, $\{0, \pm 1, \pm 2, \pm 3, ...\}$
$\mathbb{Z}^+$	the set of positive integers, $\{1, 2, 3, ...\}$
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Q}^+$	the set of positive rational numbers, $\{x \mid x \in \mathbb{Q}, x > 0\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers, $\{x \mid x \in \mathbb{R}, x > 0\}$
$\mathbb{C}$	the set of complex numbers, $\{a + \mathrm{i}b \mid a, b \in \mathbb{R}\}$
i	$\sqrt{-1}$
z	a complex number
z^*	the complex conjugate of z
$\lvert z \rvert$	the modulus of z
$\arg z$	the argument of z
$\mathrm{Re}\, z$	the real part of z
$\mathrm{Im}\, z$	the imaginary part of z
$\mathrm{cis}\,\theta$	$\cos \theta + \mathrm{i} \sin \theta$
$\{x_1, x_2, ...\}$	the set with elements $x_1, x_2, ...$
$n(A)$	the number of elements in the finite set A
$\{x \mid \quad \}$	the set of all x such that
$\in$	is an element of
$\notin$	is not an element of
$\varnothing$	the empty (null) set
U	the universal set
$\cup$	union

$\cap$	intersection
$\subset$	is a proper subset of
$\subseteq$	is a subset of
A'	the complement of the set A
$A \times B$	the Cartesian product of sets A and B (that is, $A \times B = \{(a, b) \mid a \in A, \ b \in B\}$)
$a \mid b$	a divides b
$a^{1/n}, \ \sqrt[n]{a}$	a to the power of $\dfrac{1}{n}$, n^{th} root of a (if $a \geq 0$ then $\sqrt[n]{a} \geq 0$)
$a^{1/2}, \ \sqrt{a}$	a to the power $\dfrac{1}{2}$, square root of a (if $a \geq 0$ then $\sqrt{a} \geq 0$)
$\|x\|$	the modulus or absolute value of x, that is $\begin{cases} x \text{ for } x \geq 0, \ x \in \mathbb{R} \\ -x \text{ for } x < 0, \ x \in \mathbb{R} \end{cases}$
$\equiv$	identity
$\approx$	is approximately equal to
$>$	is greater than
$\geq$	is greater than or equal to
$<$	is less than
$\leq$	is less than or equal to
$\ngtr$	is not greater than
$\nless$	is not less than
$[a, b]$	the closed interval $a \leq x \leq b$
$]a, b[$	the open interval $a < x < b$
u_n	the nth term of a sequence or series
d	the common difference of an arithmetic sequence
r	the common ratio of a geometric sequence
S_n	the sum of the first n terms of a sequence, $u_1 + u_2 + \ldots + u_n$
S_∞	the sum to infinity of a sequence, $u_1 + u_2 + \ldots$
$\displaystyle\sum_{i=1}^{n} u_i$	$u_1 + u_2 + \ldots + u_n$
$\displaystyle\prod_{i=1}^{n} u_i$	$u_1 \times u_2 \times \ldots \times u_n$
$\dbinom{n}{r}$	$\dfrac{n!}{r!(n-r)!}$

$f : A \rightarrow B$	f is a function under which each element of set A has an image in set B
$f : x \mapsto y$	f is a function under which x is mapped to y
$f(x)$	the image of x under the function f
f^{-1}	the inverse function of the function f
$f \circ g$	the composite function of f and g
$\lim_{x \to a} f(x)$	the limit of $f(x)$ as x tends to a
$\dfrac{dy}{dx}$	the derivative of y with respect to x
$f'(x)$	the derivative of $f(x)$ with respect to x
$\dfrac{d^2 y}{dx^2}$	the second derivative of y with respect to x
$f''(x)$	the second derivative of $f(x)$ with respect to x
$\dfrac{d^n y}{dx^n}$	the nth derivative of y with respect to x
$f^{(n)}(x)$	the nth derivative of $f(x)$ with respect to x
$\int y \, dx$	the indefinite integral of y with respect to x
$\int_a^b y \, dx$	the definite integral of y with respect to x between the limits $x = a$ and $x = b$
e^x	the exponential function of x
$\log_a x$	the logarithm to the base a of x
$\ln x$	the natural logarithm of x, $\log_e x$
$\sin, \ \cos, \ \tan$	the circular functions
$\arcsin, \ \arccos, \ \arctan$	the inverse circular functions
$\csc, \ \sec, \ \cot$	the reciprocal circular functions
$A(x, y)$	the point A in the plane with Cartesian coordinates x and y
$[AB]$	the line segment with end points A and B
AB	the length of $[AB]$
(AB)	the line containing points A and B
$\hat{A}$	the angle at A

$\text{C}\hat{\text{A}}\text{B}$	the angle between $[\text{CA}]$ and $[\text{AB}]$
ΔABC	the triangle whose vertices are A, B and C
$\boldsymbol{v}$	the vector $\boldsymbol{v}$
$\overrightarrow{\text{AB}}$	the vector represented in magnitude and direction by the directed line segment from A to B
$\boldsymbol{a}$	the position vector $\overrightarrow{\text{OA}}$
$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$	unit vectors in the directions of the Cartesian coordinate axes
$\lvert \boldsymbol{a} \rvert$	the magnitude of $\boldsymbol{a}$
$\lvert \overrightarrow{\text{AB}} \rvert$	the magnitude of $\overrightarrow{\text{AB}}$
$\boldsymbol{v} \cdot \boldsymbol{w}$	the scalar product of $\boldsymbol{v}$ and $\boldsymbol{w}$
$\boldsymbol{v} \times \boldsymbol{w}$	the vector product of $\boldsymbol{v}$ and $\boldsymbol{w}$
$\boldsymbol{A}^{-1}$	the inverse of the non-singular matrix $\boldsymbol{A}$
$\boldsymbol{A}^{\text{T}}$	the transpose of the matrix $\boldsymbol{A}$
$\det \boldsymbol{A}$	the determinant of the square matrix $\boldsymbol{A}$
$\boldsymbol{I}$	the identity matrix
$\text{P}(A)$	the probability of event A
$\text{P}(A')$	the probability of the event "not A"
$\text{P}(A \mid B)$	the probability of the event A given B
$x_1, x_2, \ldots$	observations
$f_1, f_2, \ldots$	frequencies with which the observations $x_1, x_2, \ldots$ occur
P_x	the probability distribution function $\text{P}(X = x)$ of the discrete random variable X
$f(x)$	the probability density function of the continuous random variable X
$F(x)$	the cumulative distribution function of the continuous random variable X
$\text{E}(X)$	the expected value of the random variable X
$\text{Var}(X)$	the variance of the random variable X
μ	population mean
σ^2	population variance, $\sigma^2 = \dfrac{\sum\limits_{i=1}^{k} f_i (x_i - \mu)^2}{n}$, where $n = \sum\limits_{i=1}^{k} f_i$
σ	population standard deviation
$\overline{x}$	sample mean

s_n^2	sample variance, $s_n^2 = \dfrac{\sum\limits_{i=1}^{k} f_i(x_i - \bar{x})^2}{n}$, where $n = \sum\limits_{i=1}^{k} f_i$
s_n	standard deviation of the sample
s_{n-1}^2	unbiased estimate of the population variance, $s_{n-1}^2 = \dfrac{n}{n-1} s_n^2 = \dfrac{\sum\limits_{i=1}^{k} f_i(x_i - \bar{x})^2}{n-1}$, where $n = \sum\limits_{i=1}^{k} f_i$
$\mathrm{B}(n, p)$	binomial distribution with parameters n and p
$\mathrm{Po}(m)$	Poisson distribution with mean m
$\mathrm{N}(\mu, \sigma^2)$	normal distribution with mean μ and variance σ^2
$X \sim \mathrm{B}(n, p)$	the random variable X has a binomial distribution with parameters n and p
$X \sim \mathrm{Po}(m)$	the random variable X has a Poisson distribution with mean m
$X \sim \mathrm{N}(\mu, \sigma^2)$	the random variable X has a normal distribution with mean μ and variance σ^2
Φ	cumulative distribution function of the standardized normal variable with distribution $\mathrm{N}(0,1)$
ν	number of degrees of freedom
$A \setminus B$	the difference of the sets A and B (that is, $A \setminus B = A \cap B' = \{x \mid x \in A \text{ and } x \notin B\}$)
$A \Delta B$	the symmetric difference of the sets A and B (that is, $A \Delta B = (A \setminus B) \cup (B \setminus A)$)
κ_n	a complete graph with n vertices
$\kappa_{n,m}$	a complete bipartite graph with one set of n vertices and another set of m vertices
$\mathbb{Z}_p$	the set of equivalence classes $\{0, 1, 2, \ldots, p-1\}$ of integers modulo p
$\gcd(a,b)$	the greatest common divisor of integers a and b
$\mathrm{lcm}(a,b)$	the least common multiple of integers a and b
$\boldsymbol{A}_G$	the adjacency matrix of graph G
$\boldsymbol{C}_G$	the cost adjacency matrix of graph G