Precalc takeaways **Name:**

Math 3 Unit 2: Logarithmic Functions

Learning targets

2.1 I can calculate compound interest. (Modeling)
2.2 I can convert equations between exponential and logarithmic forms. (Algebra)
2.3 I can graph a logarithmic function and a transformed logarithmic function. (Geometry)
2.4 I can apply properties of logarithms. (Algebra)
2.5 I can solve exponential equations using logarithms. (Modeling)

Compound interest equations

$A = P\left(1 + \frac{r}{n}\right)^{nt}$

r = annual interest rate (APR) (written as a decimal, not a percent)
n = number of compounding periods per year
P = starting amount of money (or debt) / "before"
t = number of years since the start
A = amount of money (or debt) now / "after"

effective annual interest

1. find amount of money (or debt) after one year
2. subtract the starting amount
3. divide by the starting amount and multiply by 100 to get a percentage

continuous compounding

compounded every instant, infinite compounding periods

$A = Pe^{rt}$

$e \approx 2.71828$, a.k.a. Euler's number

$\left(1 + \frac{1}{n}\right)^n$ as n gets very large

Compound interest example(s)

① Jennifer needs \$28000 five years from now. She will invest in a savings account that pays 4.3% APR and compounds monthly. How much should she invest?

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$28000 = P\left(1+\frac{0.043}{12}\right)^{12\cdot 5}$$

$$\frac{28000}{1.239} = \frac{P(1.239)}{1.239}$$

$$\boxed{P = \$22591.84}$$

② What is the effective annual interest, if the interest rate is 14%, compounded continuously?

$$A = Pe^{rt}$$

starting amount shouldn't matter. let's say \$123

$A = 123e^{0.14\cdot 1}$ ← annual, so use $t=1$

$$A = 123e^{0.14} = 141.48$$

$$141.48 - 123 = 18.48$$

$$18.48 \div 123 = 0.150$$

$$0.150 \cdot 100 = 15.0$$

$\boxed{\text{15.0\% effective annual interest}}$

Definition of a logarithm

a logarithmic function is the inverse of an exponential function.

the log is the exponent. (the output of the log function is the exponent.)

we use logs to solve for exponents.

Logarithmic vs. exponential form

$$\log_b x = a \longleftrightarrow b^a = x$$

example: convert $\log_2 8 = 3$ to exponential form

answer: $2^3 = 8$

example: convert $4^x = y$ to logarithmic form

answer: $\log_4 y = x$

Notation and vocab: $\log_b x = a$

b is the base (same as the base of the exponent)

a is the exponent

x is the "answer" when you calculate b^a

Fun facts about $\log_b x$:

$\log_b b = 1$ because $b^1 = b$

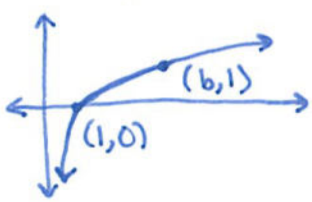
$\log_b 1 = 0$ because $b^0 = 1$

$\log_b b^n = n$ because $b^n = b^n$

b must be positive and $b \neq 1$, otherwise $\log_b x$ is undefined

Graphs of logarithmic functions

Graph of $\boxed{f(x) = \log_b x}$



like an exponential curve, but reflected.

always increasing, no upper limit, but increases slower as x gets bigger

x-intercept is (1,0)

asymptote is y-axis, or $x=0$ (the graph gets close to it but never touches)

a larger base b makes a shaper corner

domain: $x > 0$ a.k.a. $(0, \infty)$

asymptote: $x = 0$

Graph of $\boxed{g(x) = k + \log_b (x - h)}$

Same as the graph of $f(x) = \log_b x$, except:

- translated up k units (or down if k is negative)
- translated to the right h units (or left if it's like $x + \#$)
- vertical asymptote is $x = h$
- point (1,0) moves to $(1+h, k)$
- point (b,1) moves to $(b+h, 1+k)$

Special log notation

- $\log x$ means $\log_{10} x$. This is the "common log".
- $\ln x$ means $\log_e x$. This is the "natural log".

Properties of Logarithms

Logarithm of a product:

$$\log_b(xy) = \log_b x + \log_b y$$

Logarithm of a quotient:

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

Logarithm of a power:

$$\log_b(x^k) = k \cdot \log_b x$$

Change of base

$$\log_b x = \frac{\log x}{\log b} \quad \text{or} \quad \log_b x = \frac{\ln x}{\ln b}$$

Examples

Condense into a single logarithm:

$$\log 3 - \log x - \log x$$
$$= \log 3 - 2\log x$$
$$= \log 3 - \log(x^2)$$
$$= \log\left(\frac{3}{x^2}\right)$$

Expand: $\log_5(z\sqrt[3]{xy})$

$$= \log_5(z(xy)^{1/3})$$
$$= \log_5 z + \log_5 (xy)^{1/3}$$
$$= \log_5 z + \frac{1}{3}\log_5(xy)$$
$$= \log_5 z + \frac{1}{3}(\log_5 x + \log_5 y)$$
$$= \log_5 z + \frac{1}{3}\log_5 x + \frac{1}{3}\log_5 y$$

Continuous exponential growth

in IM1: $f(t) = P \cdot b^t$

where P = starting amount

and b = common ratio/growth factor/decay factor

in precalc: any base works, but e is a popular choice, so any continuous exponential function can be modeled as

$f(t) = Pe^{rt}$

where r = the growth rate, a number that changes the base to e

r is not the percent increase

you will probably have to solve for r to create your function

Continuous exponential decay

similar to exponential growth, $f(t) = Pe^{rt}$

but expect r to be negative

How to solve for r, the growth or decay rate

1. do algebra until you have only an exponential expression e^{rt} on one side
2. convert exponential equation to logarithmic equation
3. do algebra to solve for r

example: 15g decays to 7.5g in 3 years. write a function to model.

$f(t) = Pe^{rt}$

$\frac{7.5}{15} = \frac{15e^{r \cdot 3}}{15}$ algebra

$0.5 = e^{3r}$ convert to log

$\frac{\ln(0.5)}{3} = \frac{3r}{3}$ algebra

$-0.231 \approx r$

$\boxed{f(t) = 15e^{-0.231t}}$

How to solve for time t

similar process to solving for r

Examples

A person invests \$9000 in a bank. The bank pays 5% interest compounded twice a year. How long must the person leave the money in the bank until it reaches \$22300?

$A = P\left(1+\frac{r}{n}\right)^{nt}$

$22300 = 9000\left(1+\frac{0.05}{2}\right)^{2t}$

$\frac{22300}{9000} = \frac{9000(1.025)^{2t}}{9000}$

$2.4778 = (1.025)^{2t}$

$\log_{1.025} 2.4778 = 2t$

$\frac{36.746}{2} = \frac{2t}{2}$

$\boxed{t = 18.37 \text{ years}}$

Element X decays radioactively with a half life of 11 minutes. If there are 970 grams of Element X, how long will it take to decay to 260 grams?

First: use half life info to solve for r.

half life: the time it takes to decay from 2g to 1g.

$f(t) = Pe^{rt}$

$\frac{1}{2} = \frac{2e^{r \cdot 11}}{2}$

$0.5 = e^{11r}$

$\frac{\ln(0.5)}{11} = \frac{11r}{11}$

$-0.0630 \approx r$

Second: plug in for P & r and solve for t.

$f(t) = 970e^{-0.0630t}$

$\frac{260}{970} = \frac{970e^{-0.0630t}}{970}$

$0.268 = e^{-0.0630t}$

$\ln(0.268) = -0.0630t$

$\frac{-1.317}{-0.0630} = \frac{-0.0630t}{-0.0630}$

$\boxed{t = 20.9 \text{ min}}$