CALCULUS: Graphical, Numerical, Algebraic by Finney, Demana, Watts and Kennedy
Chapter 9: Error and Series 9.3:

What you'll Learn About
How to find the error of a series that does not alternate

Lagrange Error Bound/Taylors Inequality/Remainder Estimation Theorem

1. Give the first term of the series for $f(x) = e^x$ centered at $x = 0$
$P_1(x) = 1$
2. Find the approximation for P(.1) $= 1$
3. Find f(.1) $= e^{.1} = 1.105170918$
4. How accurate is the approximation. $|f(x) - P_1(x)| = .105170918$
5. What is the value of the next term of the polynomial at x =.1
next term $= x \rightarrow .1$

1. Give the first two terms of the series for $f(x) = e^x$ centered at $x = 0$
$1 + x$
2. Find the approximation for P(.1) $= 1 + .1 = 1.1$
3. Find f(.1) $= 1.105170918$
4. How accurate is the approximation. $|f(.1) - P_1(.1)| = .005170918$
5. What is the value of the next term of the polynomial at x =.1
next term $= \frac{x^2}{2!} \rightarrow \frac{.1^2}{2} = .005$

1. Give the first three terms of the series for $f(x) = e^x$ centered at $x = 0$
$1 + x + \frac{x^2}{2}$
2. Find the approximation for P(.1) $= 1 + .1 + \frac{.1^2}{2} = 1.105$
3. Find f(.1) $= 1.105170918$
4. How accurate is the approximation. $|f(.1) - P_2(.1)| = .000170918$
$f^3(0) = 1$
5. What is the value of the next term of the polynomial at x =.1

$\frac{f^3(0)x^3}{3!} = $ next term $= \frac{x^3}{3!} \rightarrow \frac{.1^3}{6} = \frac{.001}{6} = .000166666$

$\lvert f(x)-P(x)\rvert \le R$ *Where* *R =* $\left\lvert \dfrac{\left(\text{Max of the next derivative on the given interval}\right)(x-c)^{n+1}}{(n+1)!} \right\rvert$ Where **x-c** is the distance from the center Where **n** is the order We must build the next term a little bit bigger to have a good boundary for the error.	1. *Give the first 4 terms of the series for $f(x)=e^x$ centered at $x = 0$* $1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}$ 2. *Use Taylors Inequality to determine the error bound* $\lvert f(x)-P(x)\rvert \le R$ $f(x)=e^x$ $f'(x)=e^x$ $f''(x)=e^x$ $f'''(x)=e^x$ $\boxed{f^4(x)=e^x}$ Build next term at $x=0$: $\frac{f^4(0)x^4}{4!}$, $\frac{x^4}{4!}$ $x=.1$: $\frac{f^4(.1)x^4}{4!}$, $\frac{e^{.1}x^4}{4!} = \frac{1.105x^4}{4!}$ 1. *Find the 3rd order polynomial of the series for* $f(x)=\frac{1}{(1-x)^2}$ *centered at* $x = 0$ $f(x)=\frac{1}{(1-x)^2}=(1-x)^{-2}$ $\quad f(0)=1$ $f'(x)=+2(1-x)^{-3}$ $\quad f'(0)=2$ $f''(x)=+6(1-x)^{-4}$ $\quad f''(0)=6$ $f'''(x)=24(1-x)^{-5}$ $\quad f^3(0)=24$
Remember, whenever you see this, $\lvert f(x)-P(x)\rvert \le R$, *you are finding error bound* whenever you see this, $\lvert f(x)-P(x)\rvert$, *you are finding the actual error between the function and the approximation from the polynomial*	$\boxed{f^4(x)=120(1-x)^{-6}}$ $P_3(x)=1+2x+\frac{6x^2}{2!}+\frac{24x^3}{3!}$ Actual Difference error bound 2. *Find the Lagrange error bound* $\lvert f(x)-P(x)\rvert \le R$ *for the series between* $0 \le x \le .2$ Build next term at $\quad \frac{f^4(0)x^4}{4!}$ $x=0$: $\frac{120x^4}{4!}$ $x=.2$: $\frac{120(.8)^{-6}x^4}{4!} = \frac{\frac{120}{(.8)^6}x^4}{4!}$ distance from center

Actual difference is less than or equal to $R = .030517\ldots$

$R = \dfrac{\frac{120}{(.8)^6}(.2)^4}{4!}$

4) Write the 2nd order Taylor Polynomial for $f(x) = \cos x$ at $x = \frac{\pi}{4}$.
Then use Taylors Inequality to determine the error bound at $x = 42°$

5) Write the 1st degree Taylor Polynomial for $f(x) = \arcsin x$ at $x = 0$.
Then use Taylors Inequality to determine the error bound at $x = .2$

$f(x) = \arcsin x \qquad f(0) = 0$

$f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} \qquad f'(0) = 1$

$P_1 = \frac{0x^0}{0!} + \frac{1x^1}{1!}$

$P_1(x) = x$

Next Term

$\frac{f^2(0)x^2}{2!}$

$f^2(x) = f''(x) = -\frac{1}{2}(1-x^2)^{-3/2} \cdot (-2x)$

$= \frac{x}{(1-x^2)^{3/2}}$

Build next term at

$x = 0$

$\frac{0x^2}{2!}$

$x = .2$

$\boxed{\frac{\frac{.2}{(1-.2^2)^{3/2}} x^2}{2!}}$

$|f(x) - P(x)| \leq R$

R = Error Bound

$\boxed{R = \frac{\frac{.2}{(1-.2^2)^{3/2}}(.2)^2}{2!}}$

.05 away from center on both sides of center	9) Given that $\boxed{P_1(x) = x}$ represents the first order polynomial for sinx centered at x = 0. Use the Lagrange Error Bound to find the error when $\lvert x\rvert \le .05$ $f(x) = \sin x$ $f'(x) = \cos x$ $f''(x) = -\sin x$ $\boxed{f'''(x) = -\cos x}$ Build next term at $x = -.05$: $\frac{-\cos(-.05)x^3}{3!}$, $\frac{-.998x^3}{3!}$ $x = 0$: $\frac{f^3(0)x^3}{3!}$, $\boxed{\frac{-1x^3}{3!}}$ $x = .05$: $\frac{-\cos(+.05)x^3}{3!}$, $\frac{-.998x^3}{3!}$ error bound $\le \left\lvert \frac{-(.05)^3}{3!} \right\rvert$ 14) Given that $P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$ represents the third order Taylor polynomial for ln(x) centered at x = 1. Use the Lagrange Error Bound to find the error when $\lvert x-1\rvert \le .1$

Summary of Error Bound

For an Alternating Series – Use the next term

For a series that is Not Alternating

1. Write down the formula for the next derivative.

2. Find the value of the next derivative at the ends of the interval and the center.

3. Whichever value is bigger is the value you use to build your error bound term