2.7 Ratios and Proportions

Learning Objectives

- Write and understand a ratio.
- Write and solve a proportion.
- Solve proportions using cross products.

Introduction

Nadia is counting out money with her little brother. She gives her brother all the nickels and pennies. She keeps the quarters and dimes for herself. Nadia has four quarters (worth 25 cents each) and six dimes (worth 10 cents each). Her brother has fifteen nickels (worth 5 cents each) and five pennies (worth one cent each) and is happy because he has more coins than his big sister. How would you explain to him that he is actually getting a bad deal?

Write a ratio

A **ratio** is a way to compare two numbers, measurements or quantities. When we write a ratio, we divide one number by another and express the answer as a fraction. There are two distinct ratios in the problem above. For example, the ratio of the **number** of Nadias coins to her brothers is:

$$\frac{4+6}{15+5}=\frac{10}{20}$$

When we write a ratio, the correct way is to simplify the fraction.

$$\frac{10}{20}=\frac{\cancel{2}.\cancel{5}}{2.\cancel{2}.\cancel{5}}=\frac{1}{2}$$

In other words, Nadia has half the number of coins as her brother.

Another ratio we could look at in the problem is the **value** of the coins. The value of Nadias coins is $(4\times 25)+(6\times 10)=160$ cents . The value of her brothers coins is $(15\times 5)+(5\times 1)=80$ cents . The ratio of the **value** of Nadias coins to her brothers is:

$$\frac{160}{80}=\frac{2}{1}$$

So the value of Nadias money is twice the value of her brothers.

Notice that even though the denominator is one, it is still written. A ratio with a denominator of one is called a **unit rate**. In this case, it means Nadia is gaining money at twice the rate of her brother.

Example 1

The price of a Harry Potter Book on Amazon.com is $10.00. *The same book is also available used for* $6.50. *Find two ways to compare these prices.*

Clearly, the cost of a new book is greater than the used book price. We can compare the two numbers using a difference equation:

Difference in price = 10.00 − \$6.50 = \$3.50

We can also use a ratio to compare the prices:

$$\frac{new\ price}{used\ price} = \frac{\$10.00}{\$6.50} \qquad \text{We can cancel the units of\$ as they are the same.}$$

$$\frac{10}{6.50} = \frac{1000}{650} = \frac{20}{13} \qquad \text{We remove the decimals and simplify the fraction.}$$

Solution

The new book is \$3.50 more than the used book.

The new book costs $\frac{20}{13}$ times the cost of the used book.

Example 2

The State Dining Room in the White House measures approximately 48*feet long by* 36*feet wide. Compare the length of room to the width, and express your answer as a ratio.*

Solution

$$\frac{48\ feet}{36\ feet} = \frac{48}{36} = \frac{4}{3}$$

Example 3

A tournament size shuffleboard table measures 30*inches wide by* 14*feet long. Compare the length of the table to its width and express the answer as a ratio.*

We could write the ratio immediately as:

$\frac{14\ feet}{30\ inches}$*Notice that we cannot cancel the units.*

Sometimes it is OK to leave the units in, but as we are comparing two lengths, it makes sense to convert all the measurements to the same units.

Solution

$$\frac{14\ feet}{30\ inches} = \frac{14 \times 12\ inches}{30\ inches} = \frac{168}{30} = \frac{28}{5}$$

Example 4

A family car is being tested for fuel efficiency. It drives non-stop for 100 miles,*and uses* 3.2*gallons of gasoline. Write the ratio of distance traveled to fuel used as a* **unit rate.**

$$Ratio = \frac{100\ miles}{3.2\ gallons}$$

A unit rate has a denominator of one, so we need to divide both numerator and denominator by 3.2.

$$Unit\ Rate \frac{\left(\frac{100}{3.2}\right)\ miles}{\left(\frac{3.2}{3.2}\right)\ gallons} = \frac{31.25\ miles}{1\ gallon}$$

Solution

The ratio of distance to fuel used is $\frac{31.25 \text{ miles}}{1 \text{ gallon}}$ or 31.25 miles per gallon.

Write and Solve a Proportion

When two ratios are equal to each other, we call it a proportion.

$$\frac{10}{15} = \frac{6}{9}$$

This statement is a proportion. We know the statement is true because we can reduce both fractions to $\frac{2}{3}$.

Check this yourself to make sure!

We often use proportions in science and business. For example, when scaling up the size of something. We use them to solve for an unknown, so we will use algebra and label our unknown variable x. We assume that a certain ratio holds true whatever the size of the thing we are enlarging (or reducing). The next few examples demonstrate this.

Example 5

A small fast food chain operates 60*stores and makes* \$1.2*million profit every year. How much profit would the chain make if it operated 250 stores?*

First, we need to write a **ratio**. This will be the ratio of profit to number of stores.

$$\text{Ratio} = \frac{\$1,200,000}{60 \text{ stores}}$$

We now need to determine our unknown, x which will be in dollars. It is the profit of 250 stores. Here is the ratio that compares unknown dollars to 250 stores.

$$\text{Ratio} = \frac{\$x}{250 \text{ stores}}$$

We now write equal ratios and solve the resulting **proportion**.

$\frac{\$1,200,000}{60\ \text{stores}} = \frac{\$x}{250\ \text{stores}}$ or $\frac{1,200,000}{60} = \frac{x}{250}$

Note that we can drop the units not because they are the same on the numerator and denominator, but because they are the same on both sides of the equation.

$$\begin{aligned} \frac{1,200,000}{60} &= \frac{x}{250} && \text{Simplify fractions.} \\ 20,000 &= \frac{x}{250} && \text{Multiply both sides by 250.} \\ 5,000,000 &= x \end{aligned}$$

Solution

If the chain operated 250 stores the annual profit would be 5 milliondollars .

Example 6

A chemical company makes up batches of copper sulfate solution by adding 250 kg*of copper sulfate powder to* 1000*liters of water. A laboratory chemist wants to make a solution of identical concentration, but only needs* 350 ml(0.35 liters)*of solution. How much copper sulfate powder should the chemist add to the water?*

First we write our ratio. The mass of powder divided by the volume of water used by the chemical company.

$$\text{Ratio} = \frac{250\ \text{kg}}{1000\ \text{liters}} \qquad \text{We can reduce this to :}\ \frac{1\ \text{kg}}{4\ \text{liters}}$$

Our unknown is the mass in kilograms of powder to add. This will be x. The volume of water will be 0.35 liters .

$$Ratio = \frac{x\ kg}{0.35\ liters}$$

Our proportion comes from setting the two ratios equal to each other:

$\frac{1\ kg}{4\ liters} = \frac{x\ kg}{0.35\ liters}$ which becomes $\frac{1}{4} = \frac{x}{0.35}$

We now solve for x.

$$\begin{aligned} \frac{1}{4} &= \frac{x}{0.35} && \text{Multiply both sides by 0.35.} \\ 0.35 \cdot \frac{1}{4} &= \frac{x}{0.35} \cdot 0.35 \\ x &= 0.0875 \end{aligned}$$

Solution

The mass of copper sulfate that the chemist should add is 0.0875 kg or 87.5 grams .

Solve Proportions Using Cross Products

One neat way to simplify proportions is to cross multiply. Consider the following proportion.

$$\frac{16}{4} = \frac{20}{5}$$

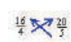
If we want to eliminate the fractions, we could multiply both sides by 4 and then multiply both sides by 5. In fact we ***could*** do both at once:

$$\begin{aligned} 4 \cdot 5 \cdot \frac{16}{4} &= 4 \cdot 5 \cdot \frac{20}{5} \\ 5 \cdot 16 &= 4 \cdot 20 \end{aligned}$$

Now comparing this to the proportion we started with, we see that the denominator from the left hand side ends up multiplying with the numerator on the right hand side.

You can also see that the denominator from the ***right*** hand side ends up multiplying the numerator on the ***left*** hand side.

In effect the two denominators have ***multiplied*** across the equal sign:



$\Rightarrow$

$$5 \cdot 16 = 4 \cdot 20$$

This movement of denominators is known as **cross multiplying**. It is extremely useful in solving proportions, especially when the unknown variable is on the denominator.

Example 7

Solve the proportion for x.

$$\frac{4}{3} = \frac{9}{x}$$

Cross multiply:

$$\begin{aligned} x \cdot 4 &= 9 \cdot 3 \\ \frac{4x}{4} &= \frac{27}{4} && \text{Divide both sides by4.} \end{aligned}$$

Solution

$$x = 6.75$$

Example 8

Solve the following proportion for x.

$$\frac{0.5}{3} = \frac{56}{x}$$

Cross multiply:

$$x \cdot 0.5 = 56 \cdot 3$$
$$\frac{0.5x}{0.5} = \frac{168}{0.5} \qquad \text{Divide both sides by } 0.5.$$

Solution:

$$x = 336$$

Solve Real-World Problems Using Proportions

When we are faced with a word problem that requires us to write a proportion, we need to identify both the unknown (which will be the quantity we represent as x) and the ratio which will stay fixed.

Example 9

A cross-country train travels at a steady speed. It covers 15 miles in20 minutes . *How far will it travel in* 7 hours*assuming it continues at the same speed?*

This example is a $\text{Distance} = \text{speed} \times \text{time}$ problem. We came across a similar problem in Lesson 3.3. Recall that the speed of a body is the quantity distance/time. This will be our ratio. We simply plug in the known quantities. We will, however convert to hours from minutes.

$$\text{Ratio} = \frac{15 \text{ miles}}{20 \text{ minutes}} = \frac{15 \text{ miles}}{\frac{1}{3}\text{hour}}$$

This is a very awkward looking ratio, but since we will be cross multiplying we will leave it as it is. Next, we set up our proportion.

$$\frac{15 miles}{\frac{1}{3}\text{hour}} = \frac{x\ miles}{7\ hours}$$

Cancel the units and cross-multiply.

$$7 \cdot 15 = \frac{1}{3} \cdot x \qquad \text{Multiply both sides by 3.}$$
$$3 \cdot 7 \cdot 15 = 3 \cdot \frac{1}{3} \cdot x$$
$$315 = x$$

Solution

The train will travel 315 miles in7 hours .

Example 10

Rain is falling at 1*inch every* 1.5 hours. *How high will the water level be if it rains at the same rate for* 3 hours*?*

Although it may not look it, this again uses the Distance = speed × time relationship. The distance the water rises in inches will be our x. The ratio will again be $\frac{distance}{time}$.

$$\begin{aligned} \frac{1 \text{ inch}}{1.5 \text{ hours}} &= \frac{x \text{ inch}}{3 \text{ hours}} && \text{Cancel units and cross multiply.} \\ \frac{3(1)}{1.5} &= \frac{1.5x}{1.5} && \text{Divide by } 1.5 \\ 2 &= x \end{aligned}$$

Solution

The water will be 2 inches high if it rains for 3 hours .

Example 11

In the United Kingdom, Alzheimers disease is said to affect one in fifty people over 65*years of age. If approximately* 250000*people over* 65*are affected in the UK, how many people over* 65*are there in total?*

The fixed ratio in this case will be the 1 person in 50. The unknown (x) is the number of persons over 65. Note that in this case, the ratio does not have units, as they will cancel between the numerator and denominator.

We can go straight to the proportion.

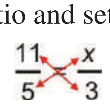
$$\begin{aligned} \frac{1}{50} &= \frac{250000}{x} && \text{Cross multiply :} \\ 1 \cdot x &= 250000 \cdot 50 \\ x &= 12,500,000 \end{aligned}$$

Solution

There are approximately 12.5 million people over the age of 65.

Lesson Summary

- A **ratio** is a way to compare two numbers, measurements or quantities by dividing one number by the other and expressing the answer as a fraction. $\frac{2}{3}$, $\frac{32 \text{ miles}}{1.4 \text{ gallons}}$, and $\frac{x}{13}$ are all ratios.
- **A proportion** is formed when two ratios are set equal to each other.
- **Cross multiplication** is useful for solving equations in the form of proportions. To cross multiply, multiply the bottom of each ratio by the top of the other ratio and set them equal. For instance, cross multiplying



results in $11 \cdot 3 = 5 \cdot x$.

Review Questions

1. Write the following comparisons as ratios. Simplify fractions where possible.
   (a) $150 to $3
   (b) 150 boys to 175 girls
   (c) 200 minutes to 1 hour
   (d) 10 days to 2 weeks
2. Write the following ratios as a unit rate.
   (a) 54 hotdogs to 12 minutes
   (b) 5000 lbs to $250in^2$
   (c) 20 computers to 80 students
   (d) 180 students to 6 teachers
   (e) 12 meters to 4 floors
   (f) 18 minutes to 15 appointments
3. Solve the following proportions.
   (a) $\frac{13}{6} = \frac{5}{x}$
   (b) $\frac{1.25}{7} = \frac{3.6}{x}$
   (c) $\frac{6}{19} = \frac{x}{11}$
   (d) $\frac{1}{x} = \frac{0.01}{5}$
   (e) $\frac{300}{4} = \frac{x}{99}$
   (f) $\frac{2.75}{9} = \frac{x}{\left(\frac{2}{9}\right)}$
   (g) $\frac{1.3}{4} = \frac{x}{1.3}$
   (h) $\frac{0.1}{1.01} = \frac{1.9}{x}$
4. A restaurant serves 100 people per day and takes $908. If the restaurant were to serve 250 people per day, what might the taking be?
5. The highest mountain in Canada is Mount Yukon. It is $\frac{298}{67}$ the size of Ben Nevis, the highest peak in Scotland. Mount Elbert in Colorado is the highest peak in the Rocky Mountains. Mount Elbert is $\frac{220}{67}$ the height of Ben Nevis and $\frac{44}{48}$ the size of Mont Blanc in France. Mont Blanc is 4800 meters high. How high is Mount Yukon?
6. At a large high school it is estimated that two out of every three students have a cell phone, and one in five of all students have a cell phone that is one year old or less. Out of the students who own a cell phone, what proportion owns a phone that is more than one year old?

Review Answers

1. (a) $\frac{50}{1}$
   (b) $\frac{6}{7}$
   (c) $\frac{10}{3}$
   (d) $\frac{5}{7}$
2. (a) 4.5 hot-dogs per minute
   (b) 20 lbs per in^2
   (c) 0.25 computers per student
   (d) 30 students per teacher
   (e) 3 meters per floor
   (f) 1.2 minutes per appointment
3. (a) $x = \frac{30}{13}$
   (b) $x = 20.16$
   (c) $x = \frac{66}{19}$
   (d) $x = 500$

    (e) $x = 7425$
    (f) $x = \frac{11}{162}$
    (g) $x = 0.4225$
    (h) $x = \frac{100}{1919}$
4. \$2270
5. 5960 meters .
6. $\frac{3}{10}$ or 30%