2.2 Order of Operations

Learning Objectives

- Evaluate algebraic expressions with grouping symbols.
- Evaluate algebraic expressions with fraction bars.
- Evaluate algebraic expressions with a graphing calculator.

Introduction

Look at and evaluate the following expression:

$$2+4\times 7-1=?$$

How many different ways can we interpret this problem, and how many different answers could someone possibly find for it?

The *simplest* way to evaluate the expression is simply to start at the left and work your way across, keeping track of the total as you go:

$$\begin{aligned}2+4&=6\\6\times 7&=42\\42-1&=41\end{aligned}$$

If you enter the expression into a *non-scientific*, non-graphing calculator you will probably get 41 as the answer. If, on the other hand, you were to enter the expression into a scientific calculator or a graphing calculator you would probably get 29 as an answer.

In mathematics, the order in which we perform the various **operations** (such as adding, multiplying, etc.) is important. In the expression above, the operation of **multiplication** takes precedence over **addition** so we evaluate it first. Lets re-write the expression, but put the multiplication in brackets to indicate that it is to be evaluated first.

$$2+(4\times 7)-1=?$$

So we first evaluate the brackets: $4\times 7=28$. Our expression becomes:

$$2+(28)-1=?$$

When we have only addition and subtraction, we start at the left and keep track of the total as we go:

$$\begin{aligned}2+28&=30\\30-1&=29\end{aligned}$$

Algebra students often use the word **PEMDAS** to help remember the order in which we evaluate the mathematical expressions: **P**arentheses, **E**xponents, **M**ultiplication, **D**ivision, **A**ddition and **S**ubtraction.

Order of Operations

1. Evaluate expressions within **P**arentheses (also all brackets [ ] and braces) first.
2. Evaluate all **E**xponents (squared or cubed terms such as 3^2 or x^3) next.
3. **M**ultiplication *and* **D**ivision is next work from left to right completing **both** multiplication and division in the order that they appear.
4. Finally, evaluate **A**ddition *and* **S**ubtraction work from left to right completing **both** addition and subtraction in the order that they appear.

Evaluate Algebraic Expressions with Grouping Symbols

The first step in the order of operations is called **parentheses**, but we include all **grouping symbols** in this step. While we will mostly use parentheses ( ) in this book, you may also see square brackets [ ] and curly braces and you should include them as part of the first step.

Example 1

Evaluate the following:

a) $4-7-11-2$

b) $4-(7-11)+2$

c) $4-[7-(11+2)]$

Each of these expressions has the same numbers and the same mathematical operations, in the same order. The placement of the various grouping symbols means, however, that we must evaluate everything in a different order each time. Let's look at how we evaluate each of these examples.

a) This expression doesn't have parentheses. PEMDAS states that we treat addition and subtraction as they appear, starting at the left and working right (its NOT addition *then* subtraction).

Solution

$$\begin{aligned} 4-7-11+2 &= -3-11+2 \\ &= -14+2 \\ &= -12 \end{aligned}$$

b) This expression has parentheses. We first evaluate $7-11=-4$. Remember that when we subtract a negative it is equivalent to adding a positive:

Solution

$$\begin{aligned} 4-(7-11)+2 &= 4-(-4)+2 \\ &= 8+2 \\ &= 10 \end{aligned}$$

c) Brackets are often used to group expressions which already contain parentheses. This expression has both brackets and parentheses. Do the innermost group first, $(11+2)=13$. Then complete the operation in the brackets.

Solution

$$\begin{aligned}4-[7-(11+2)] &= 4-[7-(13)]\\ &= 4-[-6]\\ &= 10\end{aligned}$$

Example 2

Evaluate the following:

a) $3\times 5-7\div 2$

b) $3\times(5-7)\div 2$

c) $(3\times 5)-(7\div 2)$

a) There are no grouping symbols. PEMDAS dictates that we evaluate multiplication and division first, working from left to right: $3\times 5=15$; $7\div 2=3.5$. (NOTE: Its not multiplication *then* addition) Next we perform the subtraction:

Solution

$$\begin{aligned}3\times 5-7\div 2 &= 15-3.5\\ &= 11.5\end{aligned}$$

b) First, we evaluate the expression inside the parentheses: $5-7=-2$. Then work from left to right.

Solution

$$\begin{aligned}3\times(5-7)\div 2 &= 3\times(-2)\div 2\\ &= (-6)\div 2\\ &= -3\end{aligned}$$

c) First, we evaluate the expressions inside parentheses: $3\times 5=15$, $7\div 2=3.5$. Then work from left to right.

Solution

$$\begin{aligned}(3\times 5)-(7\div 2) &= 15-3.5\\ &= 11.5\end{aligned}$$

Note that in part (c), the result was unchanged by adding parentheses, but the expression does appear easier to read. Parentheses can be used in two distinct ways:

- To alter the order of operations in a given expression
- To clarify the expression to make it easier to understand

Some expressions contain no parentheses, others contain many sets. Sometimes expressions will have sets of parentheses **inside** other sets of parentheses. When faced with **nested parentheses**, start at the innermost parentheses and work outward.

Example 3

Use the order of operations to evaluate:

$$8-[19-(2+5)-7)]$$

Follow PEMDAS first parentheses, starting with innermost brackets first:

Solution

$$\begin{aligned}8-(19-(2+5)-7) &= 8-(19-7-7)\\ &= 8-5\\ &= 3\end{aligned}$$

In algebra, we use the order of operations when we are substituting values into expressions for variables. In those situations we will be given an expression involving a variable or variables, and also the values to substitute for any variables in that expression.

Example 4

Use the order of operations to evaluate the following:

a) $2-(3x+2)$ when $x=2$

b) $3y^2+2y-1$ when $y=-3$

c) $2-(t-7)^2 \times (u^3-v)$ when $t=19, u=4$ and $v=2$

a) The first step is to substitute in the value for x into the expression. Let's put it in parentheses to clarify the resulting expression.

Solution

$$2-(3(2)+2) \qquad 3(2) \text{ is the same as } 3\times 2$$

Follow PEMDAS first parentheses. Inside parentheses follow PEMDAS again.

$$\begin{aligned}2-(3\times 2+2) = 2-(6+2) &\qquad \text{Inside the parentheses, we evaluate the multiplication first.}\\ 2-8=-6 &\qquad \text{Now we evaluate the parentheses.}\end{aligned}$$

b) The first step is to substitute in the value for y into the expression.

Solution

$$3\times(-3)^2+2\times(-3)-1$$

Follow PEMDAS: we cannot simplify parentheses.

$$\begin{aligned}&= 3\times(-3)^2+2\times(-3)-1 &&\text{Evaluate exponents}: (-3)^2=9\\ &= 3\times 9+2\times(-3)-1 &&\text{Evaluate multiplication}: 3\times 9=27; 2\times -3=-6\\ &= 27+(-6)-1 &&\text{Evaluate addition and subtraction in order from left to right.}\\ &= 27-6-1\\ &= 20\end{aligned}$$

c) The first step is to substitute the values for t, u, and v into the expression.

Solution:

$$2-(19-7)^2\times(4^3-2)$$

Follow **PEMDAS:**

$$\begin{aligned}
&=2-(19-7)^2\times(4^3-2) && \text{Evaluate parentheses : }(19-7)=12;(4^3-2)=(64-2)=62\\
&=2-12^2\times 62 && \text{Evaluate exponents : }12^2=144\\
&=2-144\times 62 && \text{Evaluate the multiplication : }144\times 62=8928\\
&=2-8928 && \text{Evaluate the subtraction.}\\
&=-8926
\end{aligned}$$

In parts (b) and (c) we left the parentheses around the negative numbers to clarify the problem. They did not affect the order of operations, but they did help avoid confusion when we were multiplying negative numbers.

Part (c) in the last example shows another interesting point. When we have an expression inside the parentheses, we use PEMDAS to determine the order in which we evaluate the contents.

Evaluating Algebraic Expressions with Fraction Bars

Fraction bars count as grouping symbols for PEMDAS, and should therefore be evaluated in the first step of solving an expression. All numerators and all denominators can be treated as if they have invisible parentheses. When **real** parentheses are also present, remember that the innermost grouping symbols should be evaluated first. If, for example, parentheses appear on a numerator, they would take precedence over the fraction bar. If the parentheses appear outside of the fraction, then the fraction bar takes precedence.

Example 5

Use the order of operations to evaluate the following expressions:

a) $\frac{z+3}{4}-1 \textit{When } z=2$

b) $\left(\frac{a+2}{b+4}-1\right)+b \textit{When } a=3$ and $b=1$

c) $2\times\left(\frac{w+(x-2z)}{(y+2)^2}-1\right) \textit{When } w=11, x=3, y=1$ and $z=-2$

a) We substitute the value for z into the expression.

Solution:

$$\frac{2+3}{4}-1$$

Although this expression has no parentheses, we will rewrite it to show the effect of the fraction bar.

$$\frac{(2+3)}{4}-1$$

Using PEMDAS, we first evaluate the expression on the numerator.

$$\frac{5}{4} - 1$$

We can convert $\frac{5}{4}$ to a mixed number:

$$\frac{5}{4} = 1\frac{1}{4}$$

Then evaluate the expression:

$$\frac{5}{4} - 1 = 1\frac{1}{4} - 1 = \frac{1}{4}$$

b) We substitute the values for *a* and *b* into the expression:

Solution:

$$\left(\frac{3+2}{1+4} - 1\right) - 1$$

This expression has nested parentheses (remember the effect of the fraction bar on the numerator and denominator). The innermost grouping symbol is provided by the fraction bar. We evaluate the numerator $(3+2)$ and denominator $(1+4)$ first.

$\left(\frac{5}{5} - 1\right) - 1$	Now we evaluate the inside of the parentheses, starting with division.
$(1-1) - 1$	Next the subtraction.
$0 - 1 = -1$	

c) We substitute the values for *w*, *x*, *y* and *z* into the expression:

Solution:

This complicated expression has several layers of nested parentheses. One method for ensuring that we start with the innermost parentheses is to make use of the other types of brackets. We can rewrite this expression, putting brackets in for the fraction bar. The outermost brackets we will leave as parentheses ( ). Next will be the *invisible brackets* from the fraction bar, these will be written as [ ]. The third level of nested parentheses will be the . We will leave negative numbers in round brackets.

$2\left(\frac{[11+\{\,3-2(-2)\,\}]}{[\{\,1+2\,\}^2]} - 1\right)$	We start with the innermost grouping sign $\{\}$.
	$\{\,1+2\,\} = 3; \{\,3-2(-2)\,\} = 3+4 = 7$
$2\left(\frac{[11+7]}{[3^2]} - 1\right)$	The next level has two square brackets to evaluate.
$2\left(\frac{18}{9} - 1\right)$	We now evaluate the round brackets, starting with division.
$2(2-1)$	Finally, we complete the addition and subtraction.
$2(1) = 2$	

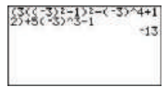
Evaluate Algebraic Expressions with a TI-83/84 Family Graphing Calculator

A graphing calculator is a very useful tool in evaluating algebraic expressions. The graphing calculator follows PEMDAS. In this section we will explain two ways of evaluating expressions with the graphing calculator.

Method 1: Substitute for the variable first. Then evaluate the numerical expression with the calculator.

Example 6

Evaluate $[3(x^2-1)^2-x^4+12]+5x^3-1$ *when* $x=-3$



Solution:

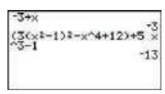
Substitute the value $x=-3$ *into the expression.*

$$[3((-3)^2-1)^2-(-3)^4+12]+5(-3)^3-1$$

Input this in the calculator just as it is and press **[ENTER]**. *(Note, use ^ for exponents)*

The answer is -13.

Method 2: Input the original expression in the calculator first and then evaluate. Lets look at the same example.

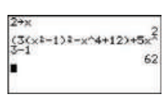


Evaluate $[3(x^2-1)^2-x^4+12]+5x^3-1$ *when* $x=-3$

First, store the value $x=-3$ *in the calculator. Type* -3**[STO]** x*(The letter* x *can be entered using the* x-**[VAR]** *button or* **[ALPHA]** + **[STO]**). *Then type in the expression in the calculator and press* **[ENTER]**.

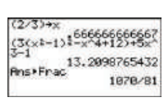
The answer is -13.

The second method is better because you can easily evaluate the same expression for any value you want. For example, lets evaluate the same expression using the values $x=2$ *and* $x=\frac{2}{3}$.



For $x=2$, store the value of x in the calculator: 2**[STO]** x. Press **[2nd] [ENTER]** twice to get the previous expression you typed in on the screen without having to enter it again. Press **[ENTER]** to evaluate.

The answer is 62.



For $x=\frac{2}{3}$, store the value of x in the calculator: $\frac{2}{3}$**[STO]** x. Press **[2nd] [ENTER]** twice to get the expression on the screen without having to enter it again. Press **[ENTER]** to evaluate.

The answer is 13.21 or $\frac{1070}{81}$ in fraction form.

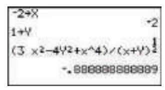
Note: On graphing calculators there is a difference between the minus sign and the negative sign. When we stored the value negative three, we needed to use the negative sign which is to the left of the **[ENTER]** button on the calculator. On the other hand, to perform the subtraction operation in the expression we used the minus sign. The minus sign is right above the plus sign on the right.

You can also use a graphing calculator to evaluate expressions with more than one variable.

Example 7

Evaluate the expression: $\frac{3x^2-4y^2+x^4}{(x+y)^{1/2}}$ *for* $x=-2, y=1$.

Solution



Store the values of x and y. -2**[STO]** x, 1**[STO]** y. The letters x and y can be entered using **[ALPHA] + [KEY]**. Input the expression in the calculator. When an expression shows the division of two expressions be sure to use parentheses: (numerator) $\div$ (denominator)

Press **[ENTER]** to obtain the answer $-.8\bar{8}$ or $-\frac{8}{9}$.

Review Questions

1. Use the order of operations to evaluate the following expressions.
   (a) $8-(19-(2+5)-7)$
   (b) $2+7\times 11-12\div 3$
   (c) $(3+7)\div(7-12)$
   (d) $\frac{2\cdot(3+(2-1))}{4-(6+2)}-(3-5)$
2. Evaluate the following expressions involving variables.
   (a) $\frac{jk}{j+k}$ when $j=6$ and $k=12$.
   (b) $2y^2$ when $x=1$ and $y=5$
   (c) $3x^2+2x+1$ when $x=5$
   (d) $(y^2-x)^2$ when $x=2$ and $y=1$
3. Evaluate the following expressions involving variables.
   (a) $\frac{4x}{9x^2-3x+1}$ when $x=2$
   (b) $\frac{z^2}{x+y}+\frac{x^2}{x-y}$ when $x=1$, $y=-2$, and $z=4$.
   (c) $\frac{4xyz}{y^2-x^2}$ when $x=3$, $y=2$, and $z=5$
   (d) $\frac{x^2-z^2}{xz-2x(z-x)}$ when $x=-1$ and $z=3$
4. Insert parentheses in each expression to make a true equation.
   (a) $5-2\cdot 6-4+2=5$
   (b) $12\div 4+10-3\cdot 3+7=11$
   (c) $22-32-5\cdot 3-6=30$
   (d) $12-8-4\cdot 5=-8$
5. Evaluate each expression using a graphing calculator.
   (a) $x^2+2x-xy$ when $x=250$ and $y=-120$
   (b) $(xy-y^4)^2$ when $x=0.02$ and $y=-0.025$

(c) $\frac{x+y-z}{xy+yz+xz}$ when $x=\frac{1}{2}$, $y=\frac{3}{2}$, and $z=-1$
(d) $\frac{(x+y)^2}{4x^2-y^2}$ when $x=3$ and $y=-5d$

Review Answers

1. (a) 3
   (b) 75
   (c) -2
   (d) -2
2. (a) 4
   (b) 300
   (c) 86
   (d) 3
3. (a) $\frac{8}{31}$
   (b) $-\frac{47}{3}$
   (c) -24
   (d) $-\frac{8}{5}$
4. (a) $(5-2)\cdot(6-5)+2=5$
   (b) $(12\div 4)+10-(3\cdot 3)+7=11$
   (c) $(22-32-5)\cdot(3-6)=30$
   (d) $12-(8-4)\cdot 5=-8$
5. (a) 93000
   (b) 0.00000025
   (c) $-\frac{12}{5}$
   (d) $\frac{4}{11}$