1.4 Multiplication of Rational Numbers

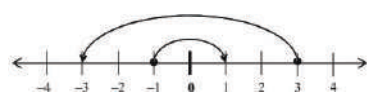
Learning Objectives

- Multiply by negative one.
- Multiply rational numbers.
- Identify and apply properties of multiplication.
- Solve real-world problems using multiplication.

Multiplying Numbers by Negative One

Whenever we multiply a number by negative one we change the sign of the number. In more mathematical words, multiplying by negative one maps a number onto its opposite. The number line below shows the process of multiplying negative one by the numbers three and negative one.

$$3\cdot -1 = -3$$
$$-1\cdot -1 = 1$$



- When we multiply a number by negative one the absolute value of the new number is the same as the absolute value of the old number. Both numbers are the same distance from zero.
- The product of a number, x, and negative one is $-x$. This does not mean that $-x$ is necessarily less than zero. If x itself is negative then $-x$ is a positive quantity because a negative times a negative is a positive.
- When we multiply an expression by negative one remember to multiply the **entire expression by** negative one.

Example 1

Multiply the following by negative one.

a) 79.5

b) π

c) $(x+1)$

d) $|x|$

a) **Solution**

$$79.5\cdot(-1) = -79.5$$

b) **Solution**

$$\pi \cdot (-1) = -\pi$$

c) **Solution**

$$(x+1) \cdot (-1) = -(x+1) = -x-1$$

d) **Solution**

$$|x| \cdot (-1) = -|x|$$

Note that in the last case the negative sign does **not** distribute into the absolute value. Multiplying the **argument** of an absolute value equation (the term between the absolute value symbol) does not change the absolute value. $|x|$ is always positive. $|-x|$ is always positive. $-|x|$ is always negative.

Whenever you are working with expressions, you can check your answers by substituting in numbers for the variables. For example you could check part *d* of example one by letting $x = -3$.

$|-3| \neq -|3|$ since $|-3| = 3$ and $-|3| = -3$.

To Multiply two numbers, multiply their absolute values.

The sign of the answer is

1. POSITIVE if both numbers have the same sign

*-5 * -5 = 25*

5 * 5 = 25

2. NEGATIVE if the numbers have opposite signs

*-5 * 5 = -25*

5 * -5 = -25

Example 2

a. $-1 \cdot -10 = 10$ b. $12 \cdot -5 = -60$ c. $-15 \cdot 5 = -75$ d. $7 \cdot 8 = 56$

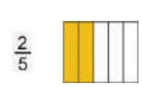
To Multiply Fractions or Rational Numbers

Cross SimplifyMultiply the numerators togetherMultiply the denominators togetherSimplify if possible.

Example 3

Simplify $\frac{1}{3} \cdot \frac{2}{5}$

One way to solve this is to think of money. For example, we know that *one third of sixty dollars* is written as $\frac{1}{3} \cdot \$60$. We can read the above problem as *one-third of two-fifths*. Here is a visual picture of the fractions ***one-third*** and ***two-fifths.***



Notice that *one-third of two-fifths* looks like the *one-third* of the shaded region in the next figure.

Here is the intersection of the two shaded regions. The whole has been divided into five pieces width-wise and three pieces height-wise. We get two pieces out of a total of fifteen pieces.

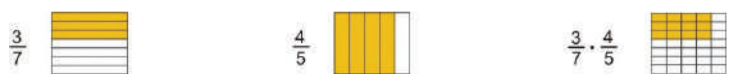
Solution

$$\frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}$$

Example 4

Simplify $\frac{3}{7} \cdot \frac{4}{5}$

We will again go with a visual representation.



We see that the whole has been divided into a total of $7 \cdot 5$ pieces. We get $3 \cdot 4$ of those pieces.

Solution

$$\frac{3}{70} \cdot \frac{4}{5} = \frac{12}{35}$$

When multiplying rational numbers, the numerators multiply together and the denominators multiply together.

When multiplying fractions $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

Even though we have shown this result for the product of two fractions, this rule holds true when multiplying multiple fractions together.

Example 5

Multiply the following rational numbers

a) $\frac{1}{2} \cdot \frac{3}{4}$

b) $-\frac{2}{5} \cdot \frac{5}{9}$

c) $-\frac{1}{3} \cdot -\frac{2}{7} \cdot \frac{2}{5}$

d) $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5}$

a) **Solution**

$$\frac{1}{2} \cdot \frac{3}{4} = \frac{1 \cdot 3}{2 \cdot 4} = \frac{3}{8}$$

b) **Solution** With this problem, we can cancel the fives.

$$\frac{-2}{5} \cdot \frac{5}{9} = \frac{-2 \cdot 5}{5 \cdot 9} = -\frac{2}{9}$$

c) **Solution** With this problem, multiply **all the numerators** and **all the denominators**.

$$\frac{-1}{3} \cdot \frac{-2}{7} \cdot \frac{2}{5} = \frac{-1 \cdot -2 \cdot 2}{3 \cdot 7 \cdot 5} = \frac{4}{105}$$

d) **Solution** With this problem, we can cancel any factor that appears as both a numerator **and** a denominator since any number divided by itself is one, according to the Multiplicative Identity Property.

$$\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{4}} \times \frac{\cancel{4}}{5} = \frac{1}{5}$$

With multiplication of fractions, we can either simplify before we multiply or after. The next example uses factors to help simplify before we multiply.

Example 6

Evaluate and simplify $\frac{12}{25} \cdot \frac{35}{42}$

We can see that 12 and 42 are both multiples of six, and that 25 and 35 are both factors of five. We write the product again, but put in these factors so that we can cancel them prior to multiplying.

$$\frac{12}{25} \cdot \frac{35}{42} = \frac{6 \cdot 2}{25} \cdot \frac{35}{6 \cdot 7} = \frac{6 \cdot 2 \cdot 5 \cdot 7}{5 \cdot 5 \cdot 6 \cdot 7} = \frac{2}{5}$$

Solution

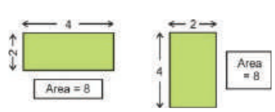
$$\frac{12}{25} \cdot \frac{35}{42} = \frac{2}{5}$$

Identify and Apply Properties of Multiplication

The four mathematical properties which involve multiplication are the **Commutative, Associative, Multiplicative Identity** and **Distributive Properties.**

- **Commutative property** When two numbers are multiplied together, the product is the same regardless of the order in which they are written:

Example $4 \cdot 2 = 2 \cdot 4$



We can see a geometrical interpretation of **The Commutative Property of Multiplication** *to the right. The Area of the shape (length* $\times$*width) is the same no matter which way we draw it.*

- **Associative Property** When three or more numbers are multiplied, the product is the same regardless of their grouping

Example $(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4)$

- **Multiplicative Identity Property:** The product of one and any number is that number.

Example $5 \cdot 1 = 5$.

- **Distributive property** The multiplication of a number and the sum of two numbers is equal to the first number times the second number plus the first number times the third number.

Example: $4(6+3) = 4 \cdot 6 + 4 \cdot 3$

Example 7

Nadia and Peter are raising money by washing cars. Nadia is charging $\$3$*per car, and she washes five cars in the first morning. Peter charges* $\$5$*per car (including a wax). In the first morning, he washes and waxes three cars. Who has raised the most money?*

Solution

Nadia raised $5 \cdot \$3$. Peter raised $3 \cdot \$5$. According to **The Commutative Property of Multiplication,** they both raised the **same amount** of money.

Example 8

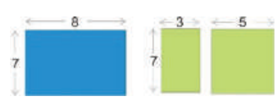
Andrew is counting his money. He puts all his money into $\$10$*piles. He has one pile. How much money does Andrew have?*

Solution

The amount of money in each pile is $\$10$. The number of piles is one. The total amount of money is $\$10 \cdot 1$. According to **The Multiplicative Identity Property,** Andrew has a total of $\$10$.

Example 8

A gardener is planting vegetables for the coming growing season. He wishes to plant potatoes and has a choice of a single 8×7 meter*plot, or two smaller plots of* 3×7 meters*and* 5×7 meter. *Which option gives him the largest area for his potatoes?*



Solution

In the first option, the gardener has a total area of (8×7).

Since $8 = (3+5)$ we have $(3+5) \cdot 7$ squaremeter, which equals $(3 \cdot 7) + (5 \cdot 7)$.

In the second option, the total area is $(3 \cdot 7) + (5 \cdot 7)$ squaremeters.

According to **The Distributive Property** both options give the gardener the same area to plant potatoes

Solve Real-World Problems Using Multiplication

Example 9

In the chemistry lab there is a bottle with two liters of a 15%*solution of hydrogen peroxide* (H_2O_2). *John removes one-fifth of what is in the bottle, and puts it in a beaker. He measures the amount of* H_2O_2 *and adds twice that amount of water to the beaker. Calculate the following measurements.*'

a) The amount of H_2O_2 left in the bottle.

b) The amount of diluted H_2O_2 in the beaker.

c) The concentration of the H_2O_2 in the beaker.

a) To determine the amount of H_2O_2 left in the bottle, we first determine the amount that was removed. That amount was $\frac{1}{5}$ of the amount in the bottle (2 liters).

$$\text{Amount removed} = \frac{1}{5} \cdot 2 \text{ liters} = \frac{2}{5} \text{ liter (or 0.4 liters)}$$
$$\text{Amount remaining} = 2 - \frac{2}{5} = \frac{10}{5} - \frac{2}{5} = \frac{8}{5} \text{ liter (or1.6 liters)}$$

Solution

There is 1.6 liters left in the bottle.

b) We determined that the amount of the $15\% H_2O_2$ solution removed was $\frac{2}{5}$ liter. The amount of water added was twice this amount.

$$\text{Amount of water} = 2 \cdot \frac{2}{5} = \frac{4}{5} \text{ liter.}$$
$$\text{Total amount} = \frac{4}{5} + \frac{2}{5} = \frac{6}{5} \text{ liter (or1.2 liters)}$$

Solution

There are 1.2 liters of diluted H_2O_2 in the beaker.

c) The new concentration of H_2O_2 can be calculated.

Initially, with $\frac{2}{5}$ of undiluted H_2O_2 there is 15% of $\frac{2}{5}$ liters of pure H_2O_2:

Amount of pure H_2O_2 = $0.15 \cdot \frac{2}{5} = 0.15 \cdot 0.4 = 0.06$ liter of **pure** H_2O_2.

After dilution, this H_2O_2 is dispersed into 1.2 liters of solution. The concentration = $\frac{0.06}{1.2} = 0.05$.

To convert to a percent we multiply this number by 100.

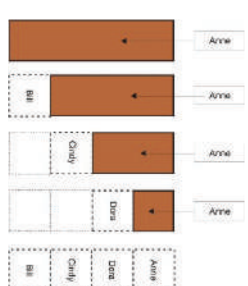
Solution

The final of diluted H_2O_2in the bottle is 5%.

Example 10

Anne has a bar of chocolate and she offers Bill a piece. Bill quickly breaks off $\frac{1}{4}$of the bar and eats it. Another friend, Cindy, takes $\frac{1}{3}$of what was left. She splits the remaining candy bar into two equal pieces which she shares with a third friend, Dora. How much of the candy bar does each person get?

First, let's look at this problem visually.



Anne starts with a full candy bar.

Bill breaks of $\frac{1}{4}$ of the bar.

Cindy takes $\frac{1}{3}$ of what was left.

Dora gets half of the remaining candy bar.

We can see that the candy bar ends up being split four ways. The sum of each piece is equal to one.

Solution

Each person gets exactly $\frac{1}{4}$ of the candy bar.

We can also examine this problem using rational numbers. We keep a running total of what fraction of the bar remains. Remember, when we read a fraction followed by *of* in the problem, it means we multiply by that fraction.

We start with one full bar of chocolate	The total we begin with is 1.
Bill breaks off of the bar	We multiply the amount of bar(1) by $\frac{1}{4}$
Bill removes $\frac{1}{4} \cdot 1 - \frac{1}{4}$ of the whole bar.	The bar remaining is $1 - \frac{1}{4} = \frac{4}{4} - \frac{1}{4} = \frac{3}{4}$
Cindy takes $\frac{1}{3}$ of what is left	We multiply the amount of bar $\left(\frac{3}{4}\right)$ by $\frac{1}{3}$
Cindy removes $\frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$ of a whole bar.	The bar remaining is $\frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$
Anne and Dora get two equal pieces	Dora gets $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ of a whole bar.
	Anne gets the remaining $\frac{1}{2} - \frac{1}{4} = \frac{2}{4} - \frac{1}{4} = \frac{1}{4}$

Solution

Each person gets exactly $\frac{1}{4}$ of the candy bar.

Extension: If each piece that is left is 3oz, how much did the original candy bar weigh?

Lesson Summary

- When multiplying an expression by negative one, remember to multiply the **entire expression** by negative one.
- To multiply fractions $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
- The **multiplicative properties** are:
- **Commutative property** the product of two numbers is the same whichever order the items to be multiplied are written.

Ex: $2 \cdot 3 = 3 \cdot 2$

- **Associative Property:** When three or more numbers are multiplied, the sum is the same regardless of how they are grouped.

Ex: $(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4)$

- **Multiplicative Identity Property** The product of any number and one is the original number.

Ex: $2 \cdot 1 = 2$

- **Distributive property** The multiplication of a number and the sum of two numbers is equal to the first number times the second number plus the first number times the third number.

Ex: $4(2+3) = 4(2) + 4(3)$

Review Questions

1. Multiply the following by negative one.
   (a) 25
   (b) -105
   (c) x^2
   (d) $(3+x)$
   (e) $(3-x)$
2. Multiply the following rational numbers, write your answer in the **simplest form**.
   (a) $\frac{5}{12} \times \frac{9}{10}$
   (b) $\frac{2}{3} \times \frac{1}{4}$
   (c) $\frac{3}{4} \times \frac{1}{3}$
   (d) $\frac{15}{11} \times \frac{9}{7}$
   (e) $\frac{1}{13} \times \frac{1}{11}$
   (f) $\frac{7}{27} \times \frac{9}{14}$
   (g) $\left(\frac{3}{5}\right)^2$
   (h) $\frac{1}{11} \times \frac{22}{21} \times \frac{7}{10}$
   (i) $\frac{12}{15} \times \frac{35}{13} \times \frac{10}{2} \times \frac{26}{36}$
3. Three monkeys spend a day gathering coconuts together. When they have finished, they are very tired and fall asleep.
   The following morning, the first monkey wakes up. Not wishing to disturb his friends, he decides to divide the coconuts into three equal piles. There is one left over, so he throws this odd one away, helps himself to his share, and goes home.
   A few minutes later, the second monkey awakes. Not realizing that the first has already gone, he too divides the coconuts into three equal heaps. He finds one left over, throws the odd one away, helps himself to his fair share, and goes home.
   In the morning, the third monkey wakes to find that he is alone. He spots the two discarded coconuts, and puts them with the pile, giving him a total of twelve coconuts. How many coconuts did the first and second monkey take? [**Extension:** solve by working backward]

Review Answers

1. (a) -25
   (b) 105
   (c) $-x^2$
   (d) $-(x+3)$ or $-x-3$
   (e) $(x-3)$
2. (a) $\frac{3}{8}$
   (b) $\frac{1}{6}$
   (c) $\frac{1}{4}$

(d) $\frac{135}{77}$
(e) $\frac{1}{143}$
(f) $\frac{1}{6}$
(g) $\frac{27}{125}$
(h) $\frac{1}{15}$
(i) $\frac{70}{9}$

3. The first monkey takes eight coconuts. The second monkey takes five coconuts.