Section 2.2

The Derivative: "The Derivative Function"

All graphics are attributed to:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

The Derivative Function

- Derivatives are the primary mathematical tool that we use to study and calculate rates of change in a variety of different applications.

Representations

- There are several different formulas you can use which all look different and mean the same thing. Pick the one that you like best and stick with it.
- This is the one we used last time:
  - Derivative = $$m_{\tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$
- Another option:

2.2.1 DEFINITION The function f' defined by the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (2)$$

is called the ***derivative of f with respect to x***. The domain of f' consists of all x in the domain of f for which the limit exists.

Example – using formula from 2.1

I will use both options to find the derivative of $\sqrt{x}$:

$$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} \frac{*(\sqrt{x} + \sqrt{x_0})}{*(\sqrt{x} + \sqrt{x_0})}$$

Rationalize the numerator

$$= \lim_{x \to x_0} \frac{x - x_0}{(x - x_0) * (\sqrt{x} + \sqrt{x_0})}$$

Simplify by cancelling common factor

$$= \lim_{x \to x_0} \frac{1}{(\sqrt{x} + \sqrt{x_0})} = \frac{1}{(\sqrt{x_0} + \sqrt{x_0})} = \frac{1}{2\sqrt{x_0}}$$

Substitute

Same example using new formula from 2.2

OR

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{*(\sqrt{x+h} + \sqrt{x})}{*(\sqrt{x+h} + \sqrt{x})}$ Rationalize the numerator

$= \lim_{h\to 0} \frac{x+h-x}{h*(\sqrt{x+h} + \sqrt{x})}$ Simplify by cancelling x

$= \lim_{h\to 0} \frac{h}{h*(\sqrt{x+h} + \sqrt{x})}$ Simplify by cancelling common factor

$= \lim_{h\to 0} \frac{1}{(\sqrt{x+h} + \sqrt{x})} = \frac{1}{(\sqrt{x+0} + \sqrt{x})} = \frac{1}{2\sqrt{x}}$ Substitute

Slope of the tangent line

- As you can see, we got the same result using both methods for the derivative which is the slope of the tangent line for any value of x.
- We can use it to find the slope at any given value of x:

b) To find the slope of the tangent line to y=$\sqrt{x}$ at x = 9, we just substitute 9 into that general derivative.

Notation: m_{tan} when x is 9 = f'(9) = $\frac{1}{2\sqrt{9}}$ = $\frac{1}{2*3}$ = $\frac{1}{6}$

We can use this slope to find the equation of the tangent line at x=9

Equation for the Tangent Line

Finding an Equation for the Tangent Line to $y = f(x)$ at $x = x_0$.

Step 1. Evaluate $f(x_0)$; the point of tangency is $(x_0, f(x_0))$.

Step 2. Find $f'(x)$ and evaluate $f'(x_0)$, which is the slope m of the line.

Step 3. Substitute the value of the slope m and the point $(x_0, f(x_0))$ into the point-slope form of the line

$$y - f(x_0) = f'(x_0)(x - x_0)$$

or, equivalently,

$$y = f(x_0) + f'(x_0)(x - x_0) \quad (3)$$

- This looks intimidating, but it just means to use point slope form once you have the slope.

Previous example continued

c) To find the equation of the tangent line to y=$\sqrt{x}$ at x = 9, we just substitute the slope we found previously into the point slope form of a line which is $y - y_1 = m(x - x_1)$.

First, we must find y_1. It is just the y-value that goes with x=9.

Therefore, $y_1 = \sqrt{9} = 3$

And the equation of the tangent line to y=$\sqrt{x}$ at x = 9 is $y - 3 = \frac{1}{6}(x - 9)$.

You can leave it in this form or distribute, combine like terms, and covert to y = mx +b.

Differentiability

- It is possible that the limit that defines the derivative of a function may not exist at certain points in the domain of the function. When that happens, the derivative is undefined.

2.2.2 DEFINITION A function f is said to be ***differentiable at*** x_0 if the limit

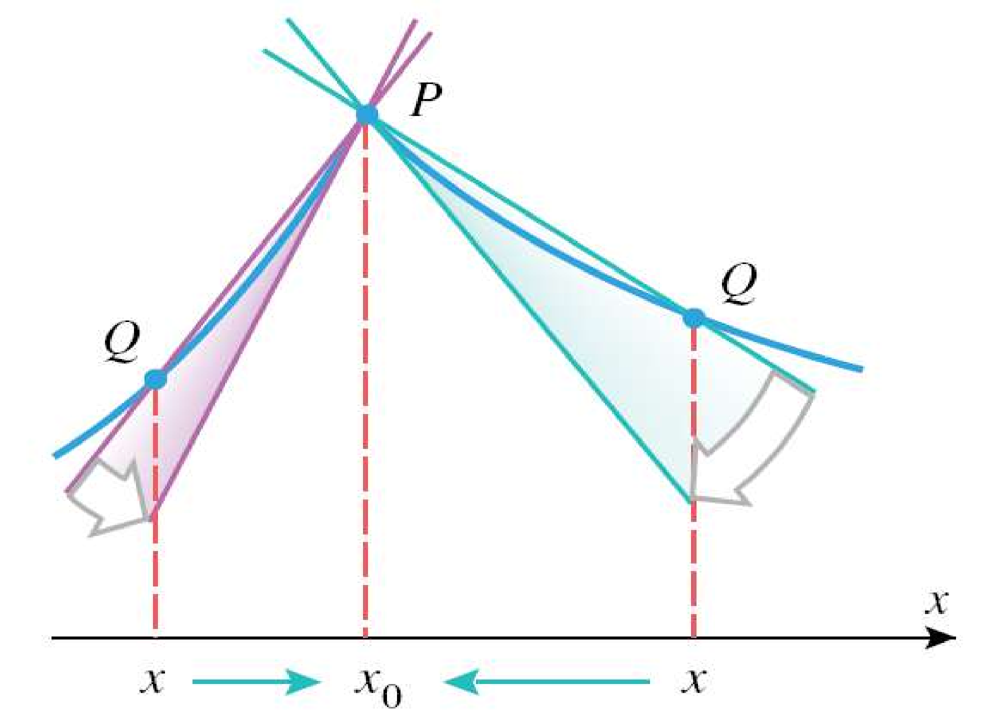
$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{5}$$

exists. If f is differentiable at each point of the open interval (a, b), then we say that it is ***differentiable on*** (a, b), and similarly for open intervals of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$. In the last case we say that f is ***differentiable everywhere***.

Differentiability Geometrically

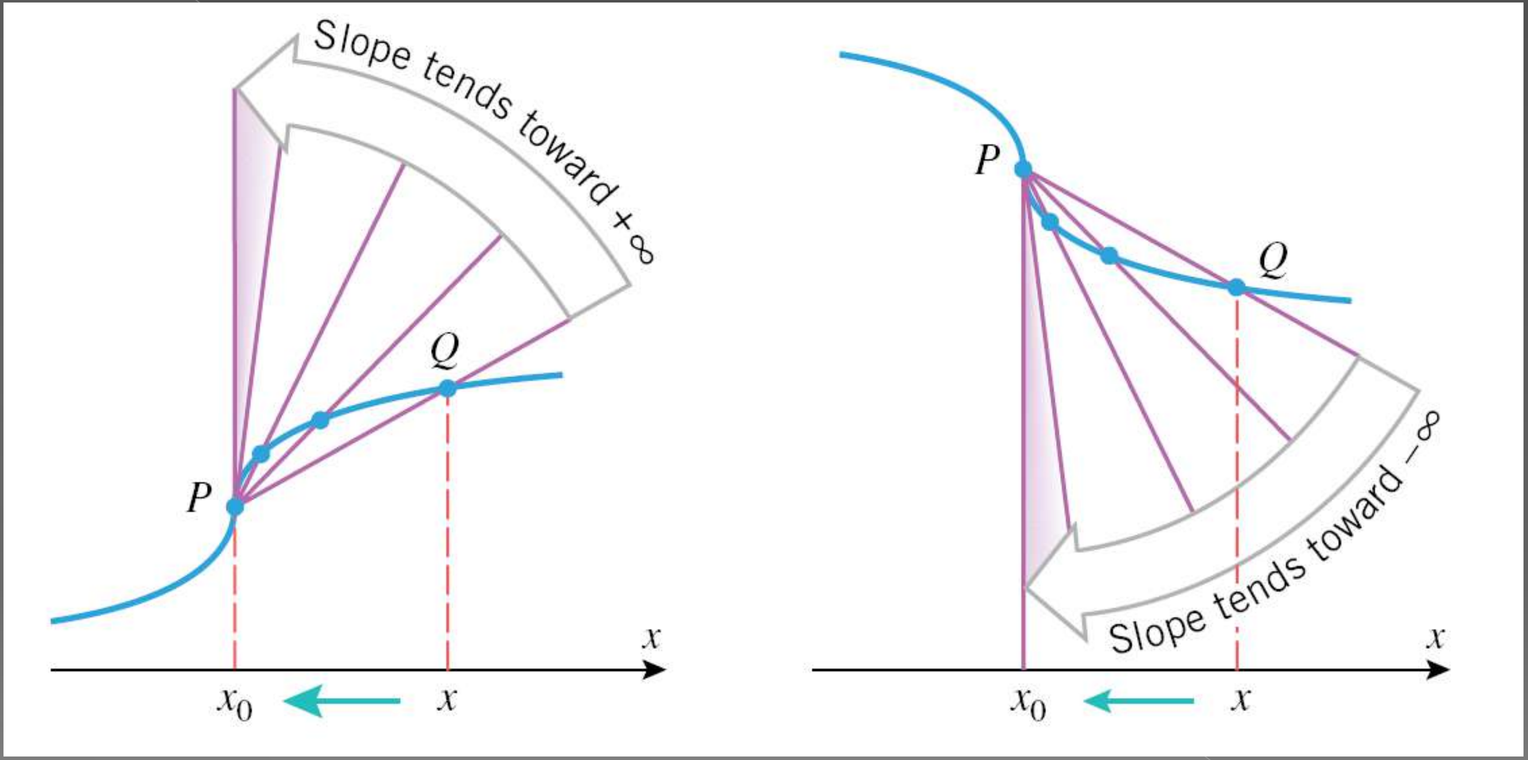
- Two common ways in which a function that is continuous can fail to be differentiable at certain values of x are when the following exist:
  - Corner points
  - Points of vertical tangency

Corner Points



- As you can see in the picture, the one sided limits of the slope of the secant line from the two sides of x_0 are not equal so the two sided limit does not exist.

Points of Vertical Tangency



- As you can see in the picture, the limits of the slope of the secant line approach +/- infinity which is not a measurable slope.

Points of Discontinuity

- The previous two slides show that functions can be continuous at certain points where they are not differentiable (meaning you cannot take the derivative there).
- It is also true that:

2.2.3 THEOREM *If a function f is differentiable at x_0, then f is continuous at x_0.*

- The proof is interesting, read about it and some related history on page 149.

Other Derivative Notations

- We will talk more about these, but you should know that these mean the same thing:

$$f'(x) = \frac{d}{dx}[f(x)] = y' = \frac{dy}{dx}$$

- Also, there are other formulas that some people use to find the derivative:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad \text{OR} \quad f'(x) = \lim_{w \to x} \frac{f(w) - f(x)}{w - x}$$

- I tend to like the last one, but make your own choice.

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