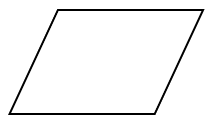
VMHS Math Circle

XIII. The Grand Geometry Review (Part III: Synthetic Geometry-Polygons)

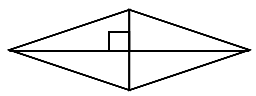
The AMC views polygons as more generic than triangles or circles. Usually when you have to deal with a polygon, you'll have to break it into triangles or draw some circles. However, there are definitely some things you should know about these. Quadrilaterals will be included here, too, but as for 3-D geometry, we'll get to that a bit later.

Types/Properties of Different Quadrilaterals

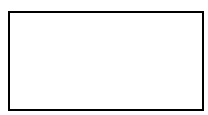
A **parallelogram** has parallel opposite edges, so opposite angles are congruent. The measures of the interior angles on one side add up to 180˚.



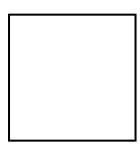
A **rhombus** is a parallelogram, except that it also has equal side lengths and its diagonals are perpendicular to each other. Here, you should be able use the Pythagorean Theorem given diagonals to find the side lengths.



A **rectangle** is a parallelogram, except that all four of its edges meet at right angles. A problem solving trick for rectangles is to use Pythagoras to find the diagonal.

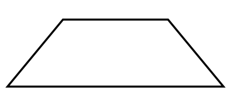


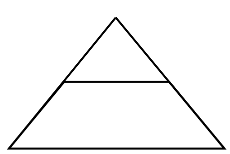
A **square** is a rectangle, but all four sides are congruent. A problem-solving trick for squares is to split them into 45-45-90 triangles.



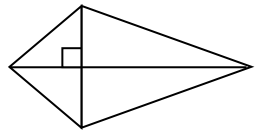
Note that for all figures above, the **diagonals bisect each other**!

A **trapezoid** is a quadrilateral with its opposite sides parallel. **One really big problem solving trick is to extend the non-parallel edges of trapezoids to get similar triangles. When you're faced with a problem concerning trapezoids, always extend them!** Note that isosceles trapezoids have non-parallel edges that have equal length.





A **kite** is a quadrilateral with two pairs of congruent adjacent sides. You don't really see problems dealing with kites on the AMC. The diagonals also meet at right angles, with the one that is either shorter or of the same length bisected.

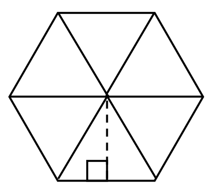


For parallelograms, rectangles, and squares (not rhombi), the area is **bh/2**, with "b" the base and "h" the height.
For the kites and rhombi, the area is **$d_1d_2/2$**, which is half the product of their diagonals.
For trapezoids, the area is **$(h/2)(b_1 + b_2)$**, with "b_1" and "b_2" being both bases, and "h" being the height.

"n-gons"

There's also stuff about convex polygons containing a larger amount of sides that you should know.

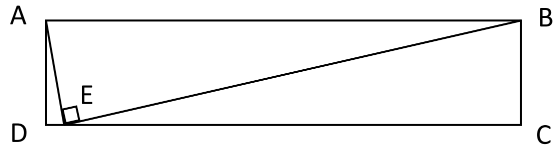
The **sum of the interior angles** of a polygon can be expressed as **180(n – 2)**, where "n" is the number of sides that the polygon has.
For **regular polygons**, the central angle of a polygon can be expressed as **360/n**, where "n" is the number of sides that the polygon has.
The area of a regular n-gon is the sum of the areas of its central triangles. With the hexagon below, if we call the distance from the center of the polygon to the midpoint of one side of the polygon an "apothem" (if this explanation is too wordy, just think of it as the height of the central triangle), we take half the product of the apothem and the side length and multiply that by 6 (since there are 6 central triangles to make).



On the AMC, you pretty much have to treat polygons instinctively. Use your intuition to pave your way toward a solution, for the AMC will most likely combine the skills you need to have under your belt.
AMC 12 takers, you also might need to break out some trig when dealing with polygons.

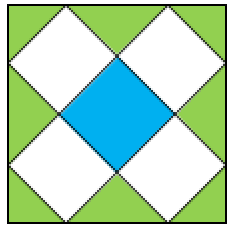
Some problems will help clarify what I'm talking about.

On rectangle ABCD, there exists a point E on CD such that angle AEB is right. If AB = 10, and BE = 9, what is the area of rectangle ABCD?



Since a triangle is basically half a rectangle, we can basically double the area of this inscribed triangle to get our answer. From the Pythagorean Theorem, AE = √(100 – 81) = √(19). Thus, our answer is 2(1/2)(√(19))(9) = 9√(19).

A tile is cut and colored, leaving green isosceles triangles, white rectangles, and a blue square in the center. If the side length of the square is 9, and the side length of the blue square is 5, what is the area of the green region?



Looking at the diagram, we know that the side length of the square is basically the hypotenuse of the smaller isosceles right triangles. Additionally, the diagonal of the square should serve as the base of the longer isosceles triangles. We have four congruent isosceles right triangles, and we have four congruent isosceles non-right triangles, as well.

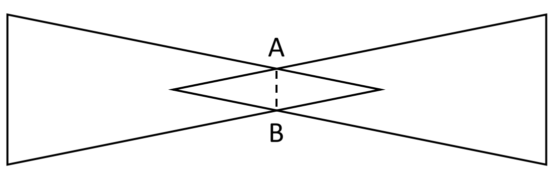
Let's find the dimensions of each. For the right triangles, simply take the square's length and divide it by √2. Thus, the dimensions of the isosceles right triangles are 9(√2)/2. The height of the isosceles non-right triangles is still 9(√2)/2 from the diagram, while the base is 9(√2) from what we said in the beginning.

Thus, the area of the green region is:

$4((9(\sqrt{2})/2)^2/2 + 4((9(\sqrt{2}))^2/2)/2 = 81 + 162 = 243$.

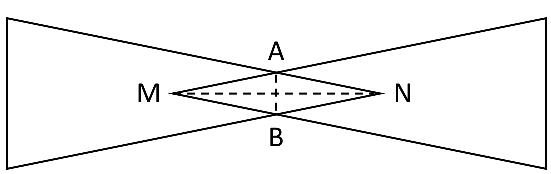
Oh, one more thing: don't worry about units in the AMC unless you're actually required to make conversions.

Two congruent isosceles triangles intersect at points A and B as shown. If the vertex angle of both triangles is 30˚, the length of AB is 2, and the bases of both triangles are 12, what is the area of the union of the regions enclosed by the two triangles?



If we call the tip of one isosceles triangle M and the other tip N, we can observe that triangle BAN is congruent to triangle BAM from symmetry. Hence, the two are basically just reflections of one another. The area of the union of the regions enclosed by the two triangles basically means the area of the figure that you see here. We don't want to over-count the area of rhombus MANB, so we add the areas of both triangles and subtract the area of the rhombus in the end.

The other two angles of both isosceles triangles are 75˚ each, so it's time to break out some trig.



Tangent half-angle for MN/2 yields:

tan(30/2) = 1/(MN/2) → tan(30/2) = 2/(MN)

→ $[1 - \cos(30)]/\sin(30) = 2/(MN)$ → $[1 - (\sqrt{3}/2)]/(1/2) = 2/(MN)$

→ $MN = 1/[1 - (\sqrt{3}/2)] = [1 + (\sqrt{3}/2)]/(1/4) = 4 + 2\sqrt{3}$.

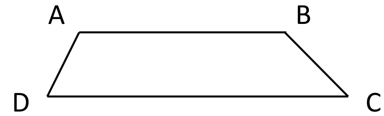
Let's do the same thing for the height "h" of one of the isosceles triangles.

$\tan(30/2) = 6/h$ → $[1 - \cos(30)]/\sin(30) = 6/h$

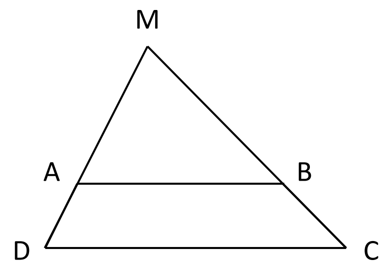
→ $2 - \sqrt{3} = 6/h$ → $h = 12 + 6\sqrt{3}$.

Now, we deal with the areas. The area of rhombus MABN is half the product of MN and AB: $(1/2)(2)(4 + 2\sqrt{3}) = 4 + 2\sqrt{3}$. The area of the two isosceles triangles is $2(1/2)(12)(12 + 6\sqrt{3}) = 144 + 72\sqrt{3}$. Thus, our answer is: $144 + 72\sqrt{3} - (4 + 2\sqrt{3}) = 140 + 70\sqrt{3}$.

Trapezoid ABCD has AB || CD, and AB = 8, BC = 3, CD = 12, and DA = 2. What is the area of this trapezoid?



It's time to extend. Do so for CB and DA until they meet at a point M.



We now see similar triangles AMB and DMC. Let's call MB "x." Using similarity ratios, we have:

$x/8 = (x + 3)/12$ → $12x = 8x + 24$ → $4x = 24$ → $x = 6$.

Doing the same for MA (let's call that "y"), we have:

$y/8 = (y + 2)/12$ → $12y = 8y + 16$ → $4y = 16$ → $y = 4$.

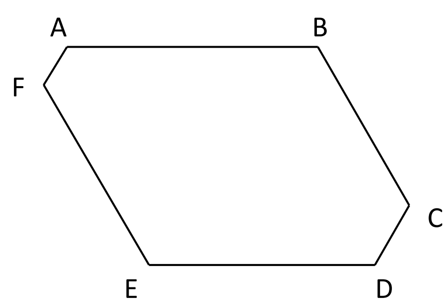
Heron's formula for the area of triangle DMC yields $\sqrt{((9 + 12 + 6)/2)(9/2)(3/2)(15/2)} = 27\sqrt{(15)}/4$.

Hence, the height "h" of triangle DMC is $(1/2)(12)(h) = 27\sqrt{(15)}/4$ → $h = 9\sqrt{(15)}/8$.

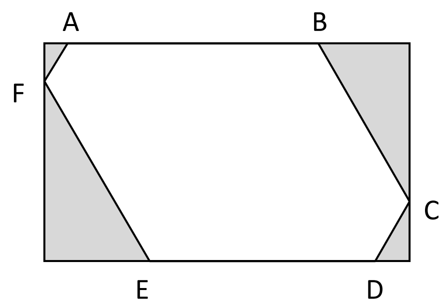
The height of triangle MAB should be two-thirds the height of triangle DMC from our similarity ratio: $3\sqrt{(15)}/4$. Subtracting the areas of triangle DMC and AMB yields:

$27\sqrt{(15)}/4 - (1/2)(8)(3\sqrt{(15)}/4) = (15\sqrt{15})/4$.

An equiangular hexagon ABCDEF has each pair of opposite sides parallel. If AB = 10, BC = 11, CD = 2, DE = 9, EF = 12, and FA = 1, what is the area of the hexagon?



Well, what I said about extending trapezoids doesn't only apply to trapezoids. It can apply to other figures too, if it's appropriate to. Let's do this.



We can extend these angles to make a big rectangle and subtract the four gray triangles here to get our answer. All gray triangles are 30-60-90 triangles because of the rectangle we made. Not only that, we made linear pairs/triples with all angles that an equiangular hexagon should have: 180(6 – 2)/6 = 120˚.

The height of this rectangle is (11√3)/2 + (2√3)/2 = (13√3)/2.

The width is 1/2 + 10 + 11/2 = 16, making the area 104√3.

Now, we subtract the areas of the four gray triangles. We have:

(1/2)(1/2)(√3/2) + (1/2)(11/2)(11√3/2) + (1/2)(12/2)(12√3/2) + (1/2)(2/2)(2√3/2) = (135√3)/4.

Subtracting gets us (281√3)/4, as desired.

TIPS:

- Not many tips here because this section is pretty short, but remember that **extending trapezoids works a LOT.**
- A lot of times, you may have to split polygons up into triangles or extend their sides. Don't be afraid to do this. AMC 12 takers, don't be afraid to use trig, either. When you have side-angle-side revealed, always think Law of Cosines.
- I didn't mention this in my triangle section, but practicing trig problems will really help you remember your trig identities. **AMC 10 takers, you can possibly try to remember the trig identities that I introduced a couple of sections ago. Doing so will save you the time to have to come up with a more creative solution.**
- The AMC has asked before what the shortest distance from the intersection of the diagonals of a rhombus/kite to a certain side was. In case this happens again, recall the section about lines in which I mentioned "dropping perpendiculars." **In simple terms, the altitude from the intersection of the diagonals to a side is the segment with the shortest length.**

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