9

Topics in Analytic Geometry

9.7

Polar Equations of Conics

What You Should Learn

- Define conics in terms of eccentricities, and write and graph equations of conics in polar form
- Use equations of conics in polar form to model real-life problems

Alternative Definition of Conics and Polar Equations

Alternative Definition of Conics and Polar Equations

To begin, consider the following alternative definition of a conic that uses the concept of eccentricity (a measure of the flatness of the conic).

Alternative Definition of Conics and Polar Equations

Alternative Definition of a Conic

The locus of a point in the plane that moves such that its distance from a fixed point (focus) is in a constant ratio to its distance from a fixed line (directrix) is a **conic.** The constant ratio is the **eccentricity** of the conic and is denoted by e. Moreover, the conic is an **ellipse** when $0 < e < 1$, a **parabola** when $e = 1$, and a **hyperbola** when $e > 1$. (See Figure 9.79.)

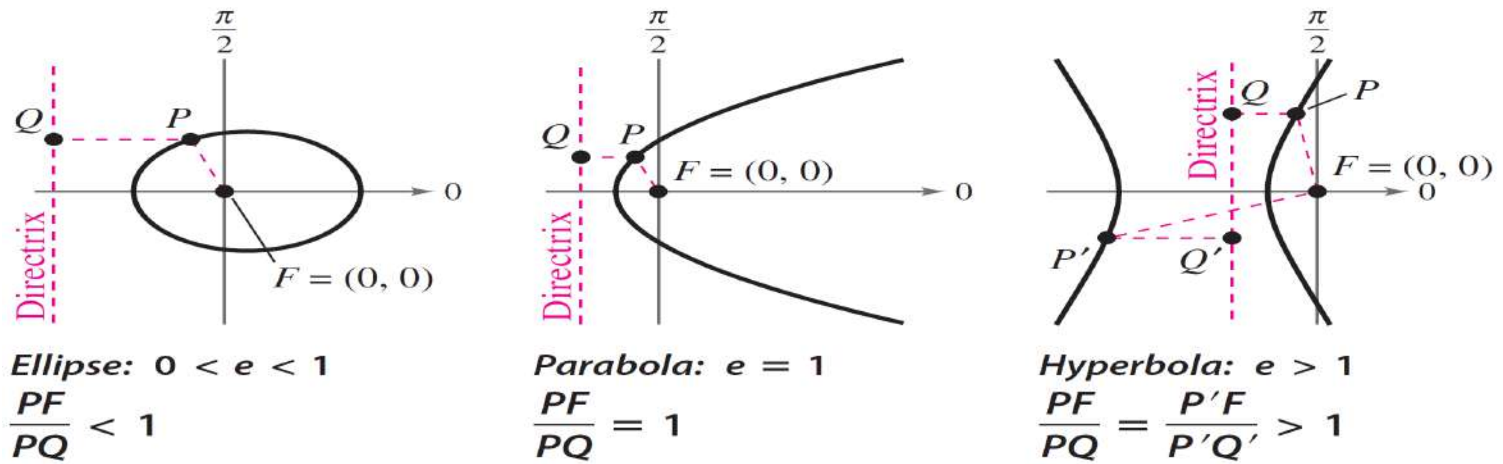

Figure 9.79

Alternative Definition of Conics and Polar Equations

The benefit of locating a focus of a conic at the pole is that the equation of the conic becomes simpler.

Polar Equations of Conics

The graph of a polar equation of the form

1. $r = \dfrac{ep}{1 \pm e \cos \theta}$

or

2. $r = \dfrac{ep}{1 \pm e \sin \theta}$

is a conic, where $e > 0$ is the eccentricity and $|p|$ is the distance between the focus (pole) and the directrix.

Alternative Definition of Conics and Polar Equations

An equation of the form

$$r = \frac{ep}{1 \pm e \cos \theta}$$

Vertical directrix

corresponds to a conic with a vertical directrix and symmetry with respect to the polar axis.

An equation of the form

$$r = \frac{ep}{1 \pm e \sin \theta}$$

Horizontal directrix

corresponds to a conic with a horizontal directrix and symmetry with respect to the line $\theta = \pi/2$.

Alternative Definition of Conics and Polar Equations

Moreover, the converse is also true—that is, any conic with a focus at the pole and having a horizontal or vertical directrix can be represented by one of the given equations.

Example 1 – *Identifying a Conic from Its Equation*

Identify the type of conic represented by the equation

$$r = \frac{15}{3 - 2\cos\theta}$$

Solution:

To identify the type of conic, rewrite the equation in the form $r = ep/(1 \pm e\cos\theta)$.

$$r = \frac{15}{3 - 2\cos\theta}$$

$$= \frac{5}{1 - (2/3)\cos\theta}$$ Divide numerator and denominator by 3.

Because $e = \frac{2}{3} < 1$, you can conclude that the graph is an ellipse.

Alternative Definition of Conics and Polar Equations

For the ellipse in Figure 9.80, the major axis is horizontal and the vertices lie at

$(r, \theta) = (15, 0)$ and $(r, \theta) = (3, \pi)$.

So, the length of the *major* axis is $2a = 18$.

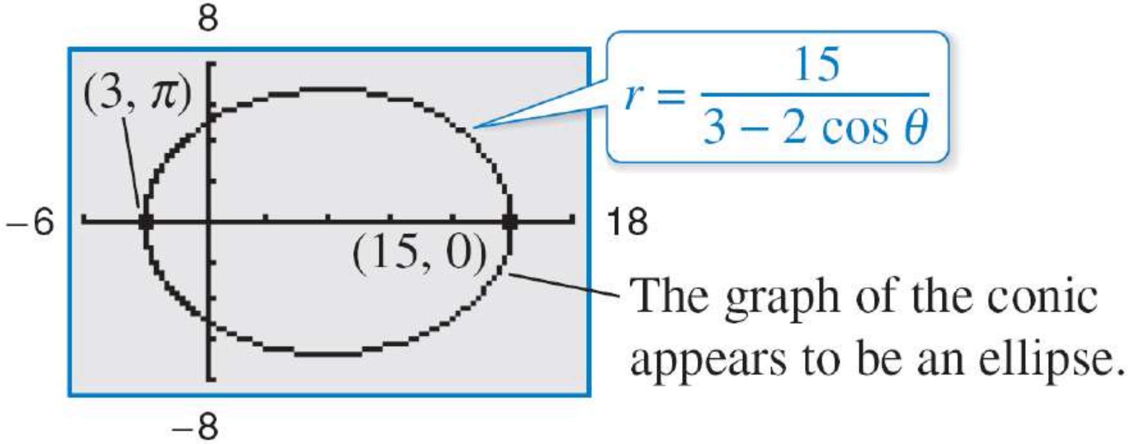

Figure 9.80

Alternative Definition of Conics and Polar Equations

To find the length of the *minor* axis, you can use the equations $e = c/a$ and $b^2 = a^2 - c^2$ to conclude that

$$b^2 = a^2 - c^2$$

$$= a^2 - (ea)^2$$

$$= a^2(1 - e^2)$$ Ellipse

Alternative Definition of Conics and Polar Equations

Because, $e = \frac{2}{3}$, you have

$$b^2 = 9^2\left[1 - \left(\tfrac{2}{3}\right)^2\right]$$

$= 45$

which implies that

$$b = \sqrt{45}$$

$$= 3\sqrt{5}$$

Alternative Definition of Conics and Polar Equations

So, the length of the minor axis is $2b = 6\sqrt{5}.$

A similar analysis for hyperbolas yields

$$b^2 = c^2 - a^2$$

$$= (ea)^2 - a^2$$

$$= a^2(e^2 - 1)$$

Hyperbola

Alternative Definition of Conics and Polar Equations

Let *p* be the distance between the pole and the directrix.

1. Horizontal directrix above the pole: $r = \dfrac{ep}{1 + e \sin \theta}$

2. Horizontal directrix below the pole: $r = \dfrac{ep}{1 - e \sin \theta}$

3. Vertical directrix to the right of the pole: $r = \dfrac{ep}{1 + e \cos \theta}$

4. Vertical directrix to the left of the pole: $r = \dfrac{ep}{1 - e \cos \theta}$

Example 4 – *Halley's Comet*

Halley's comet has an elliptical orbit with an eccentricity of $e \approx 0.967$. The length of the major axis of the orbit is approximately 35.88 astronomical units. Find a polar equation for the orbit. How close does Halley's comet come to the sun?

Solution:

Using a vertical major axis, as shown in Figure 9.83, choose an equation of the form

$$r = \frac{ep}{1 + e \sin \theta}.$$

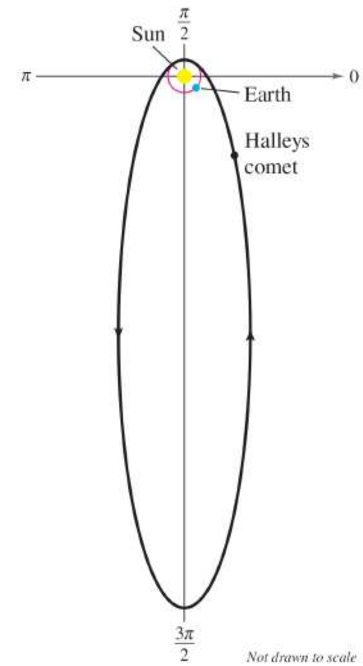

Figure 9.83

Example 4 – *Solution*

cont'd

Because the vertices of the ellipse occur at $\theta = \pi/2$ and $\theta = 3\pi/2$, you can determine the length of the major axis to be the sum of the *r*-values of the vertices.

That is,

$$2a = \frac{0.967p}{1 + 0.967} + \frac{0.967p}{1 - 0.967} \approx 29.79p \approx 35.88$$

So,

$$p \approx 1.204$$

and

$$\begin{aligned} ep &\approx (0.967)(1.204) \\ &\approx 1.164 \end{aligned}$$

Example 4 – *Solution*

cont'd

Using this value of *ep* in the equation, you have

$$r = \frac{1.164}{1 + 0.967 \sin\theta}$$

where *r* is measured in astronomical units.

To find the closest point to the sun (the focus), substitute $\theta = \pi/2$ into this equation to obtain

$$r = \frac{1.164}{1 + 0.967 \sin(\pi/2)}$$

$$\approx 0.59 \text{ astronomical units}$$

$$\approx 55{,}000{,}000 \text{ miles.}$$