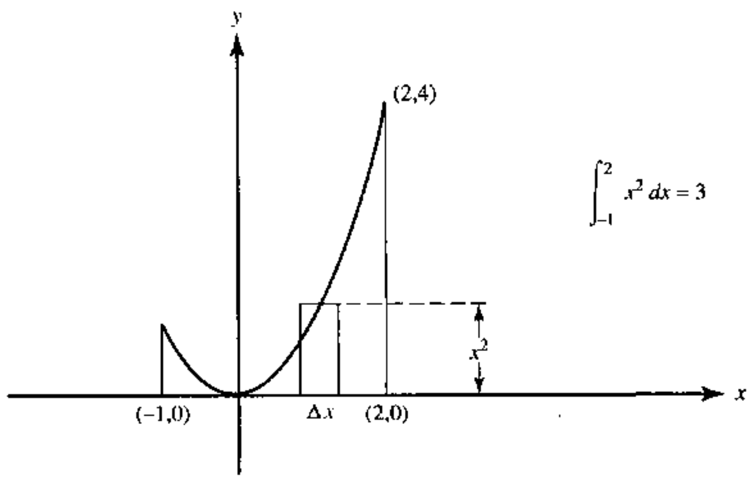
Answers for Set 7: Applications of Integration to Geometry

1. C	14. E	27. C	40. D	53. D
2. C	15. A	28. D	41. C	54. E
3. A	16. B	29. D	42. A	55. A
4. D	17. B	30. B	43. D	56. C
5. B	18. C	31. C	44. D	57. E
6. D	19. D	32. B	45. B	58. B
7. C	20. C	33. A	46. C	59. D
8. E	21. E	34. B	47. A	60. B
9. A	22. B	35. C	48. E	61. A
10. A	23. C	36. A	49. B	
11. D	24. E	37. D	50. A	
12. D	25. D	38. B	51. E	
13. C	26. A	39. E	52. C	

AREA

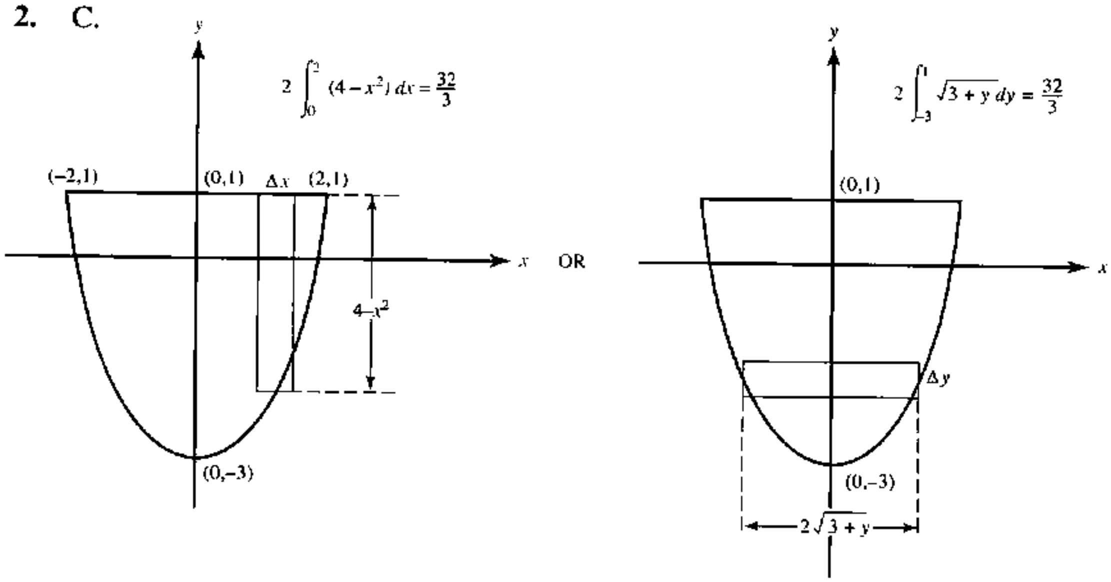
We give below, for each of Questions 1–20, a sketch of the region, and indicate a typical element of area. The area of the region is given by the definite integral. We exploit symmetry wherever possible.

1. C.

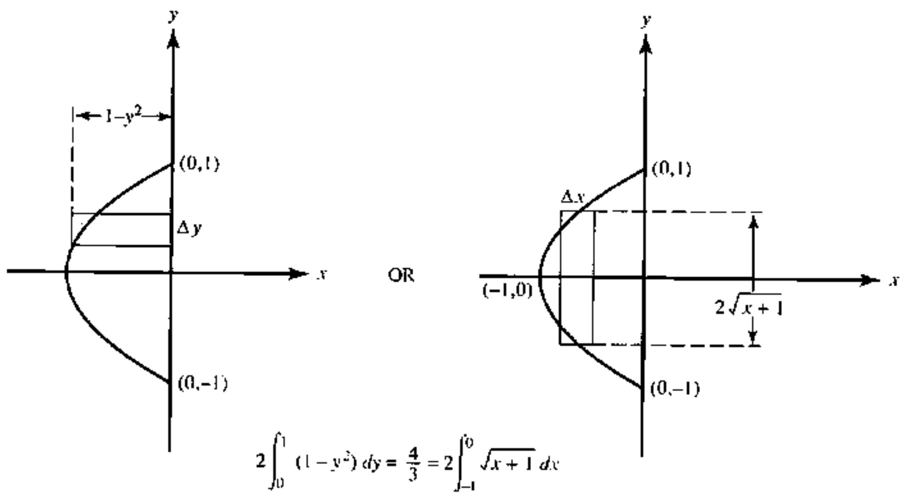


$$\int_{-1}^{2} x^2\,dx = 3$$

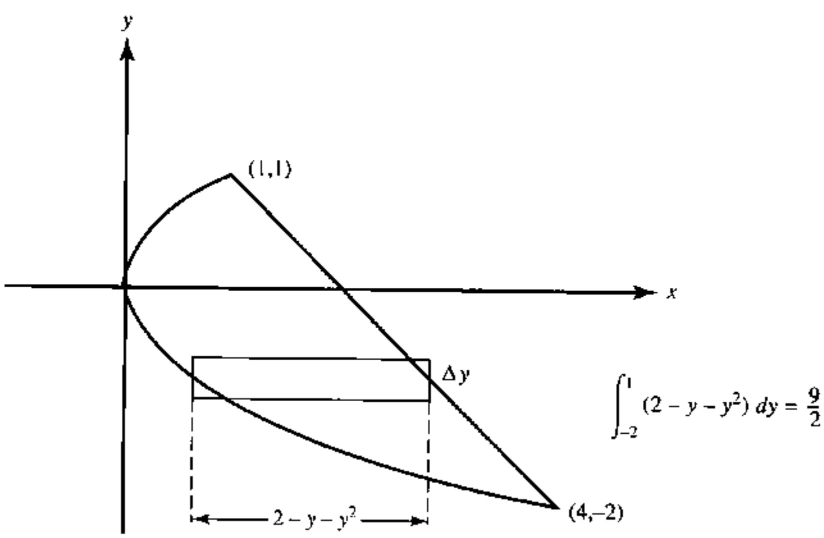
2. C.


y
$2\int_0^2 (4-x^2)\,dx = \frac{32}{3}$
(−2,1)
(0,1)
Δx
(2,1)
x
$4-x^2$
(0,−3)
OR
y
$2\int_{-3}^{1} \sqrt{3+y}\,dy = \frac{32}{3}$
(0,1)
x
Δy
(0,−3)
$2\sqrt{3+y}$

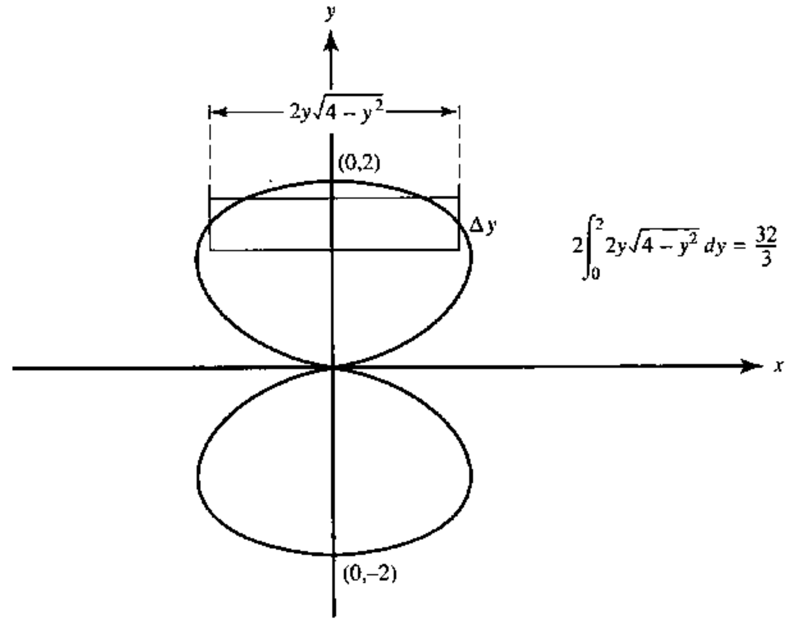
3. A.


y
$1-y^2$
(0,1)
Δy
x
(0,−1)
OR
y
(0,1)
Δx
(−1,0)
x
$2\sqrt{x+1}$
(0,−1)
$2\int_0^1 (1-y^2)\,dy = \frac{4}{3} = 2\int_{-1}^{0} \sqrt{x+1}\,dx$

4. D.

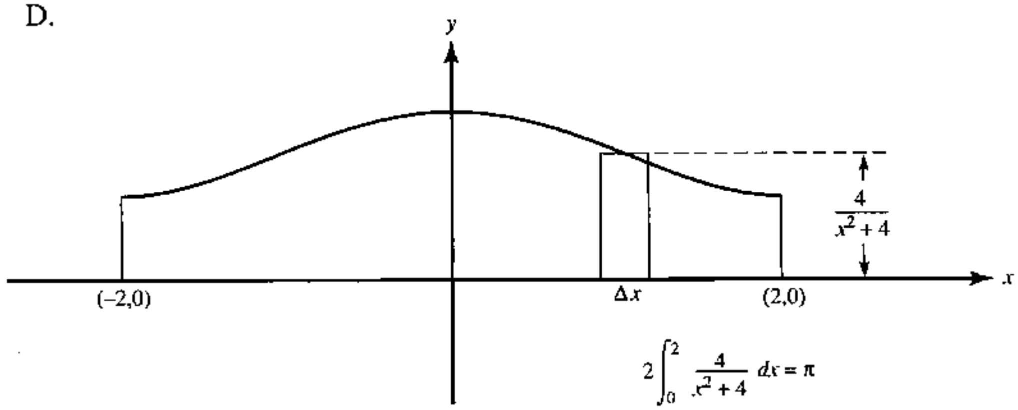

y
(1,1)
x
Δy
$\int_{-2}^{1} (2-y-y^2)\,dy = \frac{9}{2}$
$2-y-y^2$
(4,−2)

5. B.



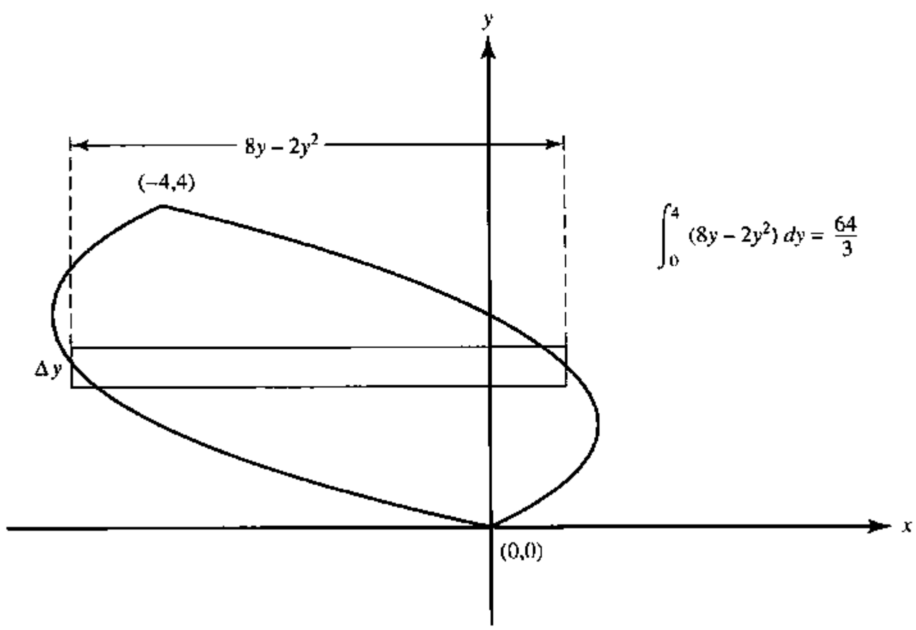
$$2\int_0^2 2y\sqrt{4-y^2}\,dy = \frac{32}{3}$$

6. D.



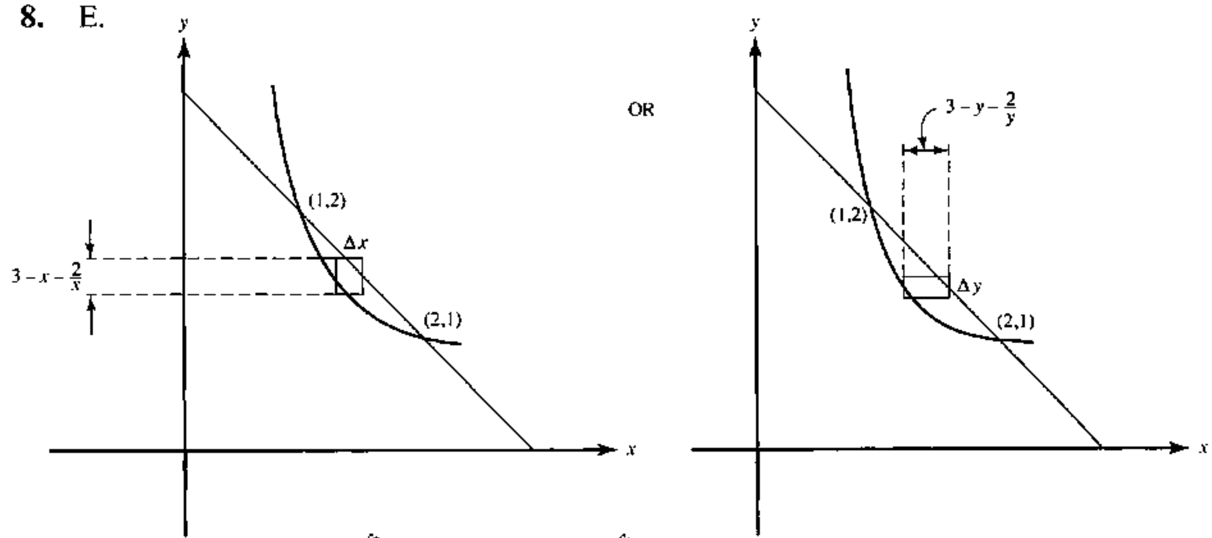
$$2\int_0^2 \frac{4}{x^2+4}\,dx = \pi$$

7. C.



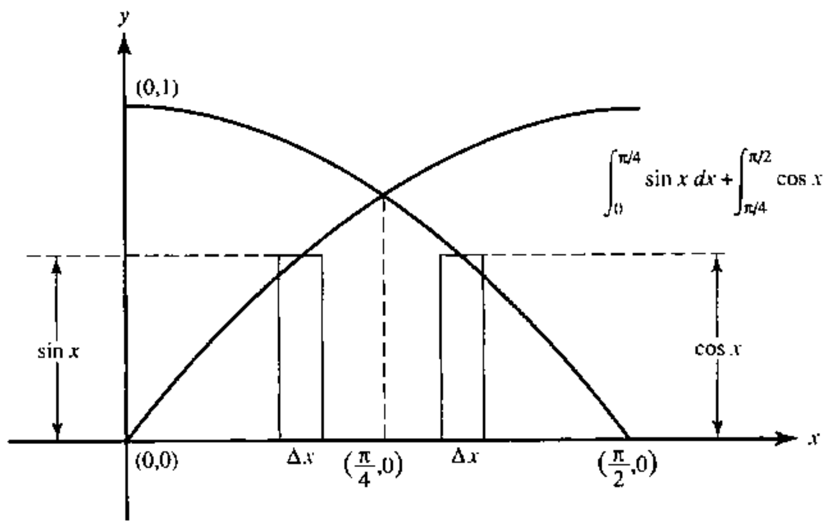
$$\int_0^4 (8y - 2y^2)\,dy = \frac{64}{3}$$

8. E.



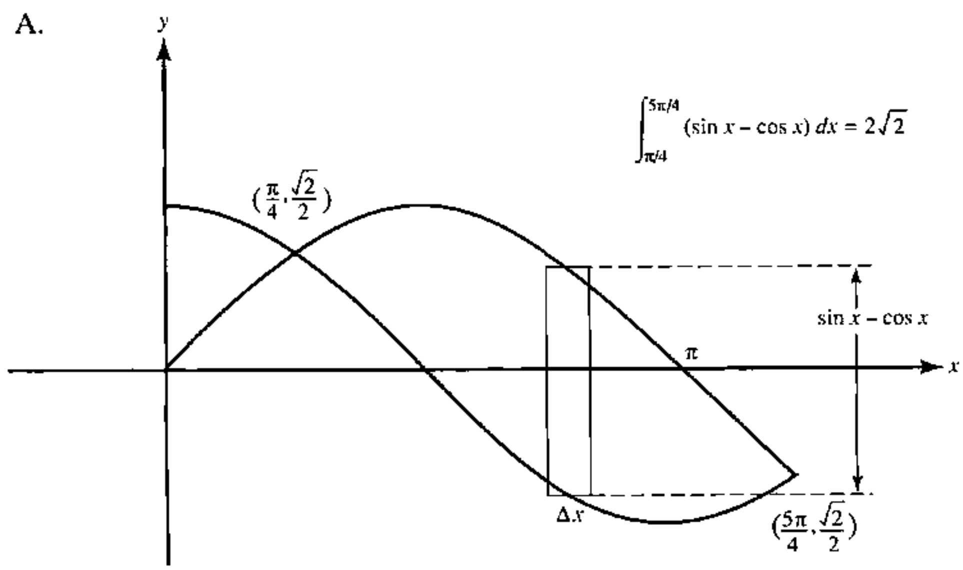
$$\int_1^2 \left(3 - x - \frac{2}{x}\right) dx = \frac{3}{2} - \ln 4 = \int_1^2 \left(3 - y - \frac{2}{y}\right) dy$$

9. A.



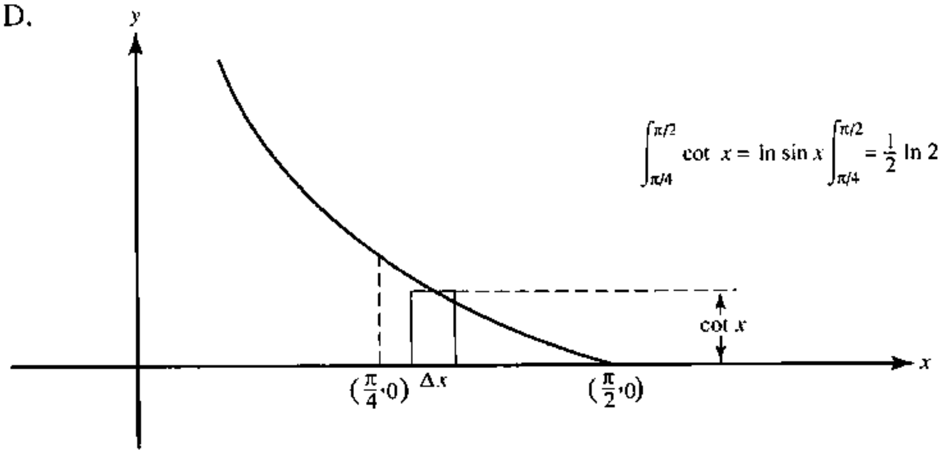
$$\int_0^{\pi/4} \sin x \, dx + \int_{\pi/4}^{\pi/2} \cos x \, dx = 2 - \sqrt{2}$$

10. A.



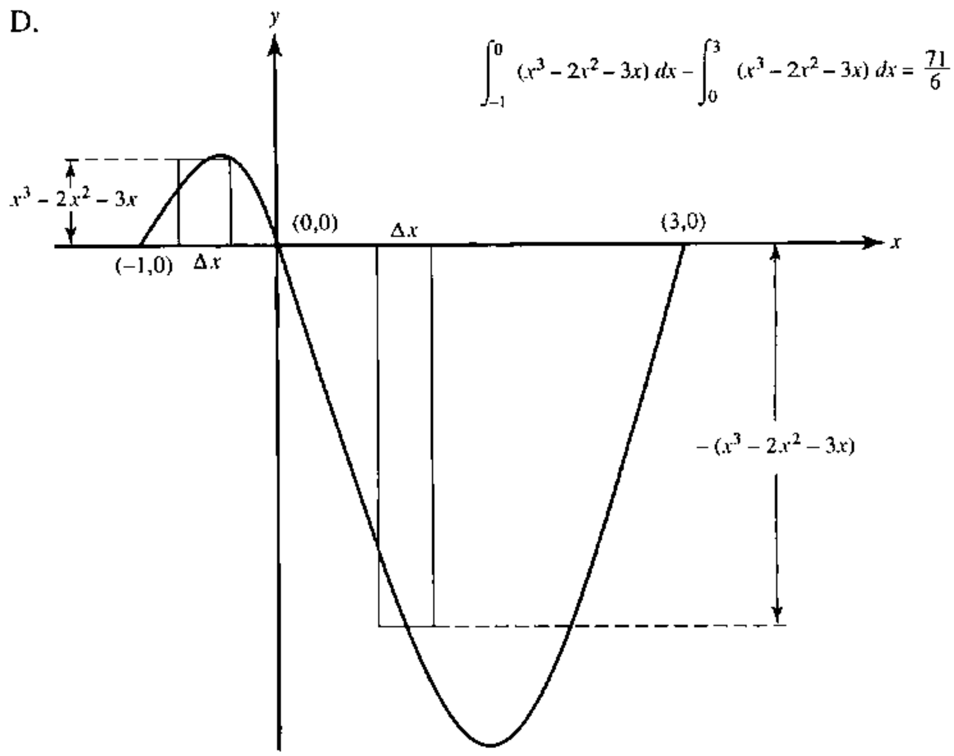
$$\int_{\pi/4}^{5\pi/4} (\sin x - \cos x) \, dx = 2\sqrt{2}$$

11. D.



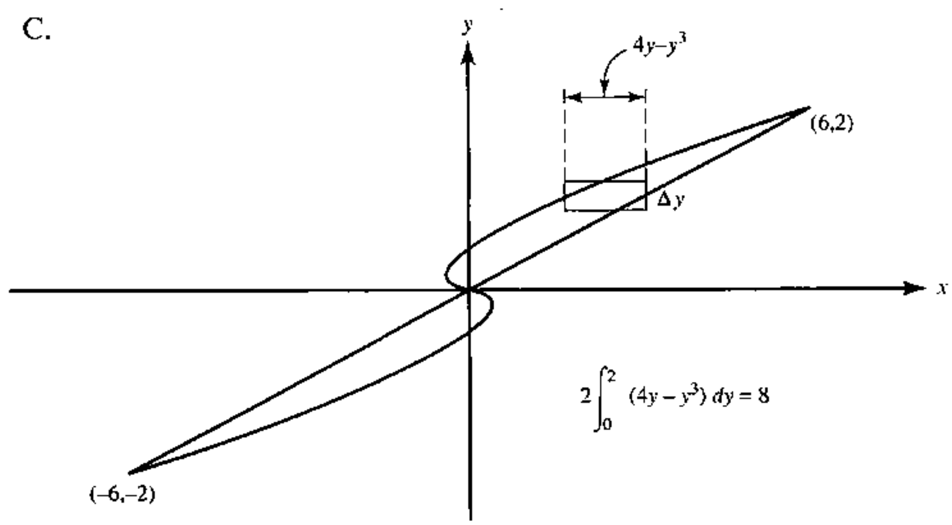
y
x
$\int_{\pi/4}^{\pi/2} \cot x = \ln \sin x \Big|_{\pi/4}^{\pi/2} = \frac{1}{2} \ln 2$
cot x
$(\frac{\pi}{4},0)$
Δx
$(\frac{\pi}{2},0)$

12. D.



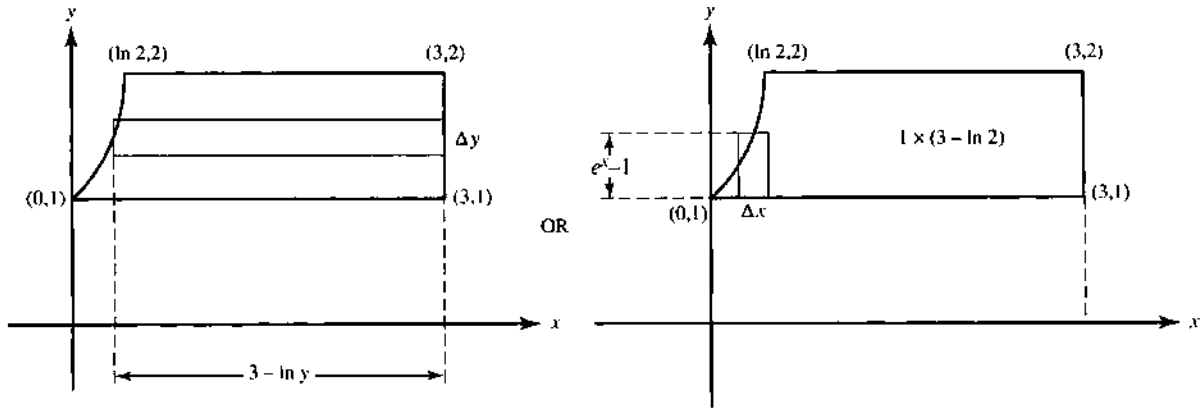
y
x
$\int_{-1}^{0} (x^3 - 2x^2 - 3x)\,dx - \int_{0}^{3} (x^3 - 2x^2 - 3x)\,dx = \frac{71}{6}$
$x^3 - 2x^2 - 3x$
(0,0)
Δx
(3,0)
(−1,0)
Δx
$-(x^3 - 2x^2 - 3x)$

13. C.



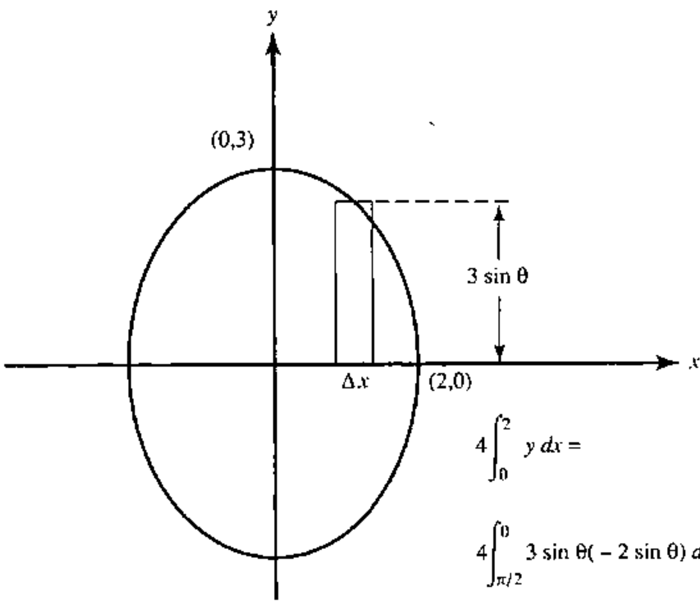
y
x
$4y-y^3$
(6,2)
Δy
$2\int_{0}^{2} (4y - y^3)\,dy = 8$
(−6,−2)

14. E.



$$\int_1^2 (3 - \ln y)\,dy = 4 - \ln 4 = \int_0^{\ln 2} (e^x - 1)\,dx + (3 - \ln 2)$$

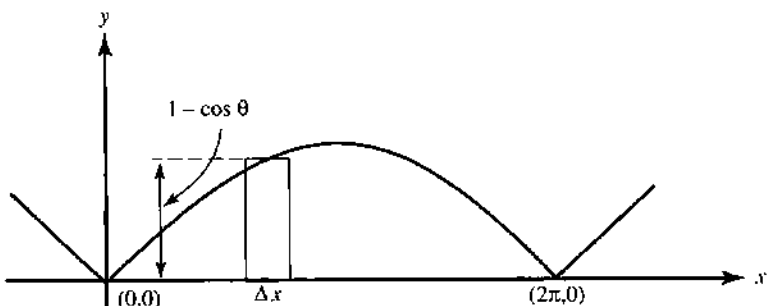
15. A.



$$4\int_0^2 y\,dx =$$

$$4\int_{\pi/2}^{0} 3\sin\theta(-2\sin\theta)\,d\theta = 6\pi$$

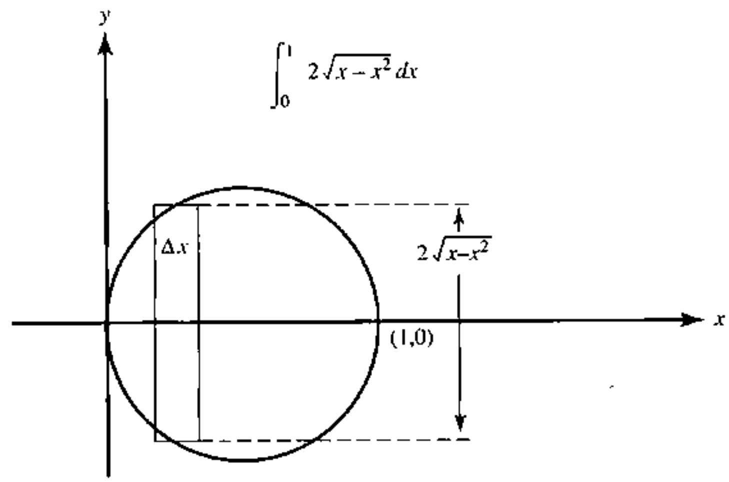
16. B.



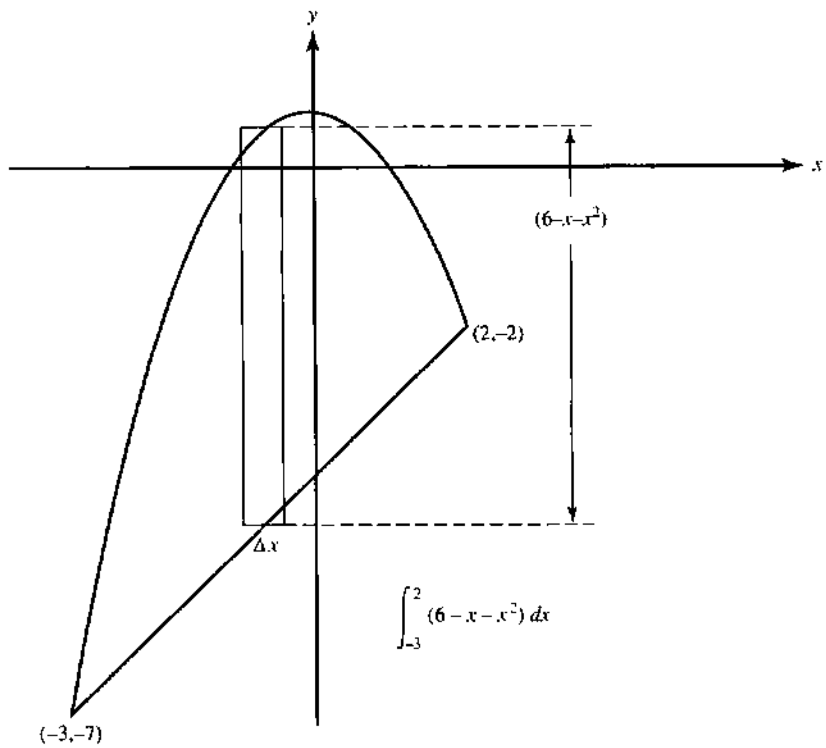
$$\int_{\theta=0}^{\theta=2\pi} y\,dx =$$

$$\int_0^{2\pi} (1-\cos\theta)(1-\cos\theta)\,d\theta = 3\pi$$

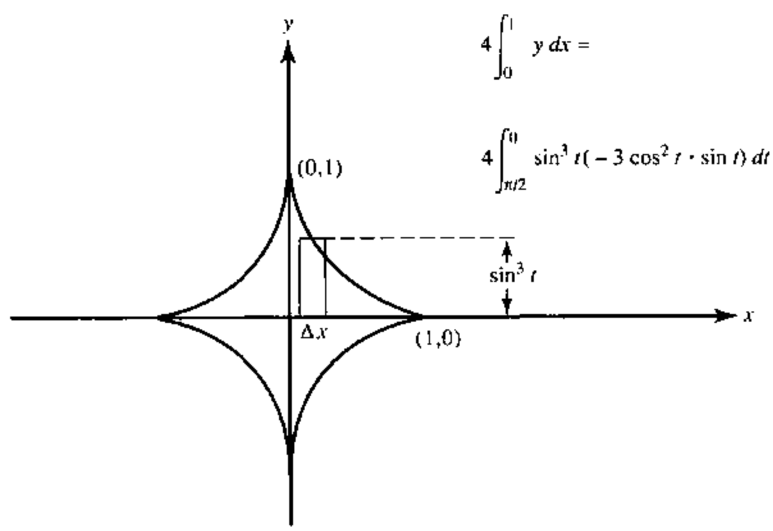
17. B.


y
$\int_0^1 2\sqrt{x-x^2}\,dx$
Δx
$2\sqrt{x-x^2}$
(1,0)
x

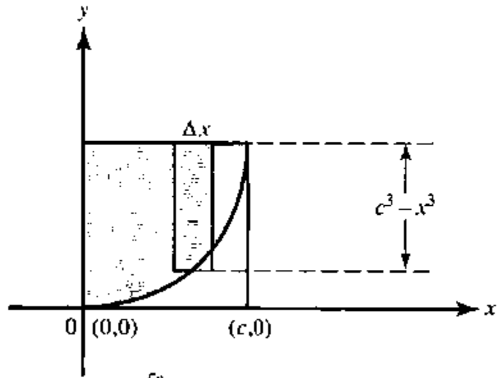
18. C.


y
x
$(6-x-x^2)$
(2,–2)
Δx
$\int_{-3}^{2} (6-x-x^2)\,dx$
(–3,–7)

19. D.


y
$4\int_0^1 y\,dx =$
$4\int_{\pi/2}^{0} \sin^3 t(-3\cos^2 t \cdot \sin t)\,dt$
(0,1)
$\sin^3 t$
Δx
(1,0)
x

20. C.



$\int_0^c (c^3 - x^3)\,dx = \frac{3}{4}c^4$; thus area of rectangle is to area of shaded region as 4 is to 3.

21. E. If $T(4)$ is used, with $\Delta x = 1$, then the area equals

$$\frac{L(4) + R(4)}{2} = 27.53$$

to two decimal places.

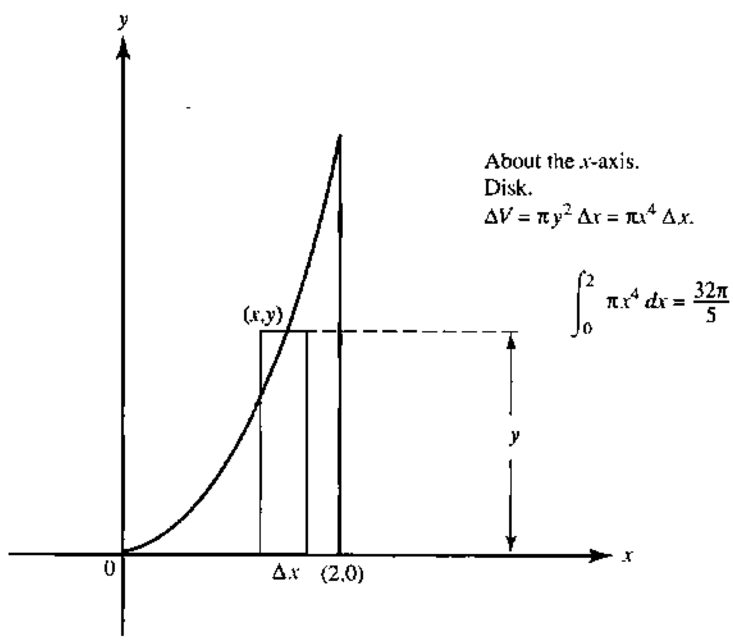
22. B. $A = 8\int_0^{\pi/4} \frac{1}{2}\cos^2 2\theta\, d\theta = 1.571$, using a graphing calculator.

23. C. The small loop is generated as θ varies from $\frac{\pi}{6}$ to $\frac{5\pi}{6}$. (C) uses the loop's symmetry.

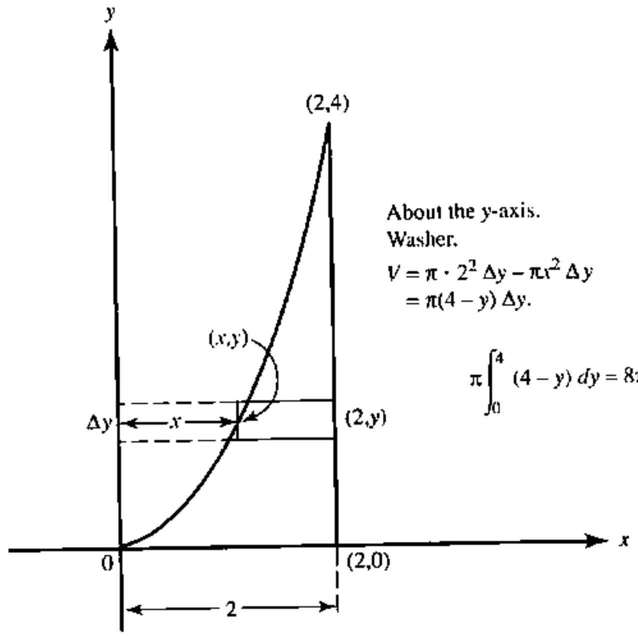
VOLUME

A sketch is given below for each question, in addition to the definite integral for each volume.

24. E.



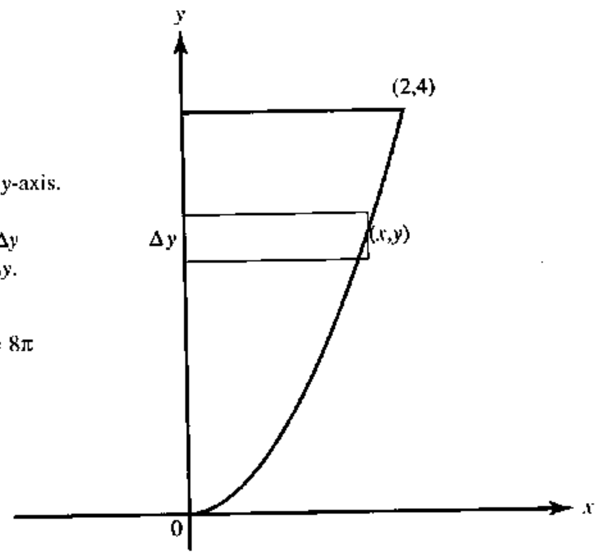
25. D.



About the y-axis.
Washer.
$V = \pi \cdot 2^2\,\Delta y - \pi x^2\,\Delta y$
$= \pi(4 - y)\,\Delta y.$

$$\pi\int_0^4 (4 - y)\,dy = 8\pi$$

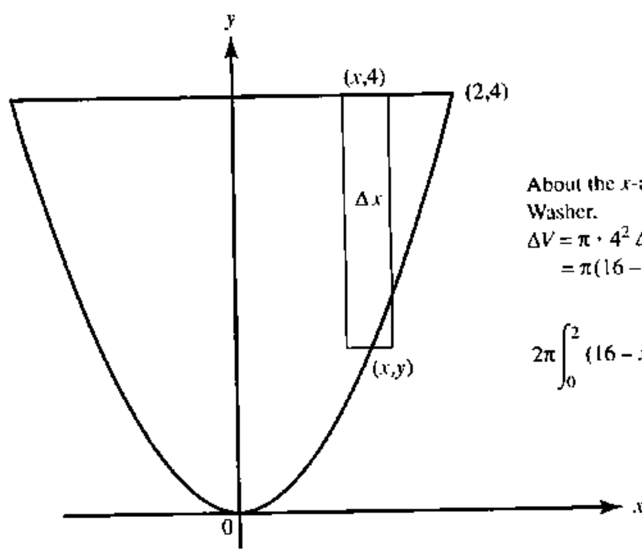
26. A.



About the y-axis.
Disk.
$\Delta V = \pi x^2\,\Delta y$
$= \pi y\,\Delta y.$

$$\pi\int_0^4 y\,dy = 8\pi$$

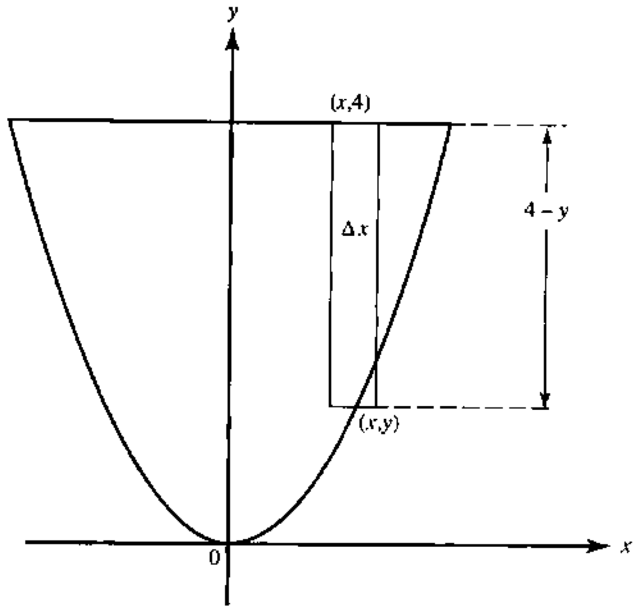
27. C.



About the x-axis.
Washer.
$\Delta V = \pi \cdot 4^2\,\Delta x - \pi y^2\,\Delta x$
$= \pi(16 - x^4)\,\Delta x.$

$$2\pi\int_0^2 (16 - x^4)\,dx = \frac{256\pi}{5}$$

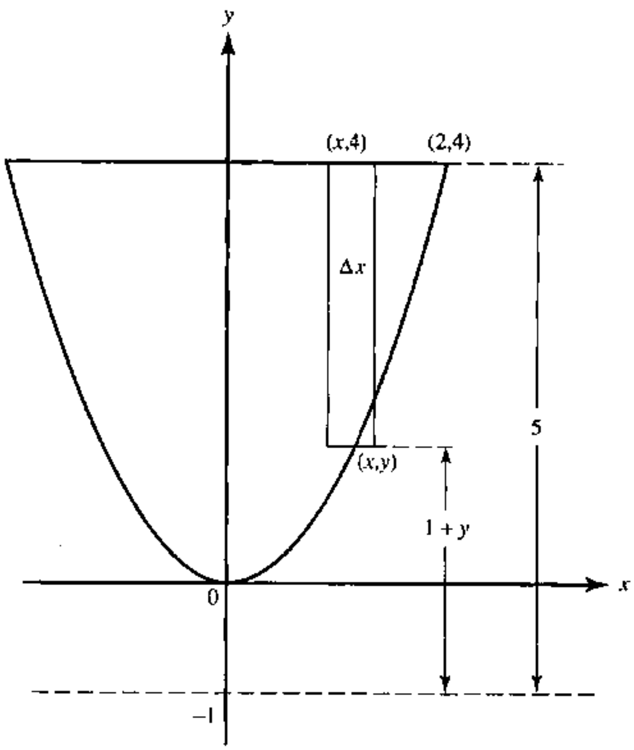
28. D.



About $y = 4$.
Disk.
$\Delta V = \pi(4 - y)^2 \Delta x$
$= \pi(4 - x^2)^2 \Delta x.$

$$2\pi \int_0^2 (4 - x^2)^2 \, dx = \frac{512\pi}{15}$$

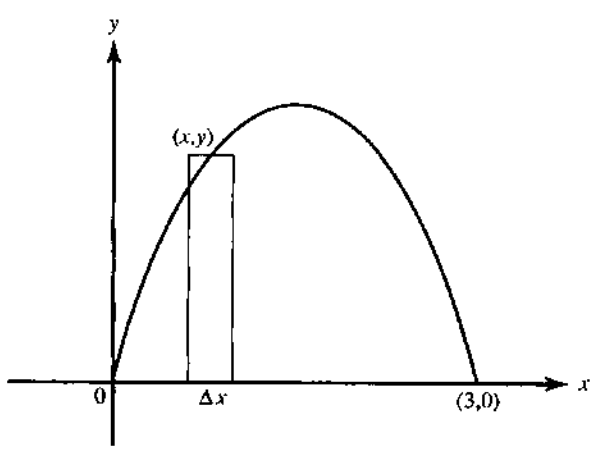
29. D.



About $y = -1$.
Washer.
$\Delta V = \pi \cdot 5^2 \Delta x - \pi(1 + y)^2 \Delta x$
$= \pi[25 - (1 + y)^2] \Delta x.$

$$2\pi \int_0^2 (24 - 2x^2 - x^4) \, dx$$

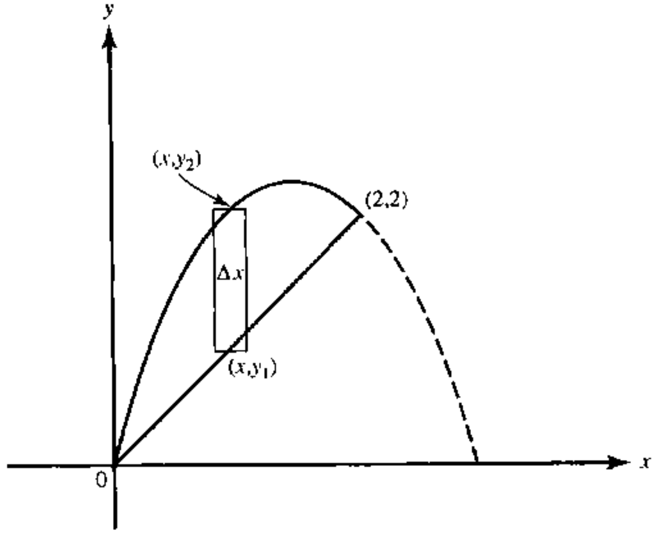
30. B.



About the x-axis.
Disk.
$\Delta V = \pi y^2 \Delta x$
$= \pi(3x - x^2)^2 \Delta x.$

$$\pi \int_0^3 (3x - x^2)^2 \, dx$$

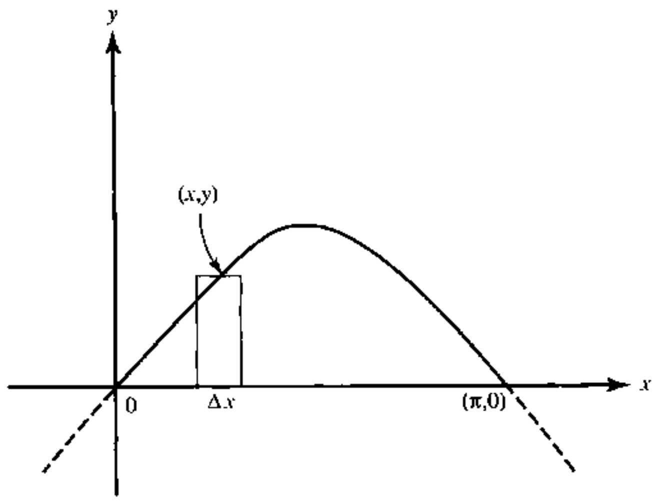
31. C.



About the x-axis.
Washer.
$\Delta V = \pi y_2^2\,\Delta x - \pi y_1^2\,\Delta x$
$= \pi[(3x - x^2)^2 - x^2]\,\Delta x.$

$$\pi\int_0^2 [(3x - x)^2 - x^2]\,dx$$

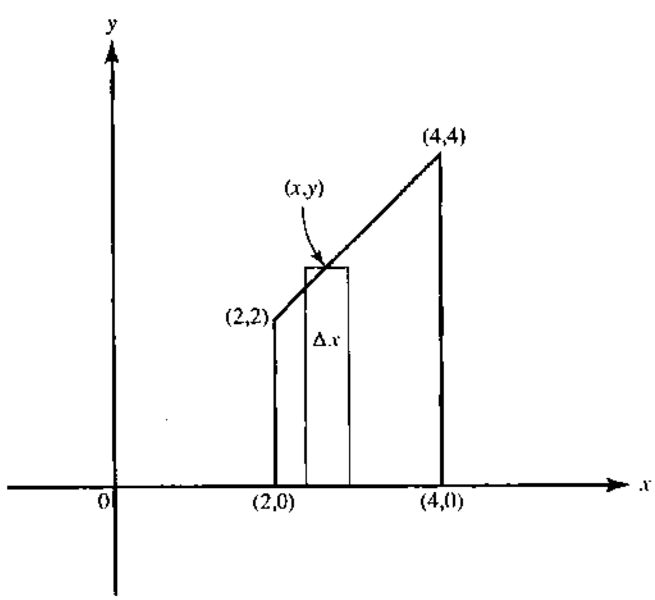
32. B.



About the x-axis.
Disk.
$\Delta V = \pi y^2\,\Delta x$
$= \pi\sin^2 x\,\Delta x.$

$$\pi\int_0^{\pi} \sin^2 x\;dx = \frac{\pi^2}{2}$$

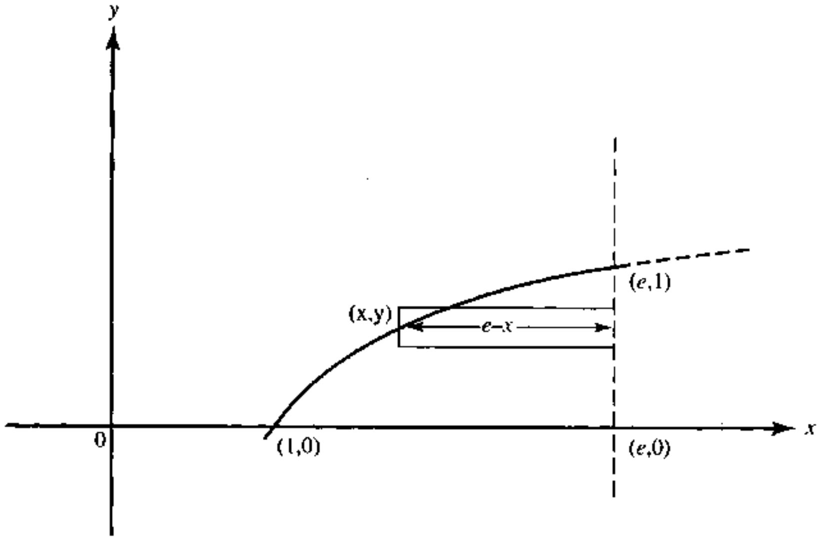
33. A.



About the x-axis.
Disk.
$\Delta V = \pi y^2\,\Delta x$

$$\pi\int_2^4 x^2\,dx = \frac{56\pi}{3}$$

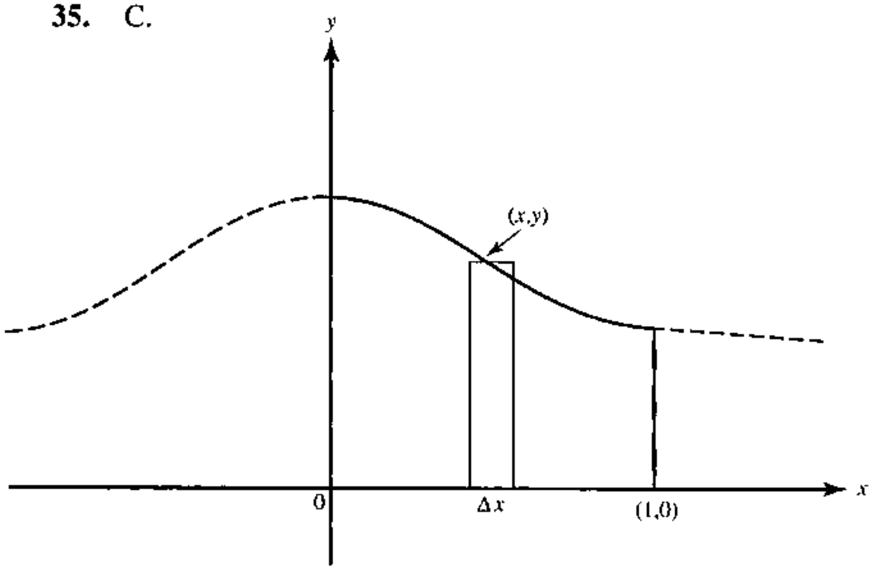
34. B.



About $x = e$.
Disk.
$\Delta V = \pi(e - x)^2 \Delta y$
$= \pi(e - e^y)^2 \Delta y.$

$$\pi \int_0^1 (e - e^y)^2 \, dy$$

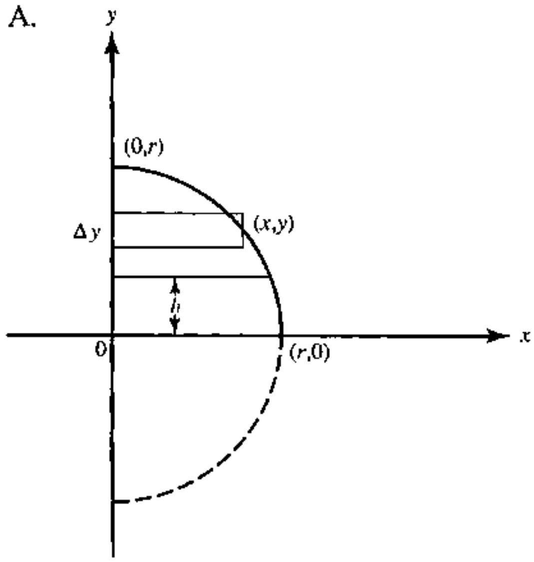
35. C.



About the x-axis.
Disk.
$\Delta V = \pi y^2 \Delta x.$

$$\pi \int_0^1 y^2 \, dx = \pi \int_0^{\pi/4} \cos^2 \theta \, d\theta$$

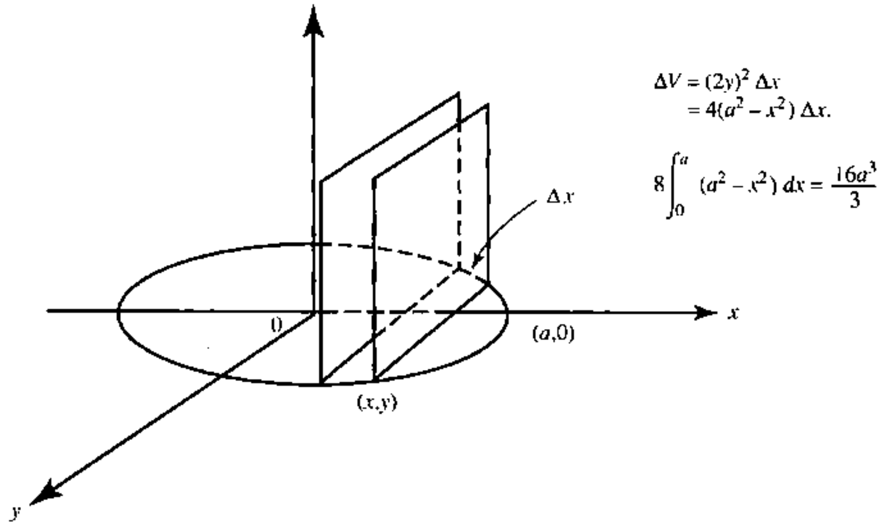
36. A.



About the y-axis.
Disk.
$\Delta V = \pi x^2 \Delta y$
$= \pi(r^2 - y^2) \Delta y.$

$$\pi \int_h^r (r^2 - y^2) \, dy = \frac{\pi}{3} (2r^3 + h^3 - 3r^2 h)$$

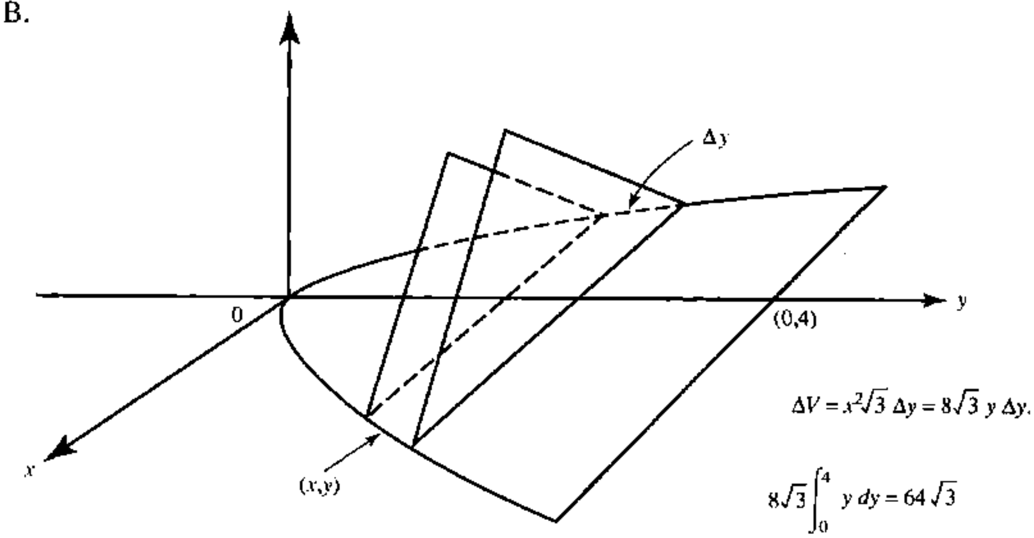
37. D.



$$\Delta V = (2y)^2\,\Delta x = 4(a^2 - x^2)\,\Delta x.$$

$$8\int_0^a (a^2 - x^2)\,dx = \frac{16a^3}{3}$$

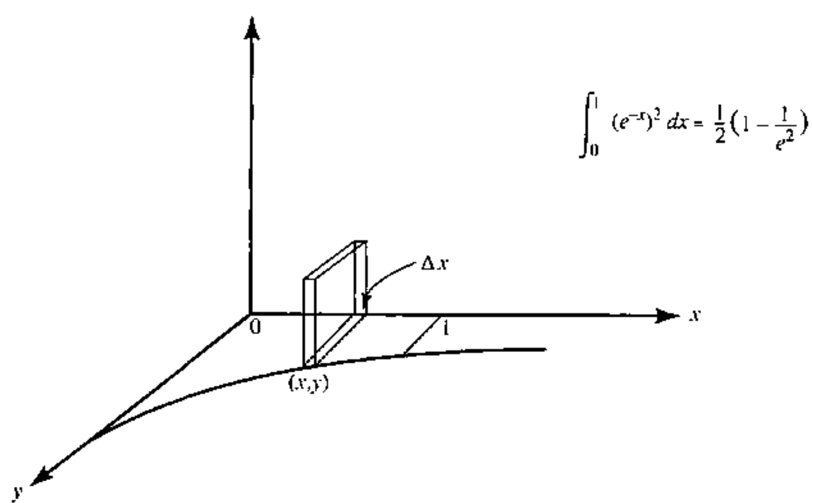
38. B.



$$\Delta V = x^2\sqrt{3}\,\Delta y = 8\sqrt{3}\,y\,\Delta y.$$

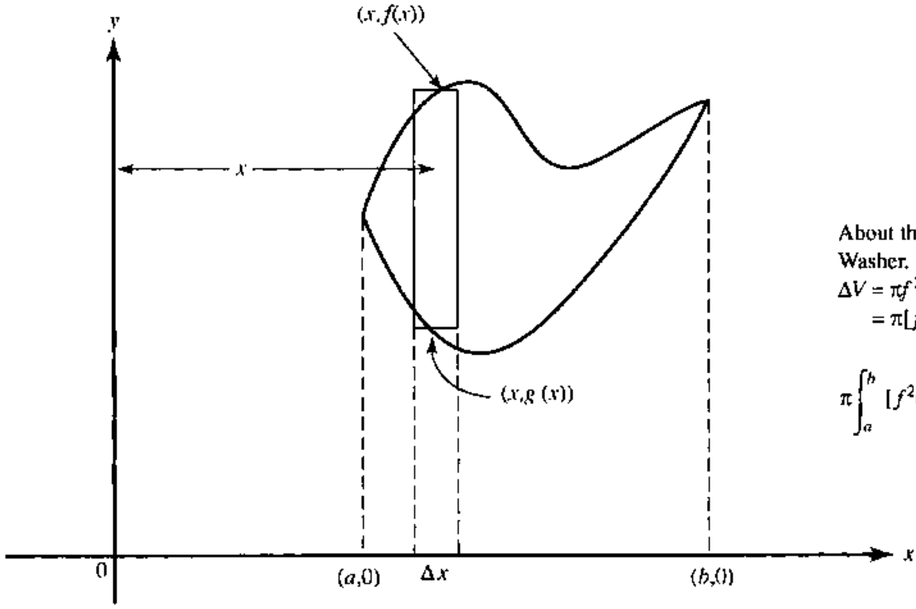
$$8\sqrt{3}\int_0^4 y\,dy = 64\sqrt{3}$$

39. E.



$$\int_0^1 (e^{-x})^2\,dx = \frac{1}{2}\left(1 - \frac{1}{e^2}\right)$$

40. D.



About the x-axis.
Washer.
$\Delta V = \pi f^2(x)\,\Delta x - \pi g^2(x)\,\Delta x$
$= \pi[f^2(x) - g^2(x)]\,\Delta x.$

$$\pi \int_a^b [f^2(x) - g^2(x)]\,dx$$

ARC LENGTH

41. C. Note that the curve is symmetric to the x-axis. The arc length equals
$$2\int_0^4 \sqrt{1+\frac{9}{4}x}\,dx.$$

42. A. Integrate $\int_{\pi/4}^{\pi/3} \sqrt{1+\tan^2 x}\,dx$. Replace the integrand by sec x. and use formula (13) on page 153 to get $\ln|\sec x + \tan x|\Big|_{\pi/4}^{\pi/3}$

43. D. From (3) on page 236 we obtain the length:
$$\int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\,dt = \int_0^{2\pi} \sqrt{2-2\cos t}\,dt$$

44. D. Note that the curve is symmetric to the x-axis. Use (2) on page 236.

45. B. Use (3) on page 236 to get the integral:
$$\int_2^3 \sqrt{(-e^t\sin t + e^t\cos t)^2 + (e^t\cos t + e^t\sin t)^2}\,dt = \sqrt{2}e^t\Big|_2^3.$$

IMPROPER INTEGRALS

46. C. The integrand is discontinuous at $x = 1$, which is on the interval of integration.

47. A. The integral equals $\lim_{b\to\infty} -\frac{1}{e^x}\Big|_0^b = -(0-1).$

48. E. $\int_0^e \frac{du}{u} = \lim_{h\to0^+} \int_h^e \frac{du}{u} = \lim_{h\to0^+} \ln|u| \Big|_h^e = \lim_{h\to0^+} (\ln e - \ln h)$. So the integral diverges to infinity.

49. B. Redefine as $\lim_{h\to0^+} \int_{1+h}^{2} (t-1)^{-1/3}\, dt$.

50. A. Rewrite as $\lim_{h\to0^+} \int_2^{3-h} (x-3)^{-2/3}\, dx + \lim_{h\to0^-} \int_{3+h}^{4} (x-3)^{-2/3}\, dx$. Each integral converges to 3.

51. E. $\int_2^4 \frac{dx}{(x-3)^2} = \int_2^3 \frac{dx}{(x-3)^2} + \int_3^4 \frac{dx}{(x-3)^2}$. Neither of the latter integrals converges; therefore the original integral diverges.

52. C. Evaluate $\lim_{h\to0^+} 2\sqrt{1-\cos x}\,\Big|_0^{(\pi/2)-h}$.

53. D. The integral in (D) is the sum of two integrals from -1 to 0 and from 0 to 1. Both diverge (see Example 24, page 242). Note that (A), (B), and (C) all converge.

54. E. Note, first, that

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \tan^{-1} x \Big|_0^b = \frac{\pi}{2}.$$

Since $Q(b) = \int_0^b \frac{dx}{x^3+1}$ increases with b and, further,

$$Q(b) \leqq \int_0^b \frac{dx}{x^2+1},$$

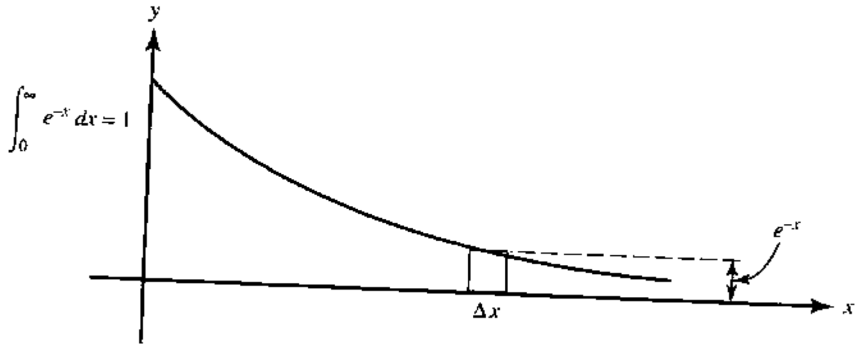
$Q(b)$ converges as $b \to \infty$. Similarly, since $\int_0^\infty \frac{dx}{e^x} = 1$ and since, further, $S(b) = \int_0^b \frac{dx}{e^x+2}$ increases with b and

$$\int_0^b \frac{dx}{e^x+2} \leqq \int_0^b \frac{dx}{e^x}.$$

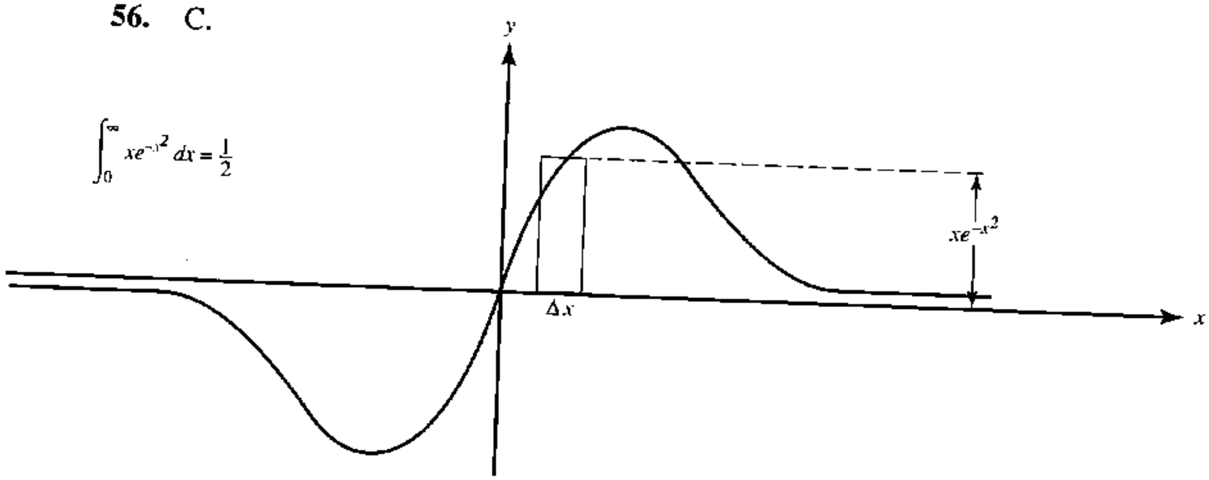
$S(b)$ also converges as $b \to \infty$.

The convergence of (B) is shown in Example 19, page 241.

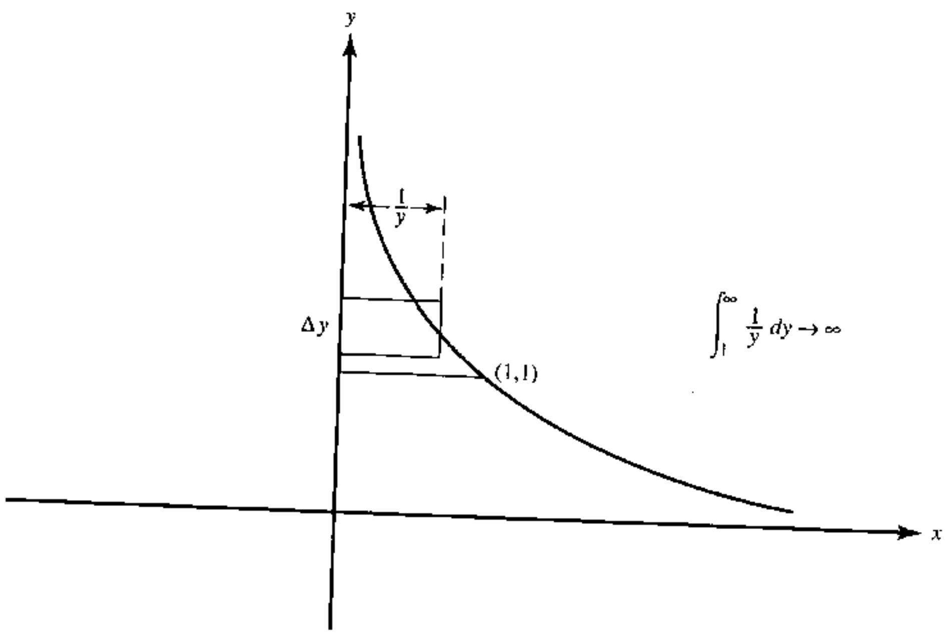
55. A.


y
$\int_0^\infty e^{-x}\,dx = 1$
e^{-x}
Δx
x

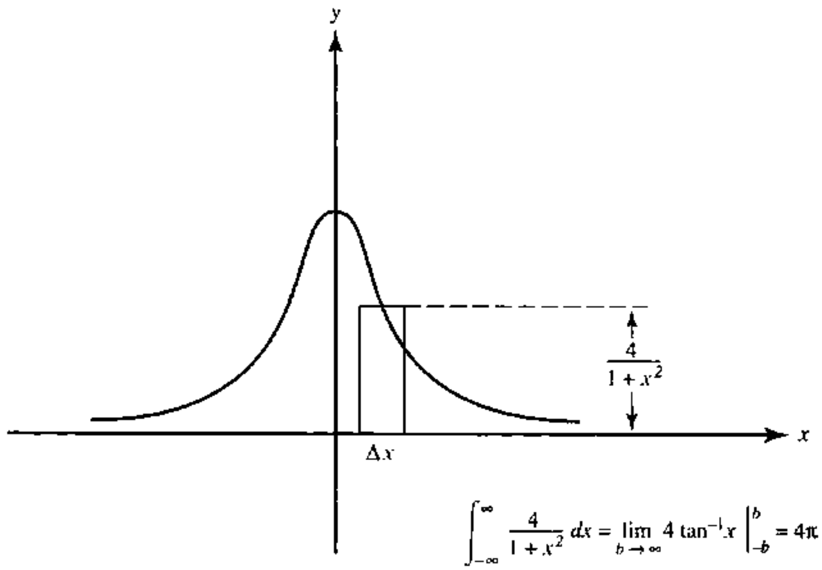
56. C.


y
$\int_0^\infty xe^{-x^2}\,dx = \frac{1}{2}$
xe^{-x^2}
Δx
x

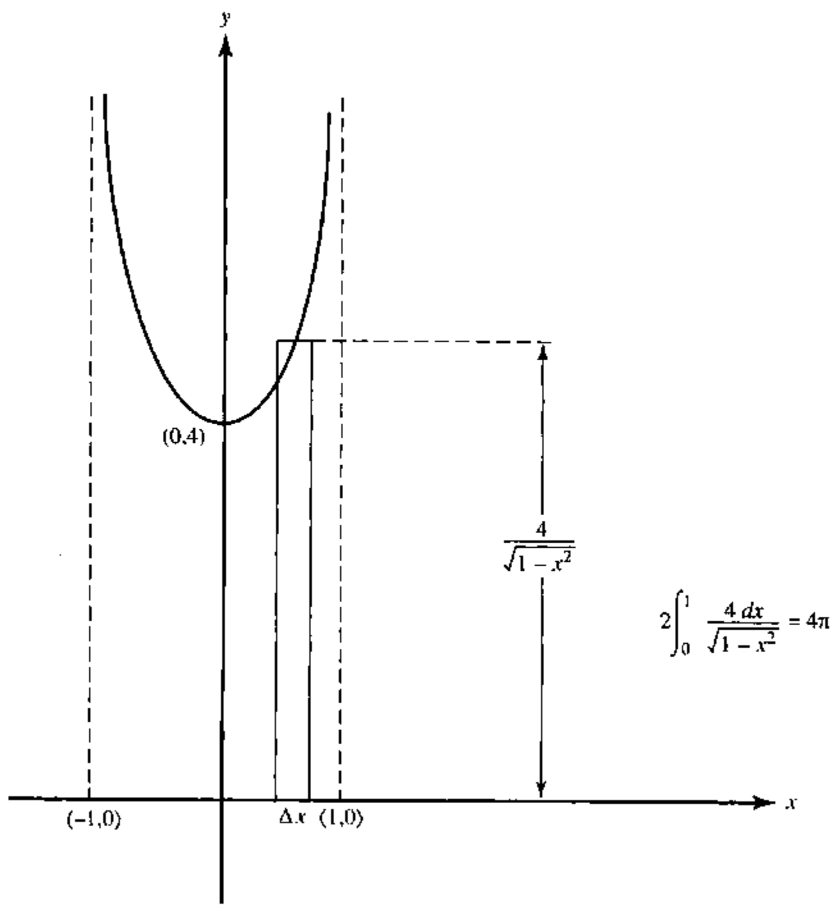
57. E.


y
$\frac{1}{y}$
Δy
(1,1)
$\int_1^\infty \frac{1}{y}\,dy \to \infty$
x

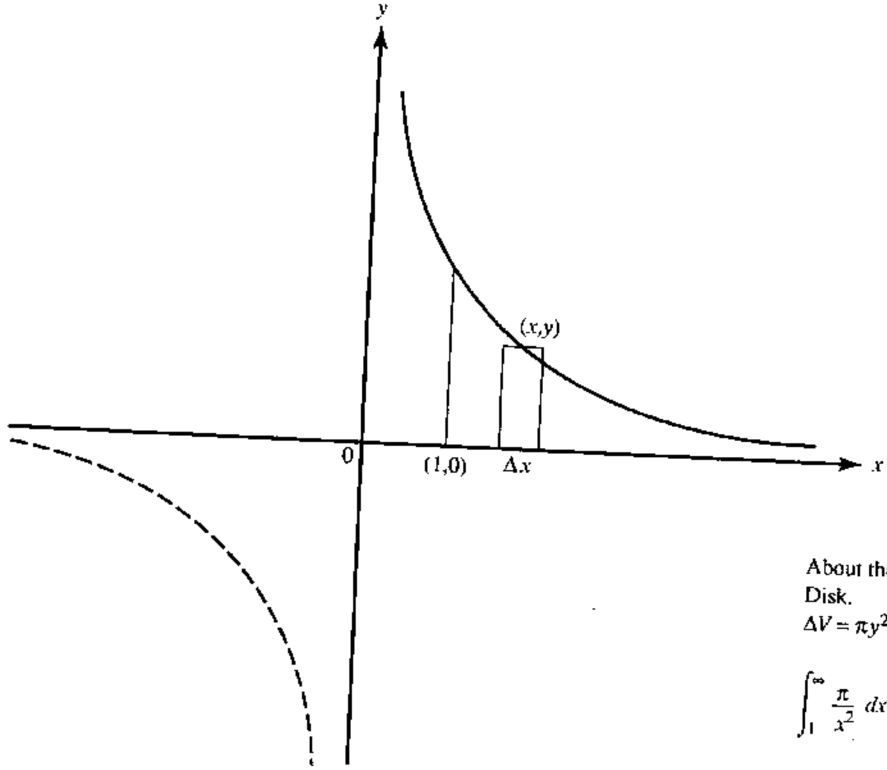
58. B.


y
x
4/(1 + x^2)
Δx
∫_{-∞}^{∞} 4/(1 + x^2) dx = lim_{b→∞} 4 tan^{-1} x |_{-b}^{b} = 4π

59. D.


y
x
(0,4)
4/√(1 − x^2)
2∫_0^1 4 dx/√(1 − x^2) = 4π
(−1,0)
Δx
(1,0)

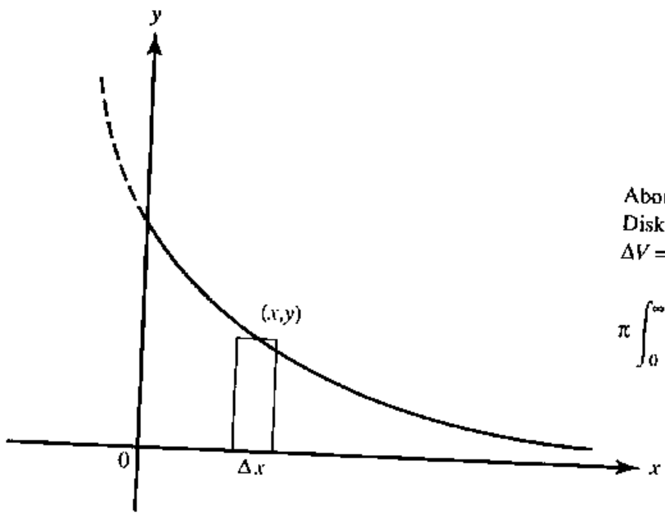
60. B.



About the x-axis.
Disk.
$\Delta V = \pi y^2 \, \Delta x = \frac{\pi}{x^2} \, \Delta x.$

$$\int_1^{\infty} \frac{\pi}{x^2} \, dx = \pi$$

61. A.



About the x-axis.
Disk.
$\Delta V = \pi y^2 \, \Delta x = \pi e^{-2x} \Delta x.$

$$\pi \int_0^{\infty} e^{-2x} \, dx = \frac{\pi}{2}$$