CK-12 Foundation

Chapter 1- Foundations of Geometry

Dillinger

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Contents

Chapter 1

Section 1.1- Points, Lines, and Planes

In this chapter, students will learn about the building blocks of geometry. We will start with the basics: point, line and plane and build upon those terms. From here, students will learn about segments, midpoints, angles, bisectors, angle relationships, and how to classify polygons.

1.1 Points, Lines, and Planes

Learning Objectives

- Understand and build upon the terms:*point, line,* and *plane.*
- Apply and use basic postulates.
- Draw and label terms in a diagram.

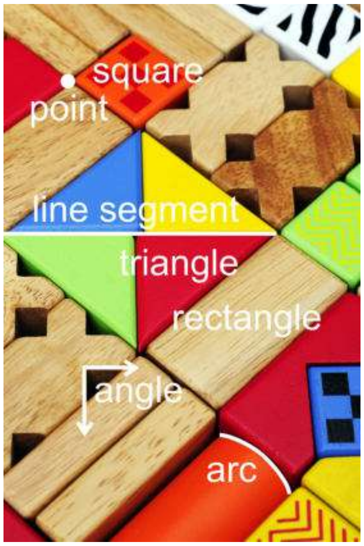
Review Queue

This subsection will provide a review of Algebra and previous lessons.

1. List and draw pictures of five geometric figures you are familiar with.
2. A plane is any flat, two-dimensional surface. List three examples of planes in real life.
3. Solve the algebraic equations.
   (a) $4x - 7 = 29$
   (b) $2(-3x + 5) - 8 = -x + 17$
   (c) $x^2 - 2x - 15 = 0$
   (d) $x^2 = 121$

Know What? Geometry is everywhere. Remember these wooden blocks that you played with as a kid? If you played with these blocks, then you have been "studying" geometry since you were a child. For example, if you were to move the four triangles at the center of the picture, so that the vertex in the middle was on the outside, what shape would you make? (Flip each triangle outward)

Geometry: The study of shapes and their spatial properties.



Building Blocks

Point: An exact location in space.

A point describes a location, but has no size. Dots are used to represent points in pictures and diagrams.

These points are said "Point A," "Point L", and "Point F." Points are labeled with a CAPITAL letter.

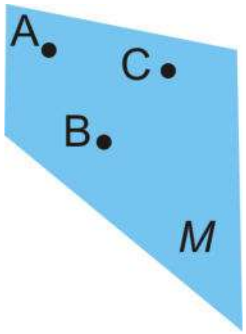
Line: Infinitely many points that extend forever in both directions.

A line, like a point, does not take up space. It has direction, location and is always straight. Lines are one-dimensional because they only have length (no width). A line can by named or identified using any two points on that line or with a lower-case, italicized letter.

This line can be labeled $\overleftrightarrow{PQ}$, $\overleftrightarrow{QP}$ or just g. You would say "line PQ," "line QP," or "line g," respectively. Notice that the line over the $\overleftrightarrow{PQ}$ and $\overleftrightarrow{QP}$ has arrows over both the P and Q. The order of P and Q does not matter.

Plane: Infinitely many intersecting lines that extend forever in all directions.

Think of a plane as a huge sheet of paper that goes on forever. Planes are considered to be two-dimensional because they have a length and a width. A plane can be classified by any three points in the plane.



This plane would be labeled Plane ABC or Plane $\mathcal{M}$. Again, the order of the letters does not matter. Sometimes, planes can also be labeled by a capital cursive letter. Typically, the cursive letter written in a corner of the plane.

Example 1: Which term best describes how San Diego, California, would be represented on a globe?

A. point

B. line

C. plane

Solution: A city is usually labeled with a dot, or point, on a globe. A.

Example 2: Which geometric object best models the surface of a movie screen?

A. point

B. line

C. plane

Solution: The surface of a movie screen extends in two dimensions: up and down and left to right. This description most closely resembles a plane, C.

Beyond the Basics

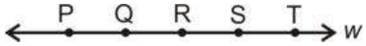
Now we can use **point, line**, and **plane** to define new terms.

Space: The set of all points expanding in ***three*** dimensions.

Think back to the plane. It extended along two different lines: up and down, and side to side. If we add a third direction, we have something that looks like three-dimensional space, or the real-world.

Collinear: Points that lie on the same line.

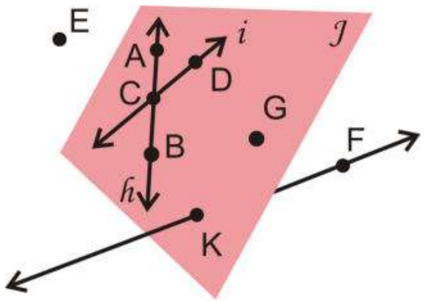
Example 3: Which points are collinear?



Solution: P, Q, R, S, and T are collinear because they are all on line w. If a point U was above or below line w, it would be **non-collinear**.

Coplanar: Points and/or lines within the same plane.

Example 4:



a) List two other ways to label Plane $\mathcal{J}$.

b) List one other way to label line h.

c) Are K and F collinear? Are they coplanar?

d) Are E, B and F coplanar?

e) List four points that are non-collinear.

Solution:

a) Plane BDG, Plane KAG, among several others. Any combination of three coplanar points that are not collinear would be correct.

b) $\overleftrightarrow{AB}$ or any combination of two of the letters A, C or B in any order.

c) Yes, they lie on the same line.Yes, you need three points to create a plane, so any two or three points are coplanar.

d) Yes, any three points are coplanar.

e) A, C, D and E would be non-collinear. Since three points define a plane, any three of the points would line in the same plane but the fourth must not.

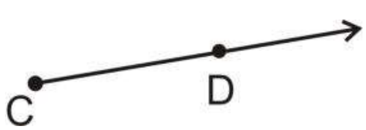
Endpoint: A point at the end of a line.

Line Segment: Part of a line with two endpoints. Or a line that stops at both ends.

Line segments are labeled by their endpoints, $\overline{AB}$ or $\overline{BA}$. Notice that the bar over the endpoints has NO arrows. Order does not matter.

Ray: Part of a line with one endpoint and extends forever in the other direction.

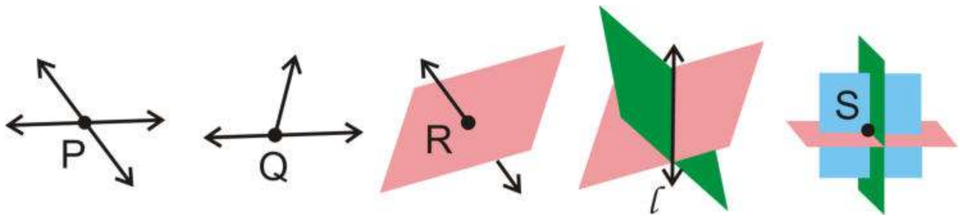
A ray is labeled by its endpoint and one other point on the line.



Of lines, line segments and rays, rays are the only one where order matters. When labeling, always write the endpoint under the side WITHOUT the arrow, $\overrightarrow{CD}$ or $\overleftarrow{DC}$.

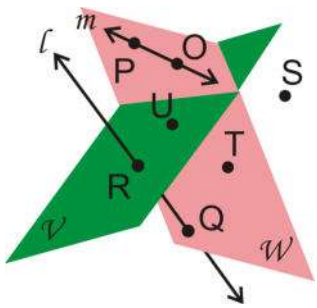
Intersection: A point or set of points where lines, planes, segments or rays cross each other.

Example 5: How do the figures below intersect?



Solution: The first three figures intersect at a point, P, Q and R, respectively. The fourth figure, two planes, intersect in a line, l. And the last figure, three planes, intersect at one point, S.

Example 6: Answer the following questions about the picture to the right.



a) How do the two planes intersect?

b) Is line l coplanar with Plane $\mathcal{V}$ or $\mathcal{W}$?

c) Are R and Q collinear?

d) What point is non-coplanar with either plane?

e) List three coplanar points in Plane $\mathcal{W}$.

Solution:

a) In a line.

b) No.

c) Yes.

d) S

e) Any combination of P, O, T and Q would be correct.

Further Beyond

With these new definitions, we can make statements and generalizations about these geometric figures. This section introduces a few basic postulates. Throughout this book we will be introducing Postulates and Theorems so it is important that you understand what they are and how they differ.

Postulates: Basic rules of geometry. We can assume that all postulates are true, much like a definition.

Theorem: A statement that can be proven true using postulates, definitions, and other theorems that have already proven.

The only difference between a theorem and postulate is that a postulate is *assumed* true because it cannot be shown to be false, a theorem must be *proven* true. We will prove theorems later in this text.

Postulate 1-1: There is exactly one (straight) line through any two points.

Postulate 1-2: There is exactly one plane that contains any three non-collinear points.

Postulate 1-3: A line with points in a plane also lies within that plane.

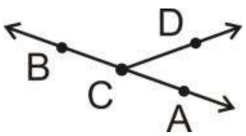
Postulate 1-4: The intersection of two distinct lines will be one point.

Postulate 1-5: The intersection of two planes is a line.

When making geometric drawings, you need to be sure to be clear and label. For example, if you draw a line, be sure to include arrows at both ends. Make sure you label your points, lines, and planes clearly, and refer to them by name when writing explanations.

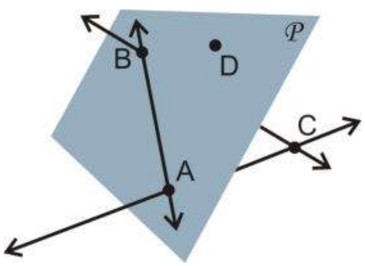
Example 7: Draw and label the intersection of line $\overleftrightarrow{AB}$ and ray $\overrightarrow{CD}$ at point C.

Solution: It does not matter the placement of A or B along the line nor the direction that $\overrightarrow{CD}$ points.

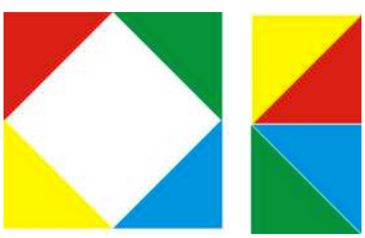


Example 8: Describe the picture below using all the geometric terms you have learned.

Solution: $\overleftrightarrow{AB}$ and D are coplanar in Plane $\mathcal{P}$, while $\overleftrightarrow{BC}$ and $\overleftrightarrow{AC}$ intersect at point C which is non-coplanar.



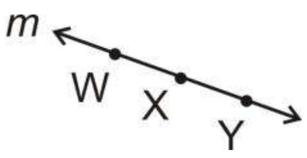
Know What? Revisited If you take the triangles and move them so that the point that met at the center of the square was on the outside, you would get the figure at the right. However, you could also argue that this is not a shape because it has a square hole in the center of it. Another shape that can be made from the four triangles is a rectangle. Part of geometry is justifying and explaining reasoning. You could reason that both of these answers are acceptable.



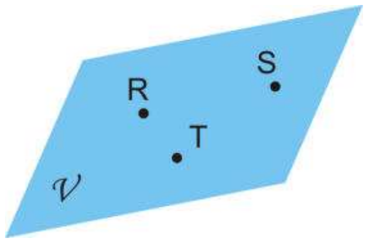
Review Questions

For questions 1-5, draw and label an image to fit the descriptions.

1. $\overrightarrow{CD}$ intersecting $\overline{AB}$ and Plane P containing $\overline{AB}$ but not $\overrightarrow{CD}$.
2. Three collinear points A, B, and C such that B is also collinear with points D and E.
3. $\overrightarrow{XY}$, $\overrightarrow{XZ}$, and $\overrightarrow{XW}$ such that $\overrightarrow{XY}$ and $\overrightarrow{XZ}$ are coplanar, but $\overrightarrow{XW}$ is not.
4. Two intersecting planes, $\mathcal{P}$ and Q, with $\overline{GH}$ where G is in plane $\mathcal{P}$ and H is in plane Q.
5. Four non-collinear points, I, J, K, and L, with line segments connecting all points to each other.
6. Name this line in five ways.

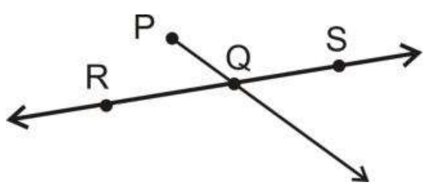


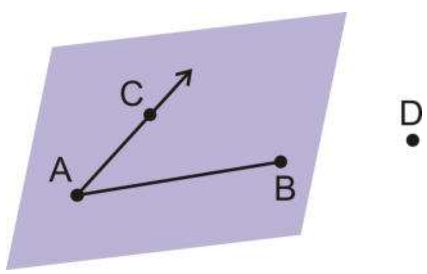
7. Name the geometric figure below in two different ways.

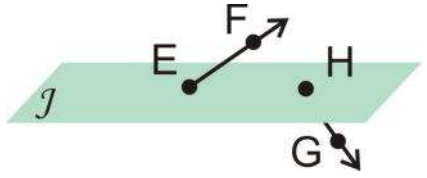


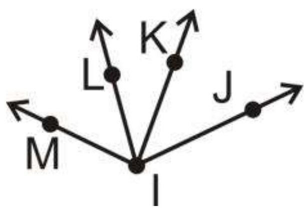
8. Draw three ways three different planes can (or cannot) intersect.
9. What type of geometric object is made by the intersection of a sphere (a ball) and a plane? Draw your answer.

For 10-13, use geometric notation to explain each picture in as much detail as possible.

10. 

11. 

12. 

13. 

For 14-23, determine if the following statements are ALWAYS true, SOMETIMES true, or NEVER true.

14. Any two distinct points are collinear.
15. Any three points determine a plane.
16. A line is composed to two rays with a common endpoint.
17. A line segment is infinitely many points between two endpoints.
18. A point takes up space.
19. A line is one-dimensional.
20. Any four distinct points are coplanar.
21. $\overrightarrow{AB}$ could be read "ray AB" or "ray BA."
22. $\overleftrightarrow{AB}$ could be read "line AB" or "line BA."
23. Theorems are proven true with postulates.
24. Two of the above statements are "never." Explain why.
25. Four of the above statements are "sometimes." Explain why.

For 26-28, describe the following real world objects in geometric terms.

26. The walls of your classroom and the intersections of these walls with each other and the floor or ceiling. What about where two walls and the floor intersect?
27. The spokes of a bicycle wheel. What about their intersection?
28. Cities on a map. What geometric figure would you draw to measure the distance between them?

In Algebra you plotted points on the coordinate plane and graphed lines. For 29-35, use graph paper and follow the steps to make the diagram on the same graph.

29. Plot the point (2, -3) and label it A.
30. Plot the point (-4, 3) and label it B.
31. Draw the segment $\overline{AB}$.
32. Locate point C, the intersection of this line with the $x-$axis.
33. Draw the ray $\overrightarrow{CD}$ with point $D(1,4)$.

Review Queue Answers

1. Examples could be triangles, squares, rectangles, lines, circles, points, pentagons, stop signs (octagons), boxes (prisms, or dice (cubes).
2. Examples of a plane would be: a desktop, the chalkboard/whiteboard, a piece of paper, a TV screen, window, wall or a door.
3. (a) $$\begin{aligned}4x-7&=29\\4x&=36\\x&=9\end{aligned}$$
   (b) $$\begin{aligned}2(-3x+5)-8&=-x+17\\-6x+10-8&=-x+17\\-6x+2&=-x+17\\-5x&=15\\x&=3\end{aligned}$$
   (c) Factor, $x=5,\ -3$
   (d) $x=\pm 11$

Chapter 2

Section 1.2-Segments and Distance

2.1 Segments and Distance

Learning Objectives

- Understand the ruler postulate.
- Understand the segment addition postulate.
- Place line segments on a coordinate grid.

Review Queue

Answer the following questions.

1. How would you label the following geometric figure? List 3 different ways

2. Draw three collinear points and a fourth that is coplanar with these points.
3. Plot the following points on the $x-y$ plane.
   (a) (3, -3)
   (b) (-4, 2)
   (c) (0, -7)
   (d) (6, 0)
4. Find the equation of the line containing the points (-4, 3) and (6, -2).

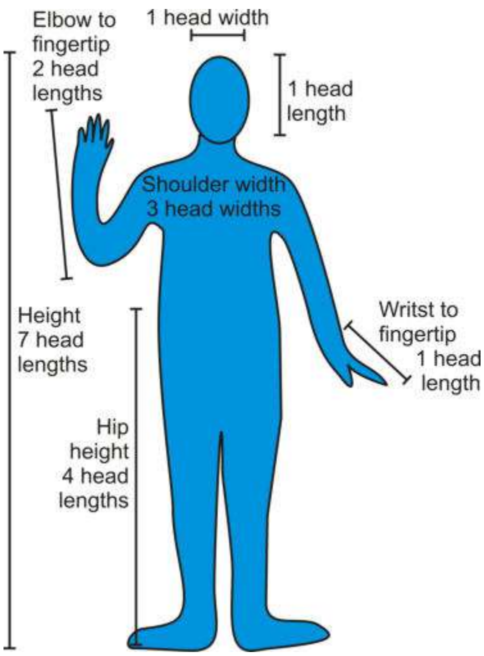
Know What? The average adult human body can be measured in "heads." For example, the average human is 7-8 heads tall. When doing this, keep in mind that each person uses their own head to measure their own body. Other interesting measurements are in the picture to the right.

After analyzing the picture, we can determine a few other measurements that aren't listed.

- The length from the wrist to the elbow

- The length from the top of the neck to the hip
- The width of each shoulder

What are these measurements?



Measuring Distances

Distance: The length between two points.

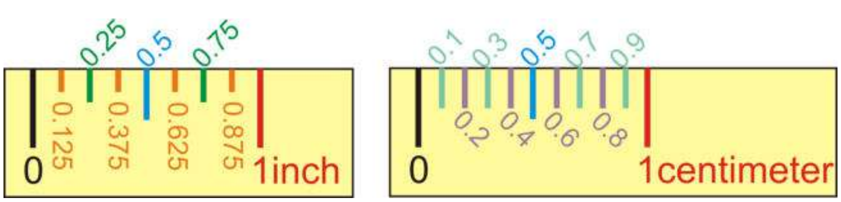
Measure: To determine how far apart two geometric objects are.

The most common way to measure distance is with a ruler. In this class we will use both inches and centimeters.

Example 1: Determine how long the line segment is, in inches. Round to the nearest quarter-inch.

Solution: To measure this line segment with a ruler, it is very important to line up the "0" with the one of the endpoints. DO NOT USE THE EDGE OF THE RULER. This segment is about 3.5 inches (in) long.

As a reminder, inch-rulers are usually divided up by $\frac{1}{8}$-in. (or 0.125 in) segments. Centimeter rulers are divided up by $\frac{1}{10}$-centimenter (or 0.1 cm) segments.



The two rulers above are NOT DRAWN TO SCALE. Anytime you see this statement, it means that the measured length is not actually the distance apart that it is labeled. Different problems and examples will be labeled this way because it can be difficult to draw problems in this text to full scale. You should never assume that objects are drawn to scale. Always rely on the measurements or markings given in a diagram.

Example 2: Determine the measurement between the two points to the nearest tenth of a centimeter.

Solution: Even though there is no line segment between the two points, we can still measure the distance using a ruler. It looks like the two points are 4.5 centimeters (cm) apart.

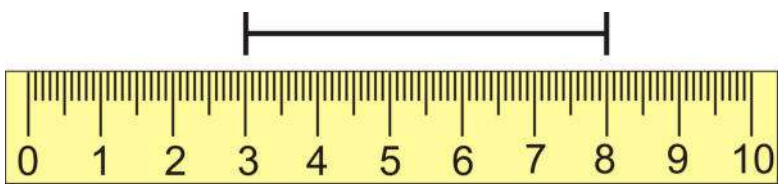
NOTE: We label a line segment, $\overline{AB}$. The ***distance*** between A and B is labeled as AB or $m\overline{AB}$, where m means measure. AB and $m\overline{AB}$ can be used interchangeably. In this text we will primarily use the first.

Ruler Postulate

Ruler Postulate: The distance between two points will be the absolute value of the difference between the numbers shown on the ruler.

The ruler postulate implies that you do not need to start measuring at "0", as long as you subtract the first number from the second. "Absolute value" is used because ***distance is always positive.***

Example 3: What is the distance marked on the ruler below? The ruler is in centimeters.



Solution: Find the absolute value of difference between the numbers shown. The line segment spans from 3 cm to 8 cm.

$$|8 - 3| = |5| = 5$$

The line segment is 5 cm long. Notice that you also could have done $|3 - 8| = |-5| = 5$.

Example 4: Draw $\overline{CD}$, such that $CD = 3.825$ *in*.

Solution: To draw a line segment, start at "0" and draw a segment to 3.825 in. Put points at each end and label.

Segment Addition Postulate

Before we introduce this postulate, we need to address what the word "between" means in geometry.

B is between A and C in this picture. As long as B is ***anywhere on the segment***, it can be considered to be ***between*** the endpoints.

Segment Addition Postulate: If B is between A and C, then $AB + BC = AC$.

The picture above illustrates the Segment Addition Postulate. If $AB = 5\ cm$ and $BC = 12\ cm$, then AC must equal $5 + 12$ or 17 cm. You may also think of this as the "sum of the partial lengths, will be equal to the whole length."

Example 5: Make a sketch of $\overline{OP}$, where Q is between O and P.

Solution: Draw $\overline{OP}$ first, then place Q somewhere along the segment.

Example 6: In the picture from Example 5, if $OP = 17$ and $QP = 6$, what is OQ?

Solution: Use the Segment Additional Postulate. $OQ + QP = OP$, so $OQ + 6 = 17$, or $OQ = 17 - 6 = 9$. So, $OQ = 9$.

Example 7: Make a sketch that matches the description: S is between T and V. R is between S and T. $TR = 6\ cm$, $RV = 23\ cm$, and $TR = SV$. Then, find SV, TS, RS and TV.

Solution: Interpret the first sentence first: S is between T and V.

Then add in what we know about R: It is between S and T.

$\underline{\text{To find } SV}$, we know it is equal to TR, so $SV = 6\ cm$.

$$\begin{aligned} \underline{For\ RS}: RV &= RS + SV \\ 23 &= RS + 6 \\ RS &= 17\ cm \end{aligned} \qquad \begin{aligned} \underline{For\ TS}: TS &= TR + RS \\ TS &= 6 + 17 \\ TS &= 23\ cm \end{aligned} \qquad \begin{aligned} \underline{For\ TV}: TV &= TR + RS + SV \\ TV &= 6 + 17 + 6 \\ TV &= 29\ cm \end{aligned}$$

Example 8: ***Algebra Connection*** For $\overline{HK}$, suppose that J is between H and K. If $HJ = 2x + 4$, $JK = 3x + 3$, and $KH = 22$, find the lengths of HJ and JK.

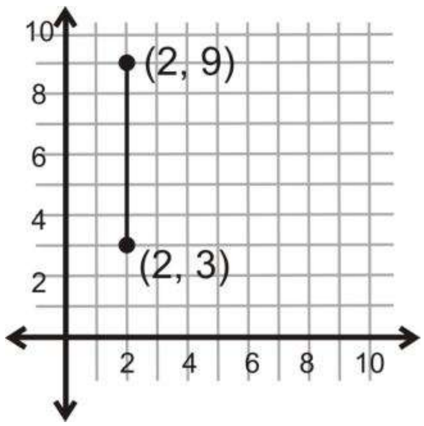
Solution: Use the Segment Addition Postulate and then substitute what we know.

$$\begin{aligned} HJ \quad + \quad JK &= KH \\ (2x + 4) + (3x + 3) &= 22 \\ 5x + 7 &= 22 \qquad \text{So, if } x = 3, \text{then } HJ = 10 \text{ and } JK = 12. \\ 5x &= 15 \\ x &= 3 \end{aligned}$$

Distances on a Grid

In Algebra, you worked with graphing lines and plotting points in the $x - y$ plane. At this point, you can find the distances between points plotted in the $x - y$ plane if the lines are horizontal or vertical. ***If the line is vertical, find the change in the $y-$coordinates. If the line is horizontal, find the change in the $x-$coordinates.***

Example 8: What is the distance between the two points shown below?

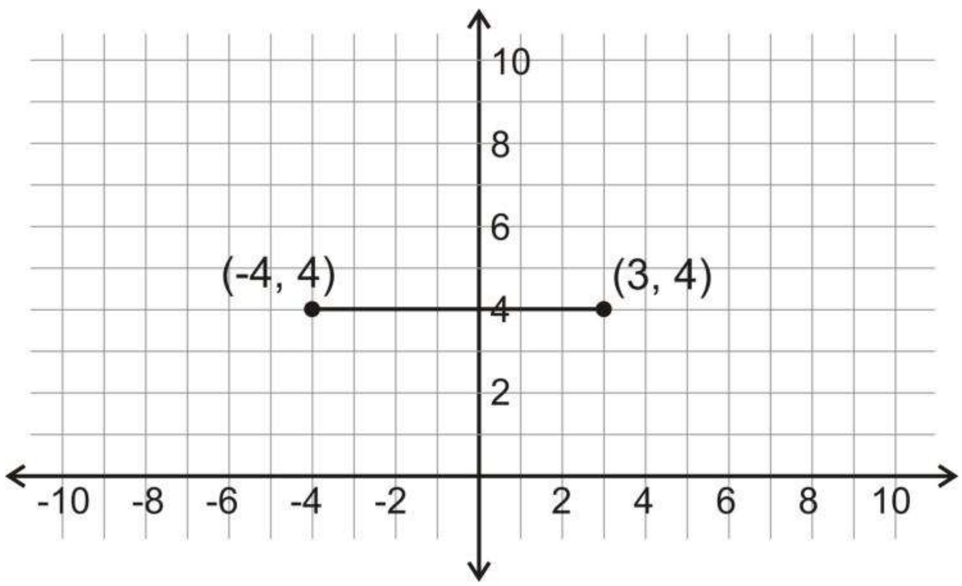


Solution: Because this line is vertical, look at the change in the $y-$coordinates.

$$|9-3| = |6| = 6$$

The distance between the two points is 6 units.

Example 9: What is the distance between the two points shown below?



Solution: Because this line is horizontal, look at the change in the $x-$coordinates.

$$|(-4)-3| = |-7| = 7$$

The distance between the two points is 7 units.

Know What? Revisited The length from the wrist to the elbow is one head, the length from the top of the neck to the hip is two heads, and the width of each shoulder one head width. There are several other interesting body proportion measurements. For example, your foot is the same length as your forearm (wrist to elbow, on the interior of the arm).There are also facial proportions. All of these proportions are what artists use to draw the human body and what da Vinci used to draw his Vitruvian Man, http://en.wikipedia.org/wiki/Vitruvian_Manhttp://en.wikipedia.org/wiki/Vitruvian_Man.

Review Questions

Find the length of each line segment in inches. Round to the nearest $\frac{1}{8}$ of an inch.

1. 

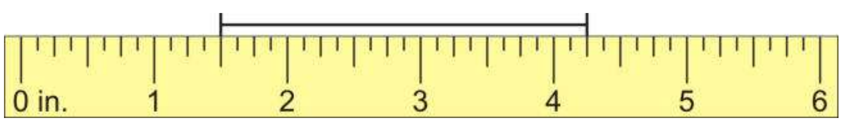
2. 

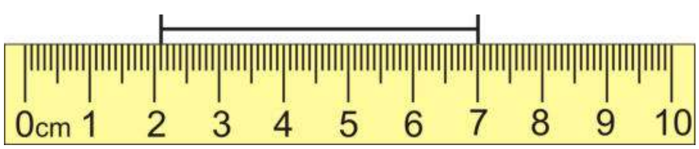
For ind the distance between each pair of points in centimeters. Round to the nearest tenth.

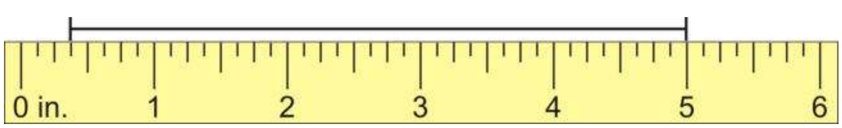
3. 

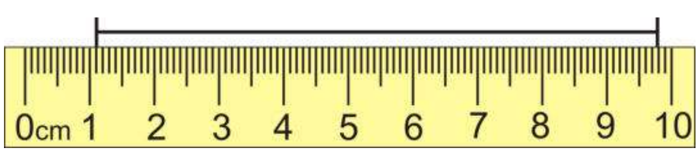
4. 

For 5-8, use the ruler in each picture to determine the length of the line segment.

5. 

6. 

7. 

8. 

9. Make a sketch of $\overline{BT}$, with A between B and T.
10. If O is in the middle of $\overline{LT}$, where exactly is it located? If $LT = 16\ cm$, what is LO and OT?
11. For three collinear points, A between T and Q.

(a) Draw a sketch.
(b) Write the Segment Addition Postulate.
(c) If $AT = 10$ *in* and $AQ = 5$ *in*, what is TQ?

12. For three collinear points, M between H and A.

(a) Draw a sketch.
(b) Write the Segment Addition Postulate.
(c) If $HM = 18$ *cm* and $HA = 29$ *cm*, what is AM?

13. Make a sketch that matches the description: B is between A and D. C is between B and D. $AB = 7$ *cm*, $AC = 15$ *cm*, and $AD = 32$ *cm*. Find BC, BD, and CD.
14. Make a sketch that matches the description: E is between F and G. H is between F and E. $FH = 4$ *in*, $EG = 9$ *in*, and $FH = HE$. Find FE, HG, and FG.
15. Make a sketch that matches the description: S is between T and V. R is between S and T. T is between R and Q. $QV = 18$, $QT = 6$, and $TR = RS = SV$.

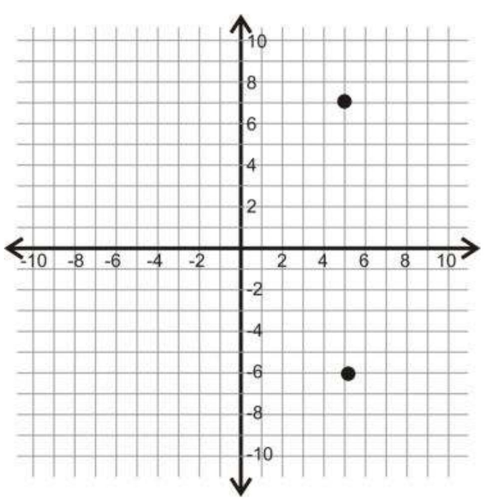
(a) Find RS.
(b) Find QS.
(c) Find TS.
(d) Find TV.

For 16-20, Suppose J is between H and K. Use the Segment Addition Postulate to solve for x. Then find the length of each segment.

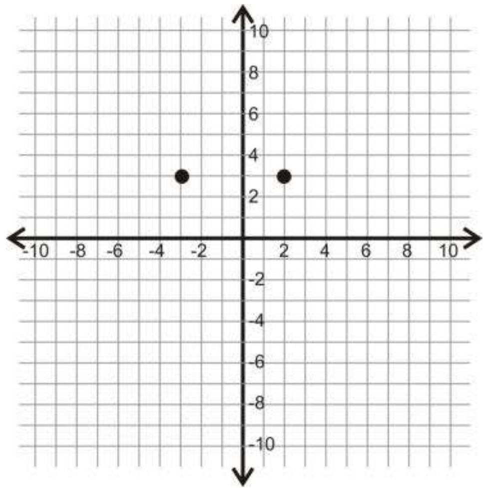
16. $HJ = 4x + 9$, $JK = 3x + 3$, $KH = 33$
17. $HJ = 5x - 3$, $JK = 8x - 9$, $KH = 131$
18. $HJ = 2x + \frac{1}{3}$, $JK = 5x + \frac{2}{3}$, $KH = 12x - 4$
19. $HJ = x + 10$, $JK = 9x$, $KH = 14x - 58$
20. $HJ = \frac{3}{4}x - 5$, $JK = x - 1$, $KH = 22$
21. Draw four points, A, B, C, and D such that $AB = BC = AC = AD = BD$ (HINT: A, B, C and D should NOT be collinear)

For 22-25, determine the vertical or horizontal distance between the two points.

22.



23.



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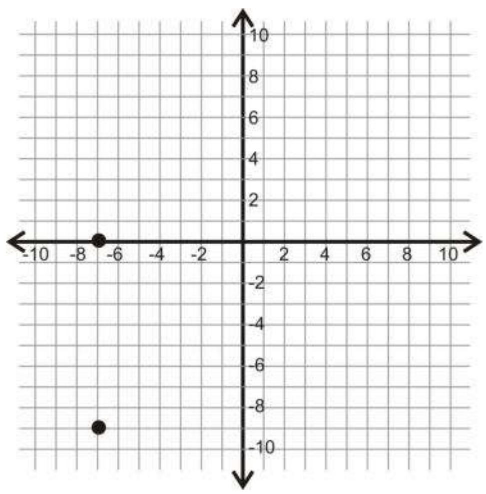
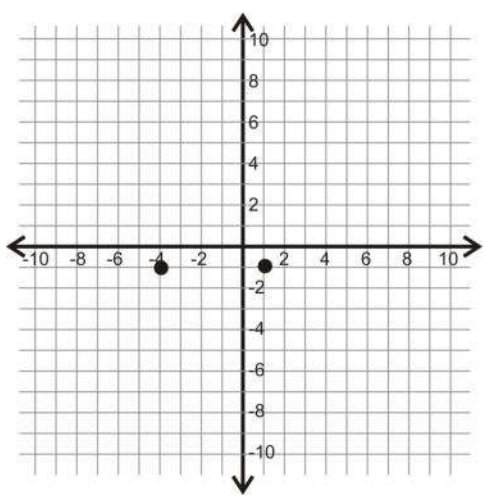


25.

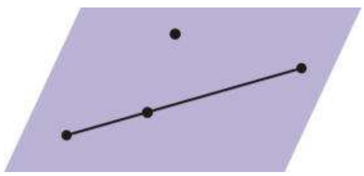
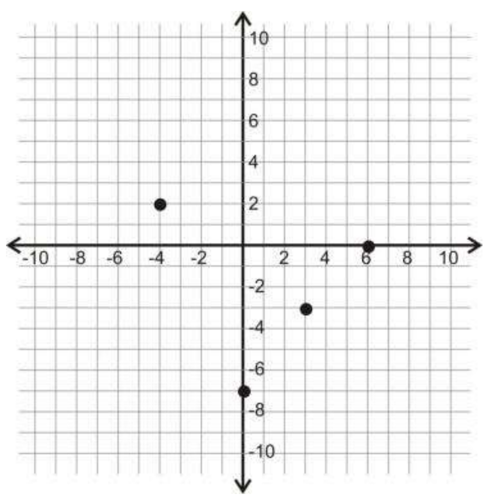


Each of the following problems presents an opportunity for students to extend their knowledge of measurement to the real world. Each of these concepts could be further developed into a mini-project.

26. Measure the length of your head and create a ''ruler'' of this length out of cardstock or cardboard. Use your ruler to measure your height. Share your height in terms of your head length with your class and compare results.

27. Describe the advantages of using the metric system to measure length over the English system. Use the examples of the two rulers (one in inches and one in centimeters) to aid in your description.
28. A speedometer in a car measures distance traveled by tracking the number of rotations on the wheels on the car. A pedometer is a device that a person can wear that tracks the number of steps a person takes and calculates the distance traveled based on the person's stride length. Which would produce a more accurate measure of distance? Why? What could you do to make the less accurate measure more precise?
29. Research the origins of ancient measurement units such as the cubit. Research the origins of the units of measure we use today such as: foot, inch, mile, meter. Why are standard units important?
30. Research the facial proportions that da Vinci used to create his Vitruvian man. Write a summary of your findings.

Review Queue Answers

1. line $l, \overline{MN}, \overline{NM}$
2. 
3. 

4. $m = \frac{3-(-2)}{-4-6} = \frac{5}{-10} = -\frac{1}{2}$
$y = -\frac{1}{2}x + b$
$-2 = -\frac{1}{2}(6) + b$
$-2 = -3 + b$
$1 = b$

So, the equation is $y = -\frac{1}{2}x + 1$

Chapter 3

Section 1.3- Angles and Measurement

3.1 Angles and Measurement

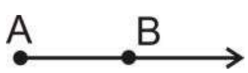
Learning Objectives

- Define and classify angles.
- Apply the Protractor Postulate and the Angle Addition Postulate.

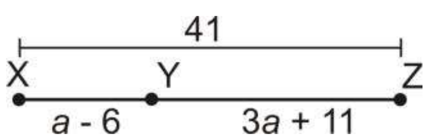
Review Queue

Answer the following questions.

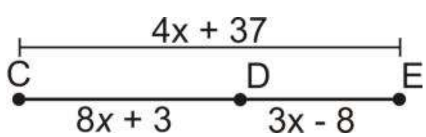
1. Label the following geometric figure. What is it called?



2. Find a, XY and YZ.



3. Find x, CD and DE.

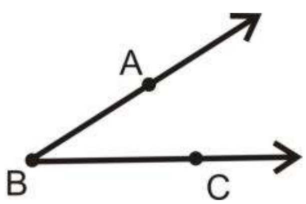


4. B is between A and C on $\overline{AC}$. If $AB = 4$ and $BC = 9$, what is AC? What postulate do you use to solve this problem?

Know What? Back to the building blocks. Every block has its own dimensions, angles and measurements. Using a protractor, find the measure of the three outlined angles in the ‘‘castle” to the right. Also, determine which other angles are equal to these measurements. Use appropriate angle markings. Do not measure any arcs.

Two Rays = One Angle

In #1 above, the figure was a ray. It is labeled $\overrightarrow{AB}$, with the arrow over the point that is NOT the endpoint. When two rays have the same endpoint, an angle is created.



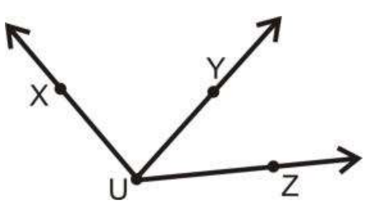
Here, $\overrightarrow{BA}$ and $\overrightarrow{BC}$ meet to form an angle. An angle is labeled with an "$\angle$" symbol in front of the three letters used to label it. This angle can be labeled $\angle ABC$ or $\angle CBA$. **Always put the vertex in the middle of the three points.** It doesn't matter which side point is written first.

Angle: When two rays have the same endpoint.

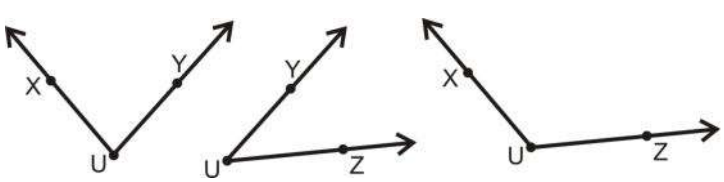
Vertex: The common endpoint of the two rays that form an angle.

Sides: The two rays that form an angle.

Example 1: How many angles are in the picture below? Label each one two different ways.



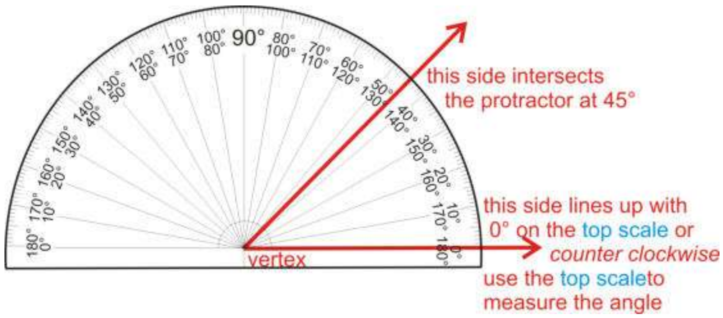
Solution: There are three angles with vertex U. It might be easier to see them all if we separate them out.



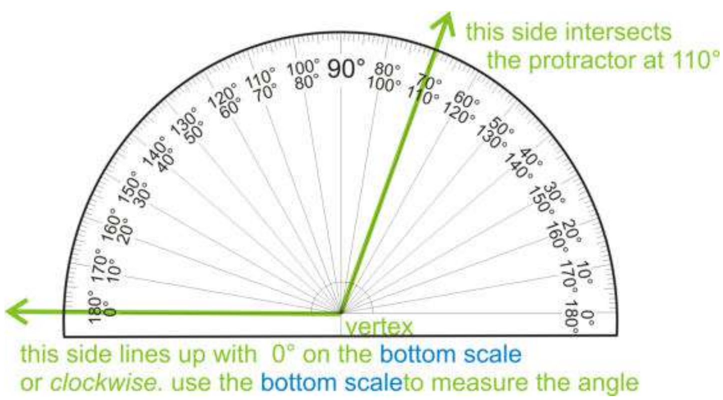
So, the three angles can be labeled, $\angle XUY$ or $\angle YUX$, $\angle YUZ$ or $\angle ZUY$, and $\angle XUZ$ or $\angle ZUX$.

Protractor Postulate

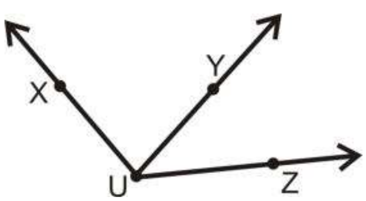
We measure a line segment's *length* with a ruler. Angles are measured with something called a ***protractor***. A protractor is a measuring device that measures how "open" an angle is. Angles are measured in degrees, and labeled with a ° symbol.



Notice that there are two sets of measurements, one opening clockwise and one opening counter-clockwise, from 0° to 180°. When measuring angles, always line up one side with 0°, and see where the other side hits the protractor. The vertex lines up in the middle of the bottom line, where all the degree lines meet.



Example 2: Measure the three angles from Example 1, using a protractor.



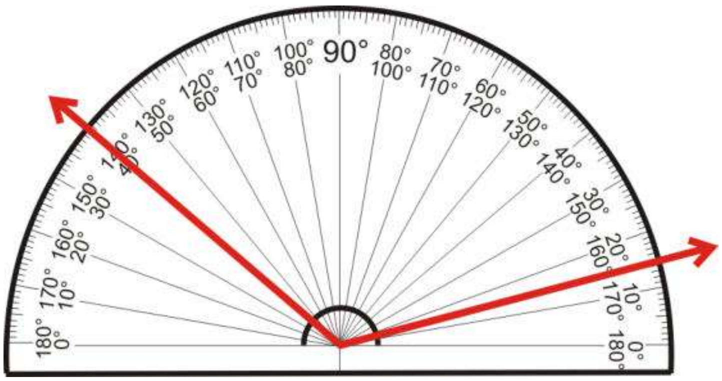
Solution: Just like in Example 1, it might be easier to measure these three angles if you separate them. With measurement, we put an m in front of the $\angle$ sign to indicate measure. So, $m\angle XUY = 84°$, $m\angle YUZ = 42°$ and $m\angle XUZ = 126°$.

In the last lesson, we introduced the Ruler Postulate. Here we introduce the Protractor Postulate.

Protractor Postulate: For every angle there is a number between 0° and 180° that is the measure of the angle in degrees. The angle's measure is then the absolute value of the difference of the numbers shown on the protractor where the sides of the angle intersect the protractor.

In other words, you do not have to start measuring an angle at 0°, as long as you subtract one measurement from the other.

Example 3: What is the measure of the angle shown below?

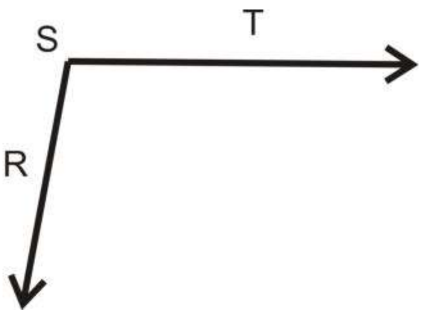


Solution: This angle is not lined up with 0°, so use subtraction to find its measure. It does not matter which scale you use.

Using the inner scale, $|140 - 25| = 125^\circ$

Using the outer scale, $|165 - 40| = 125^\circ$

Example 4: Use a protractor to measure $\angle RST$ below.

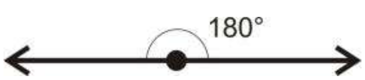


Solution: The easiest way to measure any angle is to line one side up with 0°. This angle measures 100°.

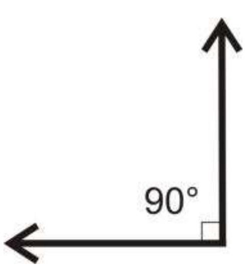
Classifying Angles

By looking at the protractor we measure angles from 0° to 180°. Angles can be classified, or grouped, into four different categories.

Straight Angle: When an angle measures 180°. The angle measure of a straight line. The rays that form this angle are called opposite rays.

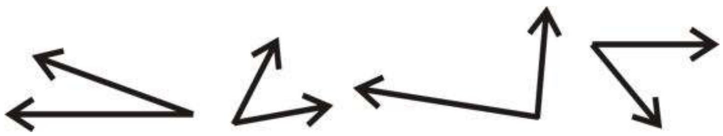


Right Angle: When an angle measures 90°.

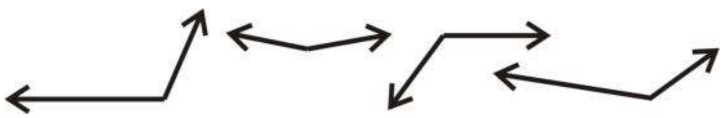


Notice the half-square, marking the angle. This marking is always used to mark right, or 90°, angles.

Acute Angles: Angles that measure between 0° and 90°.



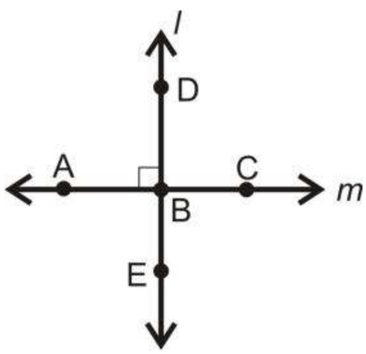
Obtuse Angles: Angles that measure between 90° and 180°.



It is important to note that 90° is NOT an acute angle and 180° is NOT an obtuse angle.

Additionally, any two lines or line segments can intersect to form four angles. If the two lines intersect to form right angles, we say the lines are perpendicular.

Perpendicular: When two lines intersect to form four right angles.

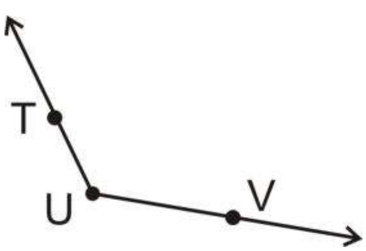


Even though all four angles are 90°, only one needs to be marked. It can be assumed thatall four are 90°.

The symbol for perpendicular is ⊥, so these two lines would be labeled $l \perp m$ or $\overleftrightarrow{AC} \perp \overleftrightarrow{DE}$.

There are several other ways to label these two intersecting lines. This picture shows **two perpendicular lines, four right angles, four** 90° **angles**, and even **two straight angles,** $\angle ABC$ and $\angle DBE$.

Example 5: Name the angle and determine what type of angle it is.



Solution: The vertex is U. So, the angle can be $\angle TUV$ or $\angle VUT$. To determine what type of angle it is, compare it to a right angle. Because it opens wider than a right angle, and less than a straight angle it is **obtuse**.

Example 6: What type of angle is 84°? What about 165°?

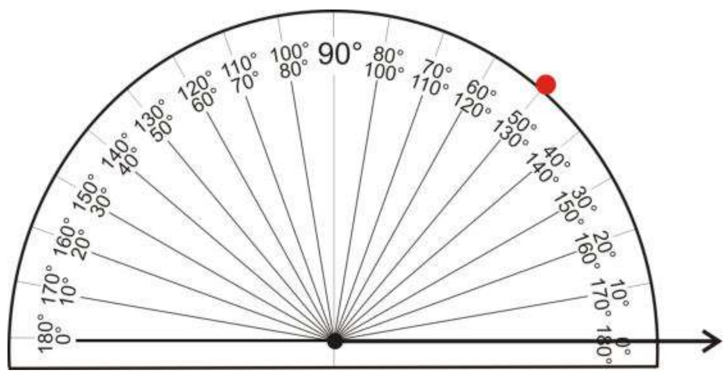
Solution: 84° is less than 90°, so it is **acute**. 165° is greater than 90°, but less than 180°, so it is **obtuse**.

Drawing an Angle

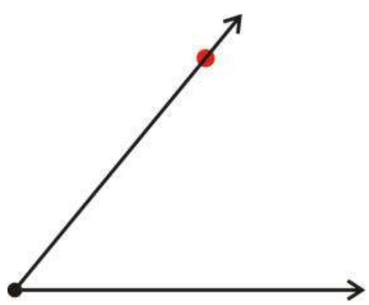
Investigation 1-1: Drawing a 50° Angle with a Protractor

1. Start by drawing a horizontal line across the page, about 2 in long.
2. Place an endpoint at the left side of your line.

3. Place the protractor on this point. Make sure to put the center point on the bottom line of the protractor on the vertex. Mark 50° on the appropriate scale.



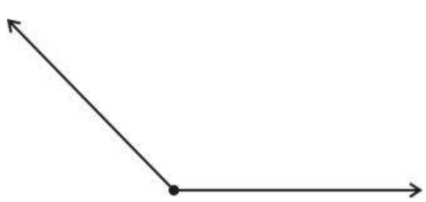
4. Remove the protractor and connect the vertex and the 50° mark.



This process can be used to draw any angle between 0° and 180°. See `http://www.mathsisfun.com/geometry/protractor-using.html`http://www.mathsisfun.com/geometry/protractor-using.html for an **animation** of this investigation.

Example 7: Draw a 135° angle.

Solution: Following the steps from above, your angle should look like this:

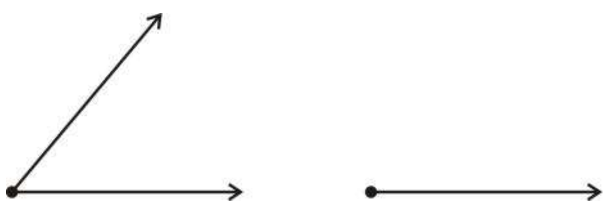


Now that we know how to draw an angle, we can also copy that angle with a compass and a straightedge, usually a ruler. Anytime we use a compass and ruler to draw different geometric figures, it called a **construction**.

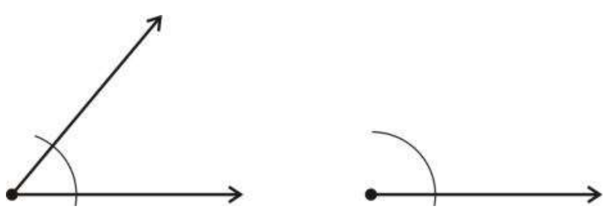
Compass: A tool used to draw circles and arcs.

Investigation 1-2: Copying an Angle with a Compass and Straightedge

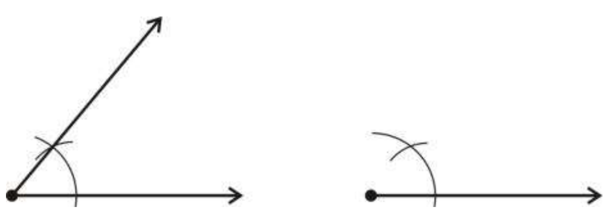
1. We are going to copy the angle created in the previous investigation, a 50° angle. First, draw a straight line, about 2 inches long, and place an endpoint at one end.



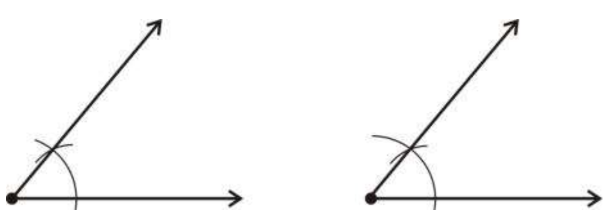
2. With the point (non-pencil side) of the compass on the vertex, draw an arc that passes through both sides of the angle. Repeat this arc with the line we drew in #1.



3. Move the point of the compass to the horizontal side of the angle we are copying. Place the point where the arc intersects this side. Open (or close) the "mouth" of the compass so you can draw an arc that intersects the other side of the arc drawn in #2. Repeat this on the line we drew in #1.



4. Draw a line from the new vertex to the arc intersections.

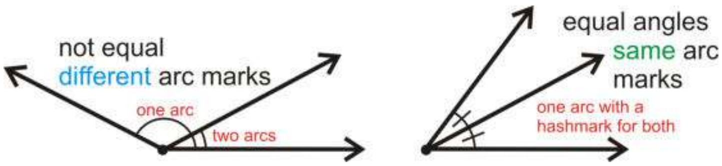


To watch an **animation** of this construction, see http://www.mathsisfun.com/geometry/construct-anglesame.htmlhttp://www.mathsisfun.com/geometry/construct-anglesame.html

Marking Angles and Segments in a Diagram

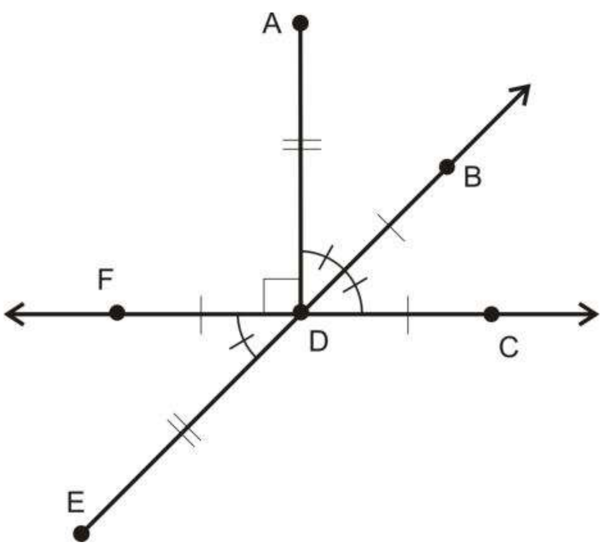
With all these segments and angles, we need to have different ways to label equal angles and segments.

Angle Markings



Segment Markings

Example 8: Interpret the picture below. Write all equal angle and segment statements.



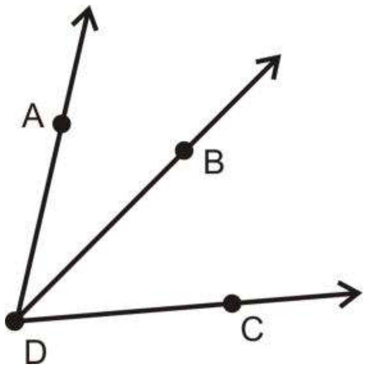
Solution:

$$\overline{AD} \perp \overleftrightarrow{FC}$$
$$m\angle ADB = m\angle BDC = m\angle FDE = 45^\circ$$
$$AD = DE$$
$$FD = DB = DC$$
$$m\angle ADF = m\angle ADC = 90^\circ$$

Angle Addition Postulate

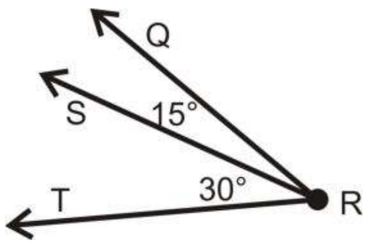
Much like the Segment Addition Postulate, there is an Angle Addition Postulate.

Angle Addition Postulate: The measure of any angle can be found by adding the measures of the smaller angles that are in it.



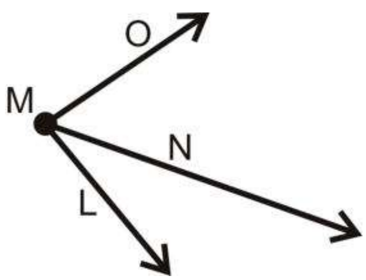
Looking at the picture, $m\angle ADC = m\angle ADB + m\angle BDC$

Example 9: What is $m\angle QRT$ in the diagram below?



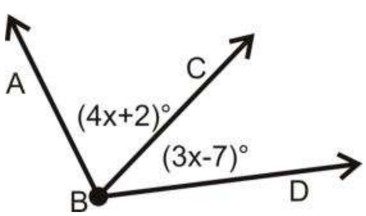
Solution: Using the Angle Addition Postulate, $m\angle QRT = 15^\circ + 30^\circ = 45^\circ$.

Example 10: What is $m\angle LMN$ if $m\angle LMO = 85^\circ$ and $m\angle NMO = 53^\circ$?



Solution: From the Angle Addition Postulate, $m\angle LMO = m\angle NMO + m\angle LMN$. Substituting in what we know, $85^\circ = 53^\circ + m\angle LMN$, so $85^\circ - 53^\circ = m\angle LMN$ or $m\angle LMN = 32^\circ$.

Example 11: ***Algebra Connection*** If $m\angle ABD = 100^\circ$, find x and $m\angle ABC$ and $m\angle CBD$?



Solution: From the Angle Addition Postulate, $m\angle ABD = m\angle ABC + m\angle CBD$. Substitute in what you know and solve the equation.

$$
\begin{aligned}
100^\circ &= (4x+2)^\circ + (3x-7)^\circ \\
100^\circ &= 7x - 5^\circ \\
105^\circ &= 7x \\
15^\circ &= x
\end{aligned}
$$

So, $m\angle ABC = 4(15^\circ) + 2^\circ = 62^\circ$ and $m\angle CBD = 3(15^\circ) - 7^\circ = 38^\circ$.

Know What? Revisited Using a protractor, the measurement marked in the red triangle is 90°, the measurement in the blue triangle is 45° and the measurement in the orange square is 90°.

All of the equal angles are marked in the picture to the right. All of the acute angles in the triangles are equal and all the other angles are right, or 90°.

Review Questions

For questions 1-10, determine if the statement is true or false. If you answered FALSE for any question, state why.

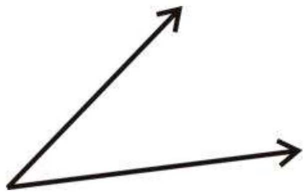
1. Two angles always add up to be greater than 90°.
2. 180° is an obtuse angle.
3. 180° is a straight angle.
4. Two perpendicular lines intersect to form four right angles.
5. A construction uses a protractor and a ruler.
6. For an angle $\angle ABC, C$ is the vertex.
7. For an angle $\angle ABC, \overline{AB}$ and $\overline{BC}$ are the sides.
8. The m in front of $m\angle ABC$ means measure.
9. Angles are always measured in degrees.
10. The Angle Addition Postulate says that an angle is equal to the sum of the smaller angles around it.

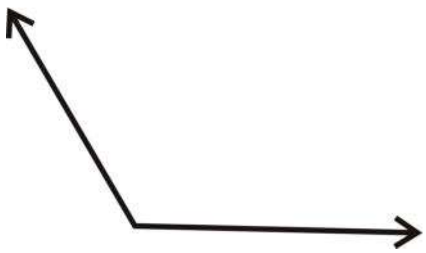
For 11-16, draw the angle with the given degree, using a protractor and a ruler. Also, state what type of angle it is.

11. 55°
12. 92°
13. 178°
14. 5°
15. 120°

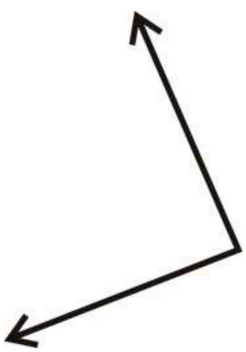
16. 73°
17. ***Construction*** Copy the angle you made from #12, using a compass and a straightedge.
18. ***Construction*** Copy the angle you made from #16, using a compass and a straightedge.

For 19-22, use a protractor to determine the measure of each angle.

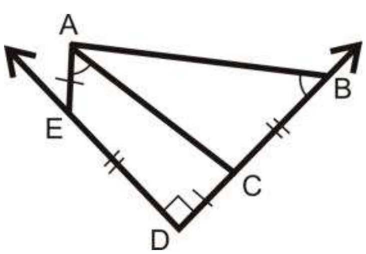
19. 

20. 

21. 

22. 

23. Interpret the picture to the right. Write down all equal angles, segments and if any lines are perpendicular.

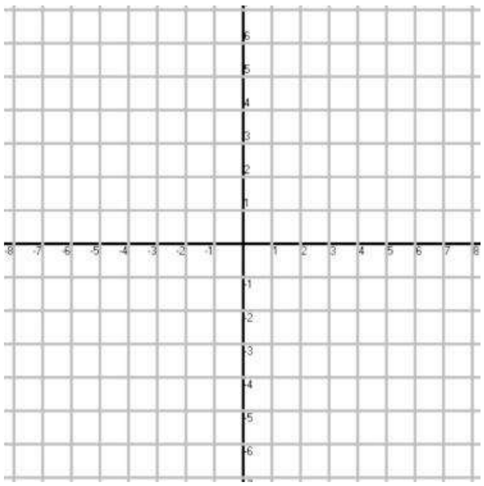


24. Draw a picture with the following requirements.

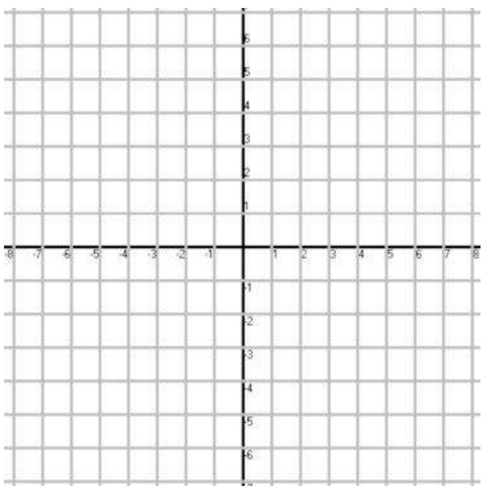
$$= BC = BD \qquad m\angle ABD = 90^\circ$$
$$\angle ABC = m\angle CBD \qquad A, B, C \text{ and } D \text{ are coplanar}$$

In 25 and 26, plot and sketch $\angle ABC$. Classify the angle. Write the coordinates of a point that lies in the interior of the angle.

25. $A(5,-3)$
$B(-3,-1)$
$C(2,2)$



26. $A(-3,0)$
$B(1,3)$
$C(5,0)$

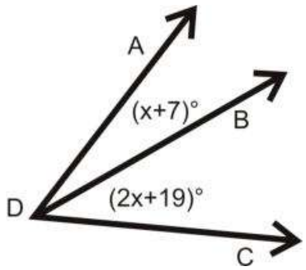


In Exercises 27-31, use the following information: Q is in the interior of $\angle ROS$. S is in the interior of $\angle QOP$. P is in the interior of $\angle SOT$. S is in the interior of $\angle ROT$ and $m\angle ROT = 160^\circ$, $m\angle SOT = 100^\circ$, and $m\angle ROQ = m\angle QOS = m\angle POT$.

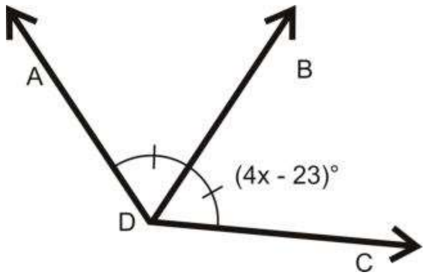
27. Make a sketch.
28. Find $m\angle QOP$
29. Find $m\angle QOT$
30. Find $m\angle ROQ$
31. Find $m\angle SOP$

Algebra Connection Solve for x.

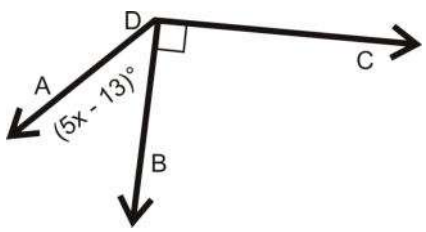
32. $m\angle ADC = 56^\circ$



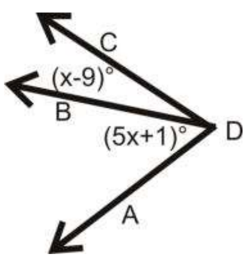
33. $m\angle ADC = 130^\circ$



34. $m\angle ADC = (16x - 55)^\circ$



35. $m\angle ADC = (9x - 80)^\circ$



36. ***Writing*** Write a paragraph about why the degree measure of a straight line is 180, the degree measure of a right angle is 90, etc. In other words, answer the question, "Why is the straight line divided into exactly 180 degrees and not some other number of degrees?"

Review Queue Answers

1. $\overrightarrow{AB}$, a ray

2. $XY = 3,\ YZ = 38$

$$\begin{aligned} a - 6 + 3a + 11 &= 41 \\ 4a + 5 &= 41 \\ 4a &= 36 \\ a &= 9 \end{aligned}$$

3. $CD = 51,\ DE = 10$

$$\begin{aligned} 8x + 3 + 3x - 8 &= 4x + 37 \\ 11x - 5 &= 4x + 37 \end{aligned}$$

$$7x = 42$$
$$x = 6$$

4. Use the Segment Addition Postulate, $AC = 13$.

Chapter 4

Section 1.4- Midpoints and Bisectors

4.1 Midpoints and Bisectors

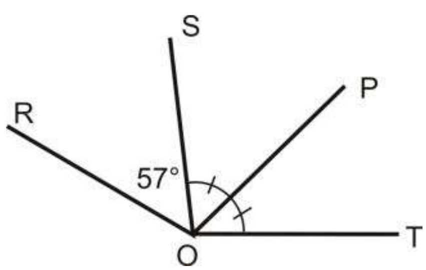
Learning Objectives

- Identify the midpoint of line segments.
- Identify the bisector of a line segment.
- Understand and the Angle Bisector Postulate.

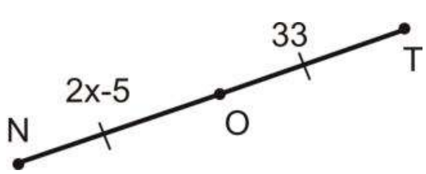
Review Queue

Answer the following questions.

1. $m\angle ROT = 165^\circ$, find $m\angle POT$



2. Find x.



3. Use the Angle Addition Postulate to write an equation for the angles in #1.

Know What? The building to the right is the TransamericaBuilding in San Francisco. This building was completed in 1972 and, at that time was one of the tallest buildings in the world. It is a pyramid with two "wings" on either side, to accommodate elevators. Because San Francisco has problems with earthquakes, there are regulations on how a building can be designed. In order to make this building as tall as it is and still abide by the building codes, the designer used this pyramid shape.

It is very important in designing buildings that the angles and parts of the building are equal. What components of this building look equal? Analyze angles, windows, and the sides of the building.

Congruence

You could argue that another word for ***equal*** is ***congruent***. However, the two differ slightly.

Congruent: When two geometric figures have the same shape and size.

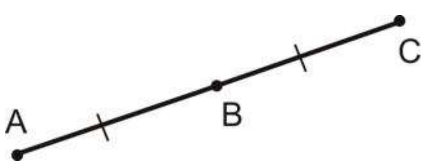
We label congruence with a $\cong$ sign. Notice the $\sim$ above the $=$ sign. $\overline{AB} \cong \overline{BA}$ means that $\overline{AB}$ is congruent to $\overline{BA}$. If we know two segments or angles are congruent, then their measures are also equal. If two segments or angles have the same measure, then, they are also congruent.

Table 4.1:

Equal	***Congruent***
$=$ used with ***measurement*** $m\overline{AB} = AB = 5\ cm$ $m\angle ABC = 60^\circ$	$\cong$ used with ***figures*** $\overline{AB} \cong \overline{BA}$ $\angle ABC \cong \angle CBA$

Midpoints

Midpoint: A point on a line segment that divides it into two congruent segments.

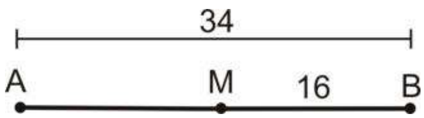


Because $AB = BC$, B is the midpoint of $\overline{AC}$.

Midpoint Postulate: Any line segment will have exactly one midpoint.

This might seem self-explanatory. However, be careful, this postulate is referring to the *midpoint,* not the lines that pass through the midpoint, which is infinitely many.

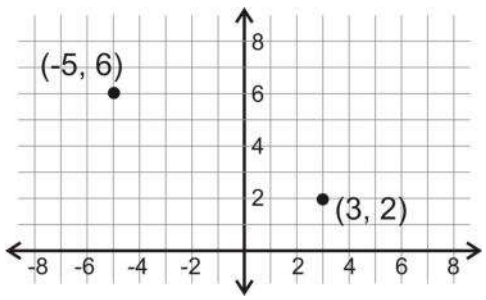
Example 1: Is M a midpoint of $\overline{AB}$?



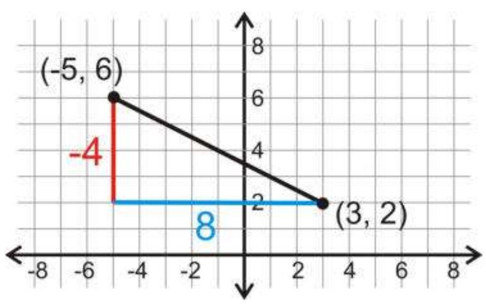
Solution: No, it is not because $MB = 16$ and $AM = 34 - 16 = 18$.

Midpoint Formula

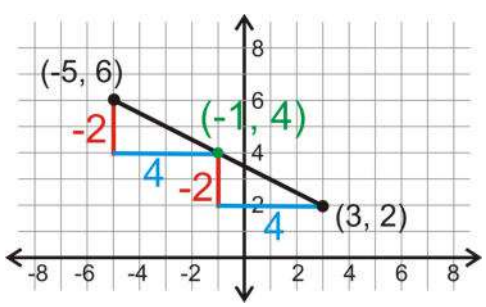
When points are plotted in the coordinate plane, you can use slope to find the midpoint between then. We will generate a formula here.



Here are two points, (-5, 6) and (3, 4). Draw a line between the two points and determine the vertical distance and the horizontal distance.



So, it follows that the midpoint is down and over half of each distance. The midpoint would then be down 2 (or -2) from (-5, 6) and over positively 4. If we do that we find that the midpoint is (-1, 4).



Let's create a formula from this. If the two endpoints are (-5, 6) and (3, 4), then the midpoint is (-1, 4). -1 is ***halfway*** between -5 and 3 and 4 is ***halfway*** between 6 and 2. Therefore, the formula for the midpoint is the average of the $x-$values and the average of the $y-$values.

Midpoint Formula: For two points, $(x_1, \ y_1)$ and $(x_2, \ y_2)$, the midpoint is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Example 2: Find the midpoint between (9, -2) and (-5, 14).

Solution: Plug the points into the formula.

$$\left(\frac{9+(-5)}{2}, \ \frac{-2+14}{2}\right) = \left(\frac{4}{2}, \ \frac{12}{2}\right) = (2,6)$$

Example 3: If $M(3,-1)$ is the midpoint of $\overline{AB}$ and $B(7,-6)$, find A.

Solution: Plug what you know into the midpoint formula.

$$\begin{aligned}\left(\frac{7+x_A}{2}, \frac{-6+y_A}{2}\right) &= (3,-1) \\ \frac{7+x_A}{2} = 3 \text{ and } \frac{-6+y_A}{2} &= -1 \qquad A \text{ is } (-1,4). \\ 7+x_A = 6 \text{ and } -6+y_A &= -2 \\ x_A = -1 \text{ and } y_A &= 4\end{aligned}$$

Another way to find the other endpoint is to find the difference between M and B and then duplicate it on the other side of M.

x−values: $7-3=4$, so 4 on the other side of 3 is $3-4=-1$

y−values: $-6-(-1)=-5$, so -5 on the other side of -1 is $-1-(-5)=4$

A is still (-1, 4). You may use either method.

Segment Bisectors

Segment Bisector: A line, segment, or ray that passes through a midpoint of another segment.

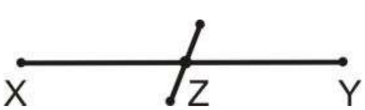
A bisector cuts a line segment into two congruent parts.

Example 4: Use a ruler to draw a bisector of the segment below.

Solution: The first step in identifying a bisector is finding the midpoint. Measure the line segment and it is 4 cm long. To find the midpoint, divide 4 by 2.

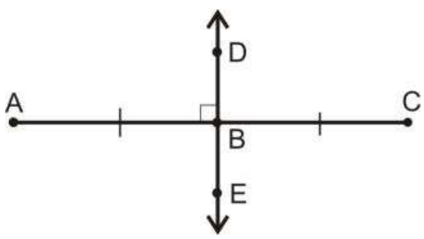
So, the midpoint will be 2 cm from either endpoint, or halfway between. Measure 2 cm from one endpoint and draw the midpoint.

To finish, draw a line that passes through the midpoint. It doesn't matter how the line intersects $\overline{XY}$, as long as it passes through Z.



A specific type of segment bisector is called a perpendicular bisector.

Perpendicular Bisector: A line, ray or segment that passes through the midpoint of another segment and intersects the segment at a right angle.

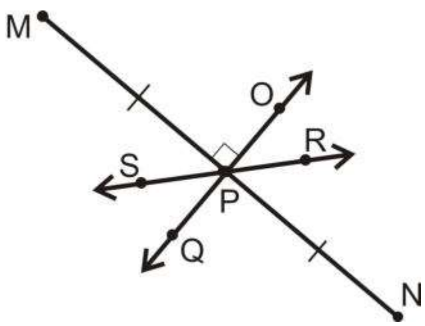


$\overleftrightarrow{DE}$ is the perpendicular bisector of $\overline{AC}$, so $\overline{AB} \cong \overline{BC}$ and $\overline{AC} \perp \overleftrightarrow{DE}$.

Perpendicular Bisector Postulate: For every line segment, there is one perpendicular bisector that passes through the midpoint.

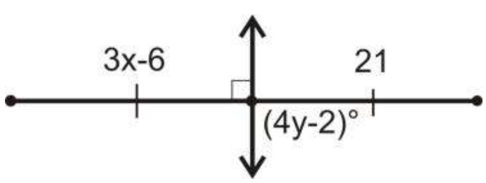
There are *infinitely many bisectors*, but only *one perpendicular bisector* for any segment.

Example 5: Which line is the perpendicular bisector of $\overline{MN}$?



Solution: The perpendicular bisector must bisect $\overline{MN}$ and be perpendicular to it. Only $\overleftrightarrow{OQ}$ satisfies both requirements. $\overleftrightarrow{SR}$ is just a bisector.

Example 6: ***Algebra Connection*** Find x and y.

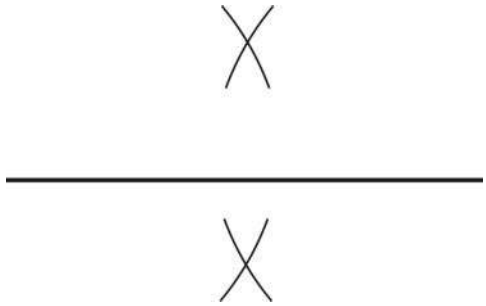


Solution: The line shown is the perpendicular bisector. So, $3x - 6 = 21,\ 3x = 27,\ x = 9$.

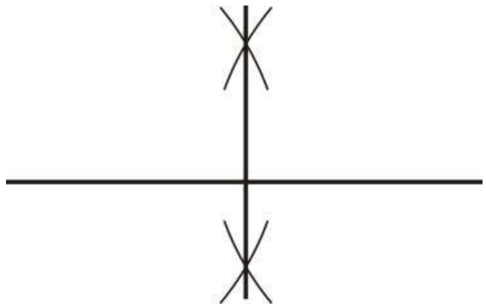
And, $(4y - 2)^\circ = 90^\circ, 4y^\circ = 92^\circ, y = 23^\circ$.

Investigation 1-3: Constructing a Perpendicular Bisector

1. Draw a line that is at least 6 cm long, about halfway down your page.
2. Place the pointer of the compass at an endpoint. Open the compass to be greater than half of the segment. Make arc marks above and below the segment. Repeat on the other endpoint. Make sure the arc marks intersect.



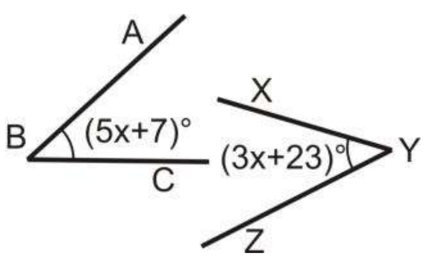
3. Use your straight edge to draw a line connecting the arc intersections.



This constructed line bisects the line you drew in #1 and intersects it at 90°. So, this construction also works to create a right angle. To see an animation of this investigation, go to http://www.mathsisfun.com/geometry/construct-linebisect.htmlhttp://www.mathsisfun.com/geometry/construct-linebisect.html.

Congruent Angles

Example 7: ***Algebra Connection*** What is the measure of each angle?



Solution: From the picture, we see that the angles are congruent, so the given measures are equal.

$$\begin{aligned}(5x+7)^\circ &= (3x+23)^\circ \\ 2x^\circ &= 16^\circ \\ x &= 8^\circ\end{aligned}$$

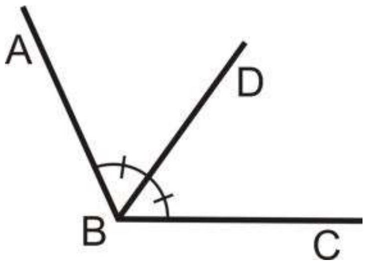
To find the measure of $\angle ABC$, plug in $x = 8^\circ$ to $(5x+7)^\circ$.

$$\begin{aligned}&(5(8)+7)^\circ \\ &(40+7)^\circ \\ &47^\circ\end{aligned}$$

Because $m\angle ABC = m\angle XYZ$, $m\angle XYZ = 47^\circ$ too.

Angle Bisectors

Angle Bisector: A ray that divides an angle into two congruent angles, each having a measure exactly half of the original angle.

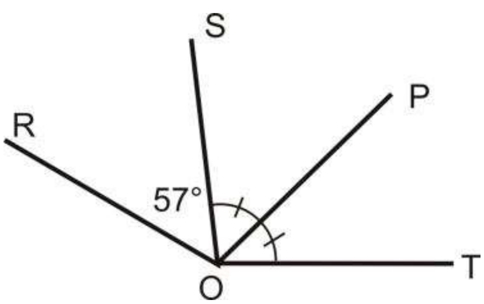


$\overline{BD}$ is the angle bisector of $\angle ABC$

$$\angle ABD \cong \angle DBC$$
$$m\angle ABD = \frac{1}{2} m\angle ABC$$

Angle Bisector Postulate: Every angle has exactly one angle bisector.

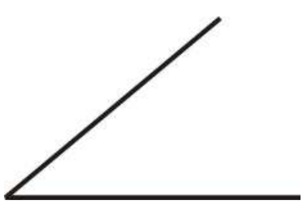
Example 8: Let's take a look at Review Queue #1 again. Is $\overline{OP}$ the angle bisector of $\angle SOT$? Recall, that $m\angle ROT = 165^\circ$, what is $m\angle SOP$ and $m\angle POT$?



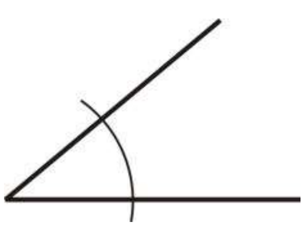
Solution: Yes, $\overline{OP}$ is the angle bisector of $\angle SOT$ according to the markings in the picture. If $m\angle ROT = 165^\circ$ and $m\angle ROS = 57^\circ$, then $m\angle SOT = 165^\circ - 57^\circ = 108^\circ$. The $m\angle SOP$ and $m\angle POT$ are each half of 108° or 54°.

Investigation 1-4: Constructing an Angle Bisector

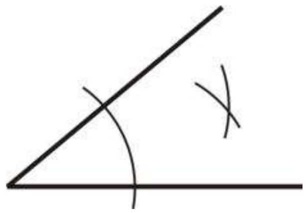
1. Draw an angle on your paper. Make sure one side is horizontal.



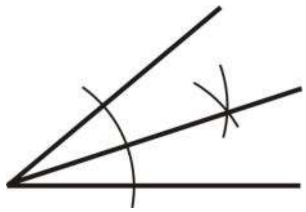
2. Place the pointer on the vertex. Draw an arc that intersects both sides.



3. Move the pointer to the arc intersection with the horizontal side. Make a second arc mark on the interior of the angle. Repeat on the other side. Make sure they intersect.



4. Connect the arc intersections from #3 with the vertex of the angle.

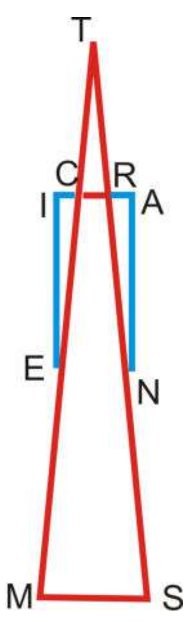


To see an animation of this construction, view http://www.mathsisfun.com/geometry/construct-anglebisect.htmlhttp://www.mathsisfun.com/geometry/construct-anglebisect.html.

Know What? Revisited The image to the right is an outline of the Transamerica Building from earlier in the lesson. From this outline, we can see the following parts are congruent:

$$\overline{TR} \cong \overline{TC} \qquad \angle TCR \cong \angle TRC$$
$$\overline{RS} \cong \overline{CM} \qquad \angle CIE \cong \angle RAN$$
$$\overline{CI} \cong \overline{RA} \quad \text{and} \quad \angle TMS \cong \angle TSM$$
$$\overline{AN} \cong \overline{IE} \qquad \angle IEC \cong \angle ANR$$
$$\overline{TS} \cong \overline{TM} \qquad \angle TCI \cong \angle TRA$$

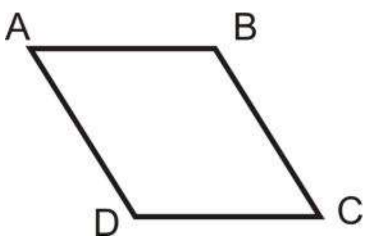
As well at these components, there are certain windows that are congruent and all four triangular sides of the building are congruent to each other.



Review Questions

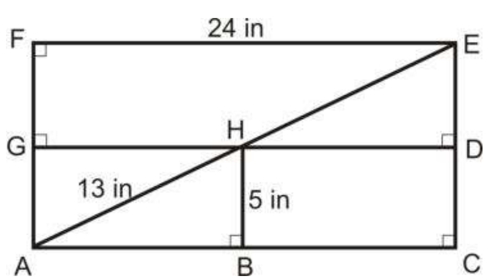
1. Copy the figure below and label it with the following information:

$$\angle A \cong \angle C$$
$$\angle B \cong \angle D$$
$$\overline{AB} \cong \overline{CD}$$
$$\overline{AD} \cong \overline{BC}$$



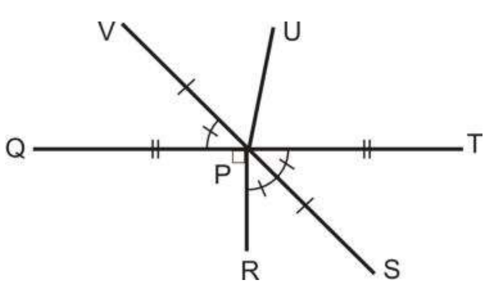
For 2-9, find the lengths, given: H is the midpoint of $\overline{AE}$ and $\overline{DG}$, B is the midpoint of $\overline{AC}$, $\overline{GD}$ is the perpendicular bisector of $\overline{FA}$ and $\overline{EC}$, $\overline{AC} \cong \overline{FE}$, and $\overline{FA} \cong \overline{EC}$.

2. AB
3. GA
4. ED
5. HE
6. $m\angle HDC$
7. FA
8. GD
9. $m\angle FED$



10. How many copies of triangle AHB can fit inside rectangle $FECA$ without overlapping?

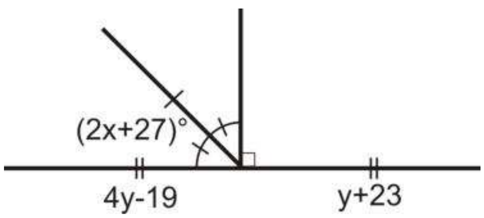
For 11-18, use the following picture to answer the questions.

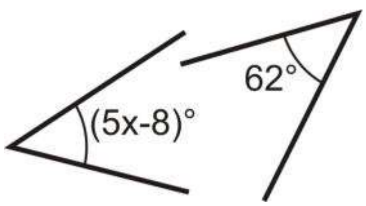


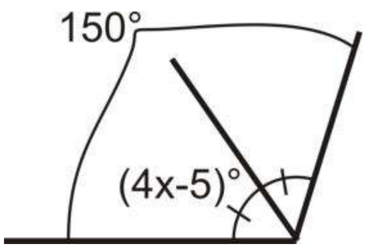
11. What is the angle bisector of $\angle TPR$?

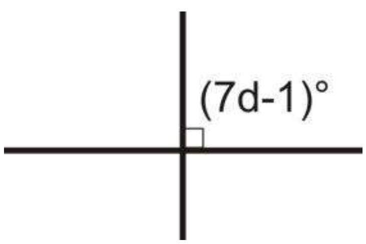
12. P is the midpoint of what two segments?
13. What is $m\angle QPR$?
14. What is $m\angle TPS$?
15. How does $\overline{VS}$ relate to $\overline{QT}$?
16. How does $\overline{QT}$ relate to $\overline{VS}$?
17. Is $\overline{PU}$ a bisector? If so, of what?
18. What is $m\angle QPV$?

Algebra Connection For 19-24, use algebra to determine the value of variable(s) in each problem.

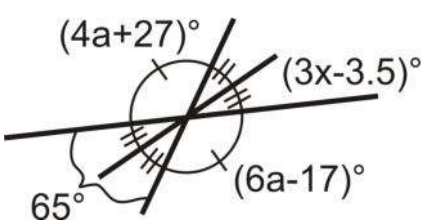
19. 

20. 

21. 

22. 

23. 

24. 

25. ***Construction*** Using your protractor, draw an angle that is 110°. Then, use your compass to construct the angle bisector. What is the measure of each angle?
26. ***Construction*** Using your protractor, draw an angle that is 75°. Then, use your compass to construct

the angle bisector. What is the measure of each angle?

27. ***Construction*** Using your ruler, draw a line segment that is 7 cm long. Then use your compass to construct the perpendicular bisector, What is the measure of each segment?
28. ***Construction*** Using your ruler, draw a line segment that is 4 in long. Then use your compass to construct the perpendicular bisector, What is the measure of each segment?
29. ***Construction*** Draw a straight angle (180°). Then, use your compass to construct the angle bisector. What kind of angle did you just construct?

For questions 30-33, find the midpoint between each pair of points.

30. (-2, -3) and (8, -7)
31. (9, -1) and (-6, -11)
32. (-4, 10) and (14, 0)
33. (0, -5) and (-9, 9)

Given the midpoint (M) and either endpoint of $\overline{AB}$, find the other endpoint.

34. $A(-1, 2)$ and $M(3, 6)$
35. $B(-10, -7)$ and $M(-2, 1)$
36. ***Error Analysis*** Erica is looking at a geometric figure and trying to determine which parts are congruent. She wrote $\overline{AB} = \overline{CD}$. Is this correct? Why or why not?
37. ***Challenge*** Use the Midpoint Formula to solve for the $x-$value of the midpoint and the $y-$value of the midpoint. Then, use this formula to solve #34. Do you get the same answer?
38. ***Construction Challenge*** Use construction tools and the constructions you have learned in this section to construct a 45° angle.
39. ***Construction Challenge*** Use construction tools and the constructions you have learned in this section to construct two 2 in segments that bisect each other. Now connect all four endpoints with segments. What figure have you constructed?
40. Describe an example of how the concept of midpoint (or the midpoint formula) could be used in the real world.

Review Queue Answers

1. See Example 6
2. $$\begin{aligned} 2x - 5 &= 33 \\ 2x &= 38 \\ x &= 19 \end{aligned}$$
3. $m\angle ROT = m\angle ROS + m\angle SOP + m\angle POT$

Chapter 5

Section 1.5- Angles Pairs

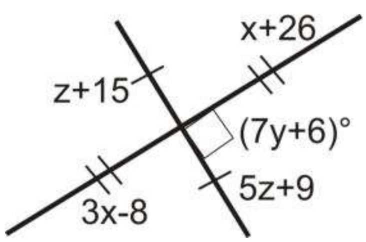
5.1 Angle Pairs

Learning Objectives

- Recognize complementary angles, supplementary angles, linear pairs and vertical angles.
- Apply the Linear Pair Postulate and the Vertical Angles Theorem.

Review Queue

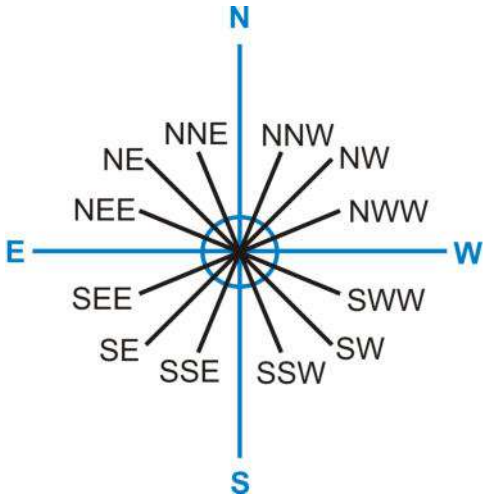
Use the picture below to answer questions 1-3.



1. Find x.
2. Find y.
3. Find z.

Know What? A compass (as seen to the right) is used to determine the direction a person is traveling in. The angles between each direction are very important because they enable someone to be more specific and precise with their direction. In boating, captains use headings to determine which direction they are headed. A heading is the angle at which these compass lines intersect. So, a heading of $45^\circ NW$, would be straight out along that northwest line.

What headings have the same angle measure? What is the angle measure between each compass line?

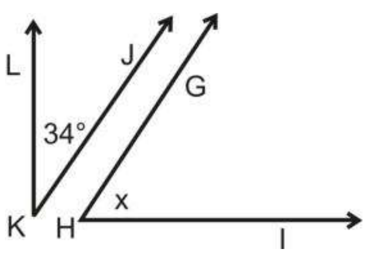


Complementary Angles

Complementary: When two angles add up to 90°.

Complementary angles do not have to be congruent to each other, nor do they have to be next to each other.

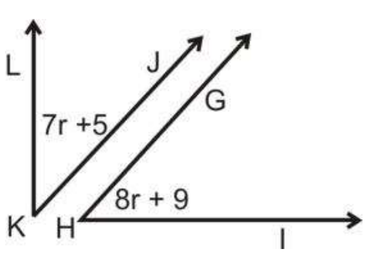
Example 1: The two angles below are complementary. $m\angle GHI = x$. What is x?



Solution: Because the two angles are complementary, they add up to 90°. Make an equation.

$$x + 34^\circ = 90^\circ$$
$$x = 56^\circ$$

Example 2: The two angles below are complementary. Find the measure of each angle.



Solution: Again, the two angles add up to 90°. Make an equation.

$$8r + 9^\circ + 7r + 5^\circ = 90^\circ$$
$$15r + 14^\circ = 90^\circ$$
$$15r = 74^\circ$$
$$r = 4.93^\circ$$

However, this is not what the question asks for. You need to plug r back into each expression to find each angle.

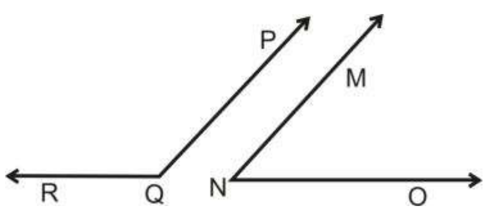
$$m\angle GHI = 8(5^\circ) + 9^\circ = 49^\circ$$
$$m\angle JKL = 7(5^\circ) + 6^\circ = 41^\circ$$

Supplementary Angles

Supplementary: When two angles add up to 180°.

Just like complementary angles, supplementary angles do not have to be congruent or touching.

Example 3: The two angles below are supplementary. If $m\angle MNO = 78^\circ$ what is $m\angle PQR$?



Solution: Just like Examples 1 and 2, set up an equation. However, instead of equaling 90°, now it is 180°.

$$78^\circ + m\angle PQR = 180^\circ$$
$$m\angle PQR = 102^\circ$$

Example 4: What is the measure of two congruent, supplementary angles?

Solution: Supplementary angles add up to 180°. Congruent angles have the same measure. Divide 180° by 2, to find the measure of each angle.

$$180^\circ \div 2 = 90^\circ$$

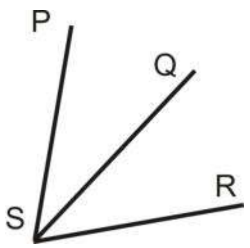
So, two congruent, supplementary angles are right angles, or 90°.

Linear Pairs

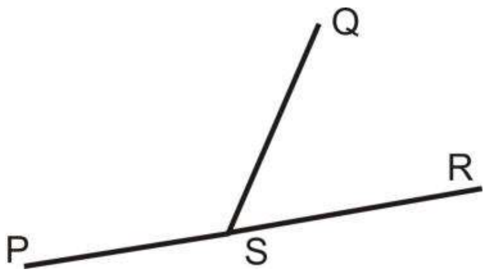
Adjacent Angles: Two angles that have the same vertex, share a side, and do not overlap.

$\angle PSQ$ and $\angle QSR$ are adjacent.

$\angle PQR$ and $\angle PQS$ are NOT adjacent because they overlap.



Linear Pair: Two angles that are adjacent and whose non-common sides form a straight line.

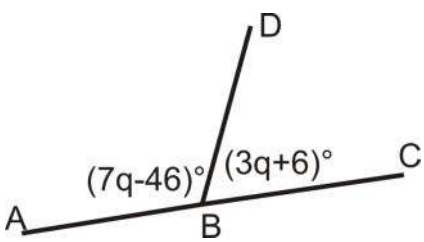


$\angle PSQ$ and $\angle QSR$ are a linear pair.

$$\begin{aligned} m\angle PSR &= 180^\circ \\ m\angle PSQ + m\angle QSR &= m\angle PSR \\ m\angle PSQ + m\angle QSR &= 180^\circ \end{aligned}$$

Linear Pair Postulate: If two angles are a linear pair, then they are supplementary.

Example 5: ***Algebra Connection*** What is the value of each angle?



Solution: These two angles are a linear pair, so they are supplementary, or add up to 180°. Write an equation.

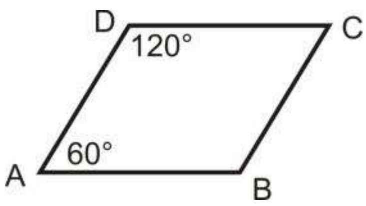
$$\begin{aligned} (7q-46)^\circ + (3q+6)^\circ &= 180^\circ \\ 10q - 40^\circ &= 180^\circ \\ 10q &= 220^\circ \\ q &= 22^\circ \end{aligned}$$

So, plug in q to get the measure of each angle.

$$m\angle ABD = 7(22^\circ) - 46^\circ = 108^\circ \quad m\angle DBC = 180^\circ - 108^\circ = 72^\circ$$

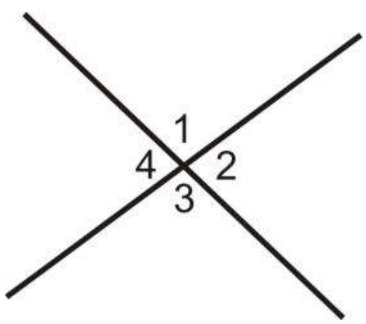
Example 6: Are $\angle CDA$ and $\angle DAB$ a linear pair? Are they supplementary?

Solution: The two angles are not a linear pair because they do not have the same vertex. However, they are supplementary, $120^\circ + 60^\circ = 180^\circ$.



Vertical Angles

Vertical Angles: Two non-adjacent angles formed by intersecting lines.



$\angle 1$ and $\angle 3$ are vertical angles

$\angle 2$ and $\angle 4$ are vertical angles

Notice that these angles are labeled with numbers. You can tell that these are labels because they do not have a degree symbol.

Investigation 1-5: Vertical Angle Relationships

1. Draw two intersecting lines on your paper. Label the four angles created $\angle 1$, $\angle 2$, $\angle 3$, and $\angle 4$. See the picture above.
2. Take your protractor and find $m\angle 1$.
3. What is the angle relationship between $\angle 1$ and $\angle 2$? Find $m\angle 2$.
4. What is the angle relationship between $\angle 1$ and $\angle 4$? Find $m\angle 4$.
5. What is the angle relationship between $\angle 2$ and $\angle 3$? Find $m\angle 3$.
6. Are any angles congruent? If so, write down the congruence statement.

From this investigation, hopefully you found out that $\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$. This is our first theorem. That means it must be proven true in order to use it.

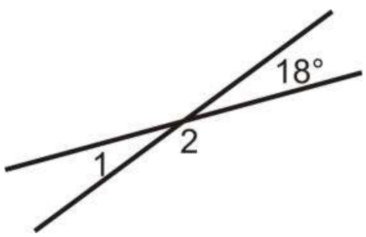
Vertical Angles Theorem: If two angles are vertical angles, then they are congruent.

We can prove the Vertical Angles Theorem using the same process we used above. However, let's not use any specific values for the angles.

From the picture above:	
$\angle 1$ and $\angle 2$ are a linear pair	$m\angle 1 + m\angle 2 = 180^\circ$
$\angle 2$ and $\angle 3$ are a linear pair	$m\angle 2 + m\angle 3 = 180^\circ$
$\angle 3$ and $\angle 4$ are a linear pair	$m\angle 3 + m\angle 4 = 180^\circ$
All of the equations $= 180^\circ$, so set the first and second equation equal to each other and the second and third.	$m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$ *AND* $m\angle 2 + m\angle 3 = m\angle 3 + m\angle 4$
Cancel out the like terms	$m\angle 1 = m\angle 3,\ m\angle 2 = m\angle 4$

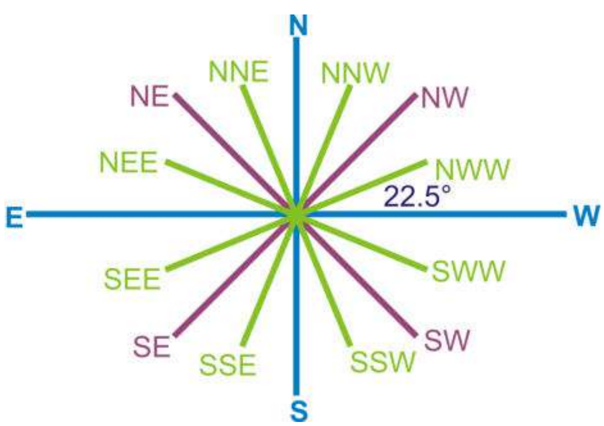
Recall that anytime the measures of two angles are equal, the angles are also congruent.

Example 7: Find $m\angle 1$ and $m\angle 2$.



Solution: $\angle 1$ is vertical angles with 18°, so $m\angle 1 = 18^\circ$. $\angle 2$ is a linear pair with $\angle 1$ or 18°, so $18^\circ + m\angle 2 = 180^\circ$. $m\angle 2 = 180^\circ - 18^\circ = 162^\circ$.

Know What? Revisited The compass has several vertical angles and all of the smaller angles are $22.5^\circ, 180^\circ \div 8$. Directions that are opposite each other, have the same angle measure, but of course, a different direction. All of the green directions have the same angle measure, 22.5°, and the purple have the same angle measure, 45°. N, S, E and W all have different measures, even though they are all 90° apart.



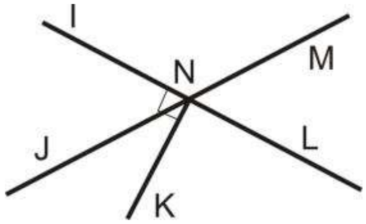
Review Questions

1. Find the measure of an angle that is complementary to $\angle ABC$ if $m\angle ABC$ is
   (a) 45°
   (b) 82°

(c) 19°
(d) $z°$

2. Find the measure of an angle that is supplementary to $\angle ABC$ if $m\angle ABC$ is
   (a) 45°
   (b) 118°
   (c) 32°
   (d) $x°$

Use the diagram below for exercises 3-7. Note that $\overline{NK} \perp \overleftrightarrow{IL}$.



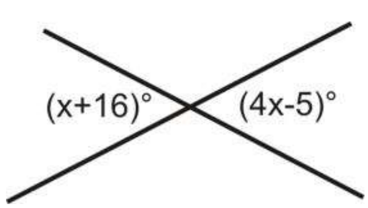
3. Name one pair of vertical angles.
4. Name one linear pair of angles.
5. Name two complementary angles.
6. Name two supplementary angles.
7. Given that $m\angle IJN = 63°$, find:
   (a) $m\angle JNL$
   (b) $m\angle KNL$
   (c) $m\angle MNL$
   (d) $m\angle MNI$

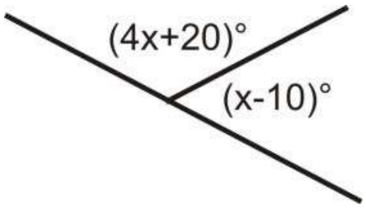
For 8-15, determine if the statement is ALWAYS true, SOMETIMES true or NEVER true.

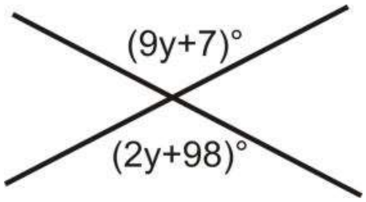
8. Vertical angles are congruent.
9. Linear pairs are congruent.
10. Complementary angles add up to 180°.
11. Supplementary angles add up to 180°
12. Adjacent angles share a vertex.
13. Adjacent angles overlap.
14. Complementary angles are 45°.
15. The complement of $x°$ is $(90 - x)°$.

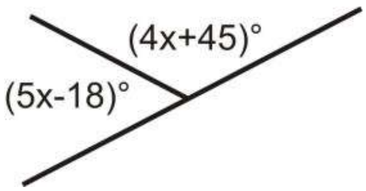
For 16-25, find the value of x or y.

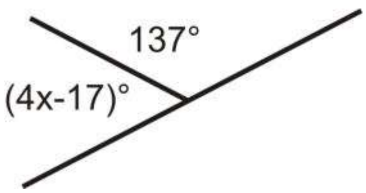
16.

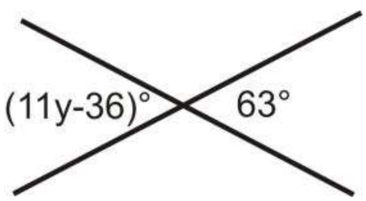


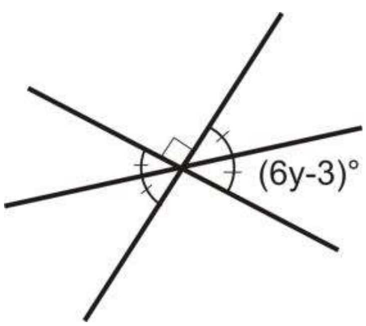
17. 

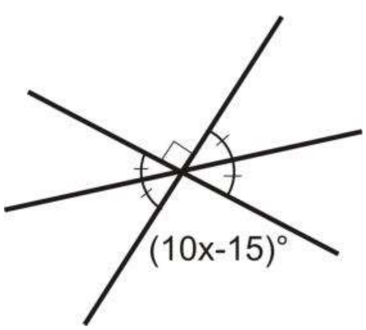
18. 

19. 

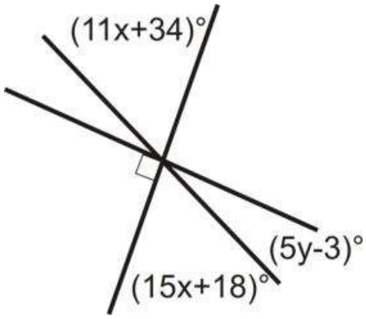
20. 

21. 

22. 

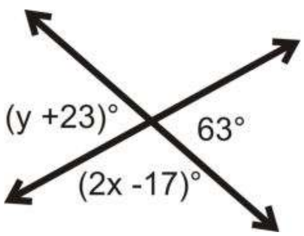
23. 

24. Find *x*.
25. Find *y*.

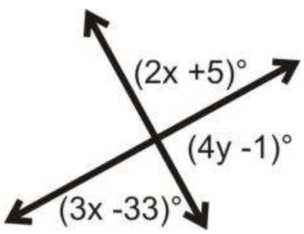


Find x and y in the following diagrams.

26.

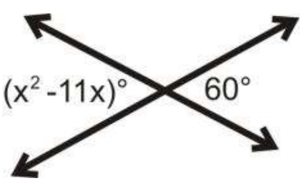


27.



Algebra Connection. Use factoring or the quadratic formula to solve for the variables.

28.



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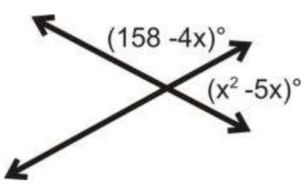
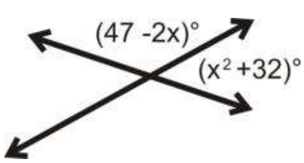
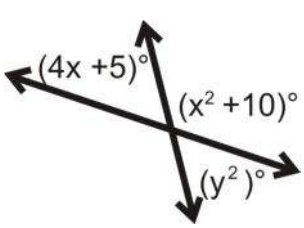


30.



31.



Review Queue Answers

1. $x + 26 = 3x - 8$
   $34 = 2x$
   $17 = x$
2. $(7y + 6)^\circ = 90^\circ$
   $7y = 84^\circ$
   $y = 12^\circ$
3. $z + 15 = 5z + 9$
   $6 = 4z$
   $1.5 = z$

Chapter 6

Section 1.6- Classifying Polgons

6.1 Classifying Polygons

Learning Objectives

- Define triangle and polygon.
- Classify triangles by their sides and angles.
- Understand the difference between convex and concave polygons.
- Classify polygons by number of sides.

Review Queue

1. Draw a triangle.
2. Where have you seen 4, 5, 6 or 8-sided polygons in real life? List 3 examples.
3. Fill in the blank.
   (a) Vertical angles are always ____________________________.
   (b) Linear pairs are ____________________________________.
   (c) The parts of an angle are called __________________________ and a __________________________.

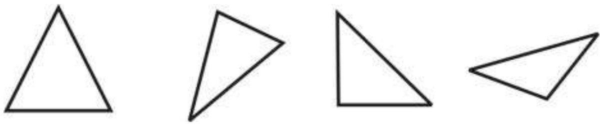
Know What? The pentagon in Washington DC is a pentagon with congruent sides and angles. There is a smaller pentagon inside of the building that houses an outdoor courtyard. Looking at the picture, the building is divided up into 10 smaller sections. What are the shapes of these sections? Are any of these division lines diagonals? How do you know?

Triangles

The first enclosed shape to examine is the triangle.

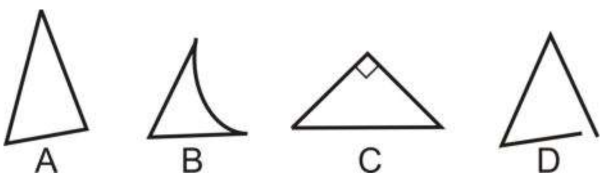
Triangle: Any closed figure made by three line segments intersecting at their endpoints.

Every triangle has three **vertices** (the points where the segments meet), three **sides** (the segments), and three **interior angles** (formed at each vertex). All of the following shapes are triangles.



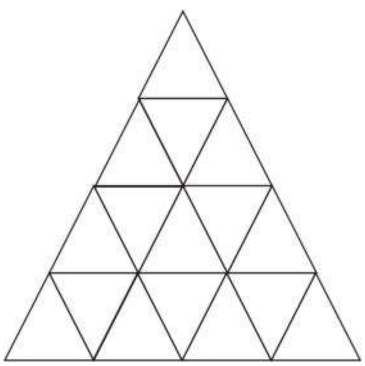
You might have also learned that the sum of the interior angles in a triangle is 180°. Later we will prove this, but for now you can use this fact to find missing angles.

Example 1: Which of the figures below are not triangles?

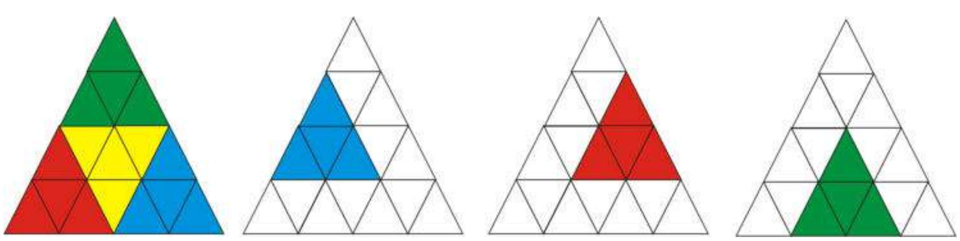


Solution: B is not a triangle because it hasone curved side. D is not a closed shape, so it is not a triangle either.

Example 2: How many triangles are in the diagram below?



Solution: Start by counting the smallest triangles, 16. Now count the triangles that are formed by four of the smaller triangles.



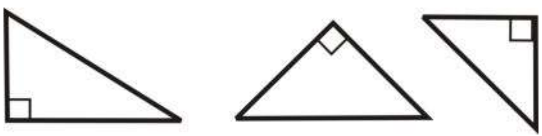
There are a total of seven triangles of this size, including the inverted one in the center of the diagram.Next, count the triangles that are formed by nine of the smaller triangles. There are three of this size. And finally, there is one triangle formed by the 16 smaller triangles. Adding these numbers together, we get $16 + 7 + 3 + 1 = 27$.

Classifying by Angles

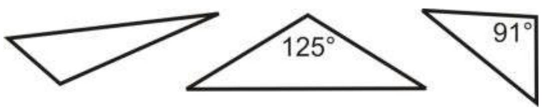
Angles can be classified by their angles;acute, obtuse or right. In any triangle, two of the angles will always be acute. The third angle can be acute, obtuse, or right.

We classify each triangle by this angle.

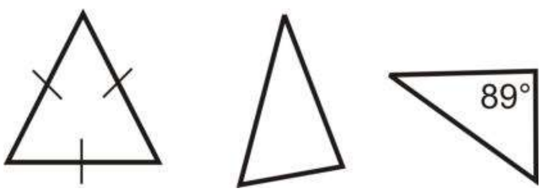
Right Triangle: When a triangle has one right angle.



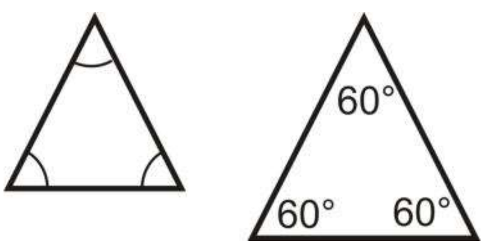
Obtuse Triangle: When a triangle has one obtuse angle.



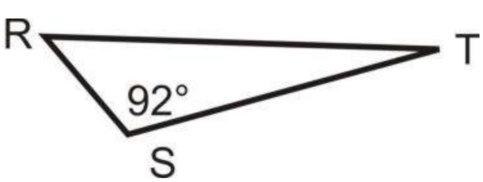
Acute Triangle: When all three angles in the triangle are acute.



Equiangular Triangle: When all the angles in a triangle are congruent.



Example 3: Which term best describes $\triangle RST$ below?

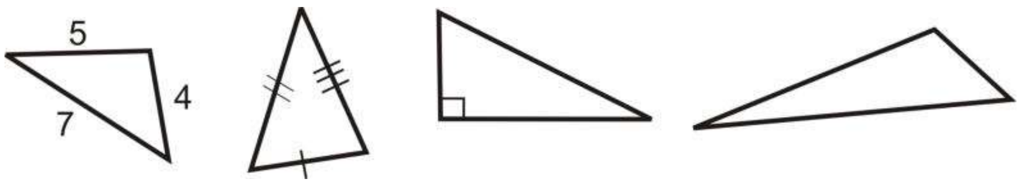


Solution: This triangle has one labeled obtuse angle of 92°. Triangles can only have one obtuse angle, so it is an obtuse triangle.

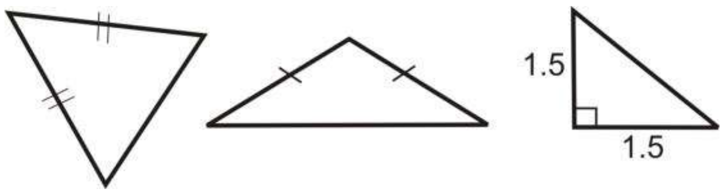
Classifying by Sides

These classifications have to do with the sides of the triangle and their relationships to each other.

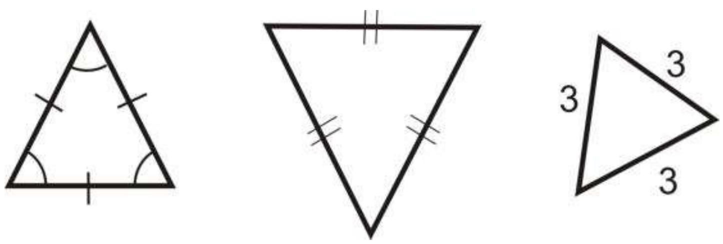
Scalene Triangle: When a triangles sides are all different lengths.



Isosceles Triangle: When *at least* two sides of a triangle are congruent.

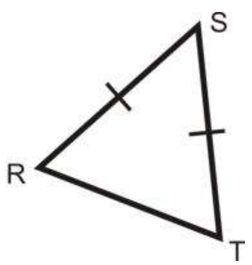


Equilateral Triangle: When all the sides of a triangle are congruent.



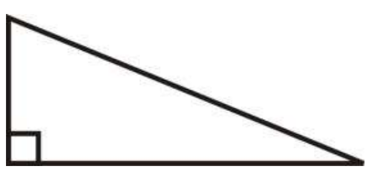
Note that by the definitions, an equilateral triangle is also an isosceles triangle.

Example 4: Classify the triangle by its sides and angles.



Solution: We are told there are two congruent sides, so it is an isosceles triangle. By its angles, they all look acute, so it is an acute triangle. Typically, we say this is an acute isosceles triangle.

Example 5: Classify the triangle by its sides and angles.



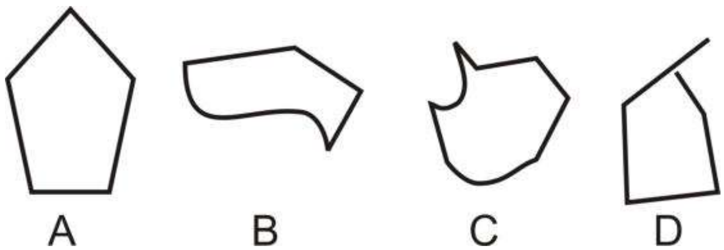
Solution: This triangle has a right angle and no sides are marked congruent. So, it is a right scalene triangle.

Polygons

Polygon: Any closed planar figure that is made entirely of line segments that intersect at their endpoints.

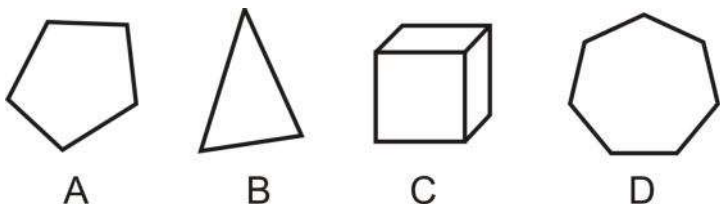
Polygons can have any number of sides and angles, but the sides can never be curved. The segments are called the **sides** of the polygons, and the points where the segments intersect are called **vertices**. The easiest way to identify a polygon is to look for a closed figure with no curved sides.

Example 6: Which of the figures below is a polygon?



Solution: The easiest way to identify the polygon is to identify which shapes are not polygons. *B* and *C* each have at least one curved side, so they cannot be polygons. *D* has all straight sides, but one of the vertices is not at the endpoint of the adjacent side, so it is not a polygon either. *A* is the only polygon.

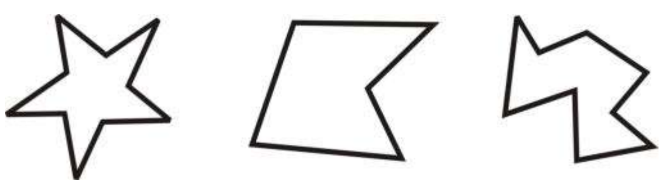
Example 7: Which of the figures below is <u>*not*</u> a polygon?



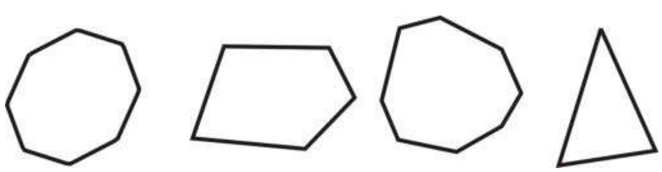
Solution: *C* is a three-dimensional shape, so it does not lie within one plane, so it is not a polygon.

Convex and Concave Polygons

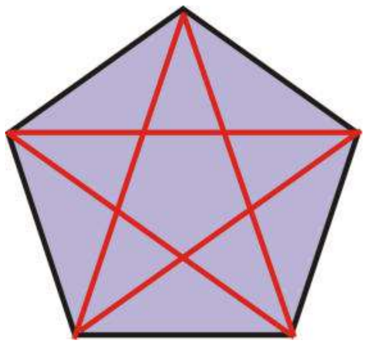
Polygons can be either **convex** or **concave**. Think of the term concave as referring to a cave, or ‘‘caving in”. A concave polygon has a section that ‘‘points inward” toward the middle of the shape. All stars are concave polygons.



A convex polygon does not share this property.



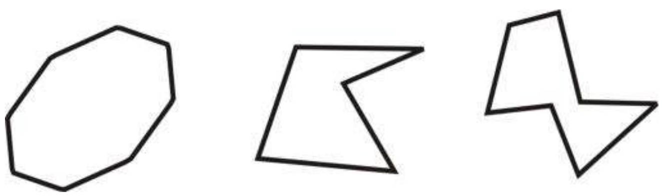
Diagonals: Line segments that connects the vertices of a convex polygon that are not sides.



The red lines are all diagonals.

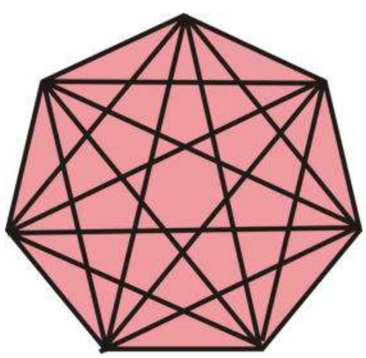
This pentagon has 5 diagonals.

Example 8: Determine if the shapes below are convex or concave.



Solution: To see if a polygon is concave, look at the polygons and see if any angle "caves in" to the interior of the polygon. The first polygon does not do this, so it is convex. The other two do, so they are concave. You could add here that concave polygons have at least one diagonal outside the figure.

Example 9: How many diagonals does a 7-sided polygon have?



Solution: Draw a 7-sided polygon, also called a heptagon. Drawing in all the diagonals and counting them, we see there are 14.

Classifying Polygons

Whether a polygon is convex or concave, it can always be named by the number of sides. See the chart below.

Table 6.1:

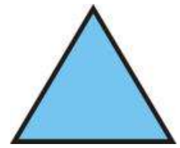
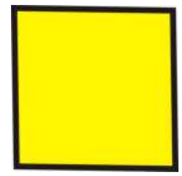
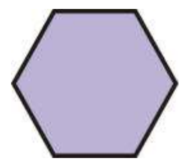
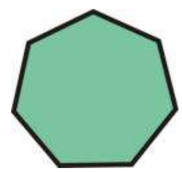
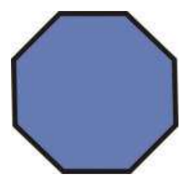
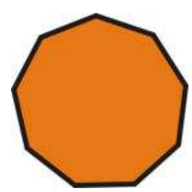
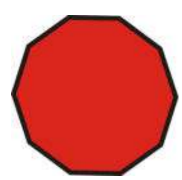
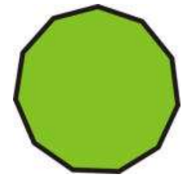
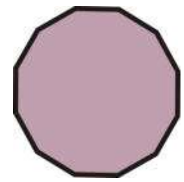
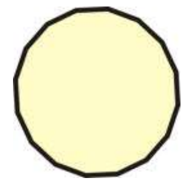
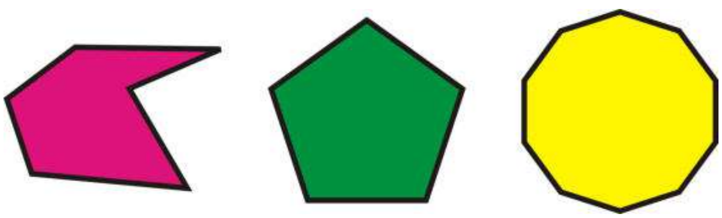
Polygon Name	**Number of Sides**	**Number of Diagonals**	**Convex Example**
Triangle	**3**	**0**	
Quadrilateral	**4**	**2**	
Pentagon	**5**	**5**	
Hexagon	**6**	**9**	
Heptagon	**7**	**14**	
Octagon	**8**	**?**	
Nonagon	**9**	**?**	
Decagon	**10**	**?**	

Table 6.1: (continued)

Polygon Name	Number of Sides	Number of Diagonals	Convex Example
Undecagon or hendecagon	**11**	**?**	
Dodecagon	**12**	**?**	
n**–gon**	n (**where** $n > 12$)	**?**	

Example 10: Name the three polygons below by their number of sides and if it is convex or concave.



Solution:

A. This shape has six sides and concave, so it is a concave hexagon.

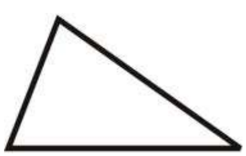
B. This shape has five sides and is convex, so it is a convex pentagon.

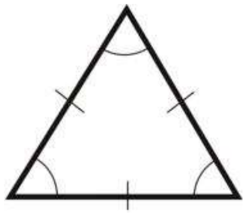
C. This shape has ten sides and is convex, so it is a convex decagon.

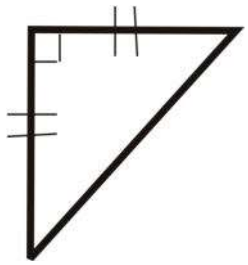
Know What? Revisited The pentagon is divided up into 10 sections, all quadrilaterals. More specifically, there are 5 rectangles and 5 kites. None of these dividing lines are diagonals because they are not drawn from vertices.

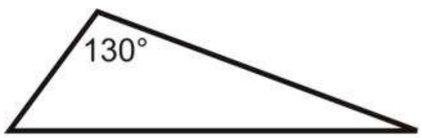
Review Questions

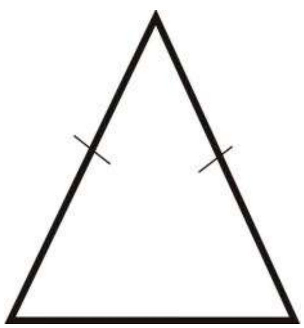
For questions 1-6, classify each triangle by its sides and by its angles.

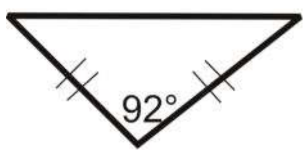
1. 

2. 

3. 

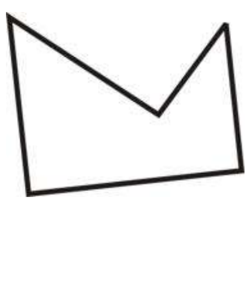
4. 

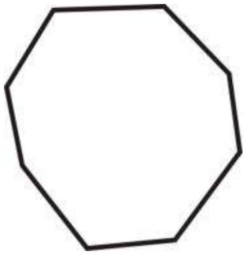
5. 

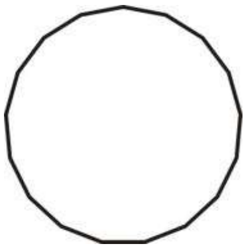
6. 

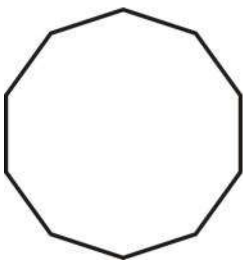
7. Can you draw a triangle with a right angle and an obtuse angle? Why or why not?
8. In an isosceles triangle, can the angles opposite the congruent sides be obtuse?
9. ***Construction*** Construct an equilateral triangle with sides of 3 cm. Start by drawing a horizontal segment of 3 cm and measure this side with your compass from both endpoints.
10. What must be true about the angles of your equilateral triangle from #9?

In problems 11-16, name each polygon in as much detail as possible.

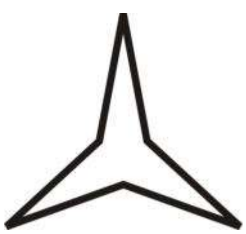
11. 

12.


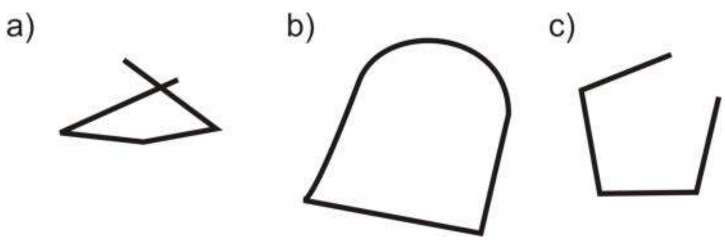
13.


14.


15.


16.


17. Explain why the following figures are NOT polygons:



18. How many diagonals can you draw from **one vertex** of a pentagon? Draw a sketch of your answer.
19. How many diagonals can you draw from **one vertex** of an octagon? Draw a sketch of your answer.
20. How many diagonals can you draw from **one vertex** of a dodecagon?
21. Use your answers from 17-19 to figure out how many diagonals you can draw from **one vertex** of an $n-$gon?

22. Determine the number of total diagonals for an octagon, nonagon, decagon, undecagon, and dodecagon. Do you see a pattern? BONUS: Find the equation of the total number of equations for an $n-$gon.

For 23-30, determine if the statement is ALWAYS true, SOMETIMES true, or NEVER true.

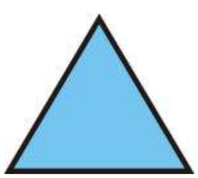
23. Obtuse triangles are isosceles.
24. A polygon must be enclosed.
25. A star is a concave polygon.
26. A right triangle is acute.
27. An equilateral triangle is equiangular.
28. A quadrilateral is a square.
29. You can draw $(n - 1)$ triangles from one vertex of a polygon.
30. A decagon is a 5-point star.

In geometry it is important to know the difference between a sketch, a drawing and a construction. A sketch is usually drawn free-hand and marked with the appropriate congruence markings or labeled with measurement. It may or may not be drawn to scale. A drawing is made using a ruler, protractor or compass and should be made to scale. A construction is made using only a compass and ruler and should be made to scale.

For 31-36, draw, sketch or construct the indicated figures.

31. Sketch a convex heptagon with two sides congruent and three angles congruent.
32. Sketch a non-polygon figure.
33. Draw a concave pentagon with exactly two right angles and at least two congruent sides.
34. Draw an equilateral quadrilateral that is NOT a square.
35. Construct a right triangle with side lengths 3 cm, 4 cm and 5 cm.
36. ***Construction Challenge*** Construct a 60° angle. (*Hint: Think about an equilateral triangle.*)

Review Queue Answers

1. 
2. Examples include: stop sign (8), table top (4), the Pentagon (5), snow crystals (6), bee hive combs (6), soccer ball pieces (5 and 6)
3. (a) congruent or equal
   (b) supplementary
   (c) sides, vertex

Chapter 7

Chapter 1 Review

7.1 Chapter 1 Review

Symbol Toolbox

$\overleftrightarrow{AB}$, $\overrightarrow{AB}$, $\overline{AB}$ Line, ray, line segment

$\angle ABC$ Angle with vertex B

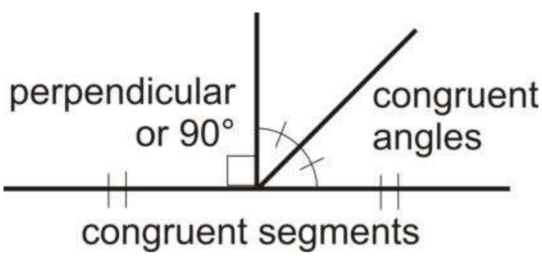
$m\overline{AB}$ or AB Distance between A and B

$m\angle ABC$ Measure of $\angle ABC$

$\perp$ Perpendicular

$=$ Equal

$\cong$ Congruent



Keywords

- Geometry
- Point
- Line
- Plane
- Space
- Collinear
- Coplanar
- Endpoint:
- Line Segment
- Ray

- Intersection
- Postulates
- Theorem
- Distance
- Measure
- Ruler Postulate
- Segment Addition Postulate
- Angle
- Vertex
- Sides
- Protractor Postulate
- Straight Angle
- Right Angle
- Acute Angles
- Obtuse Angles
- Perpendicular
- Construction
- Compass
- Angle Addition Postulate
- Congruent
- Midpoint
- Midpoint Postulate
- Segment Bisector
- Perpendicular Bisector
- Perpendicular Bisector Postulate
- Angle Bisector
- Angle Bisector Postulate
- Complementary
- Supplementary
- Adjacent Angles
- Linear Pair
- Linear Pair Postulate
- Vertical Angles
- Vertical Angles Theorem
- Triangle
- Right Triangle
- Obtuse Triangle
- Acute Triangle
- Equiangular Triangle
- Scalene Triangle
- Isosceles Triangle
- Equilateral Triangle
- Polygon
- Diagonals

Review

Match the definition or description with the correct word.

1. When three points lie on the same line. — A. Measure
2. All vertical angles are ____________. — B. Congruent
3. Linear pairs add up to ___________. — C. Angle Bisector
4. The m in from of $m\angle ABC$. — D. Ray
5. What you use to measure an angle. — E. Collinear
6. When two sides of a triangle are congruent. — F. Perpendicular
7. $\perp$ — G. Line
8. A line that passes through the midpoint of another line. — H. Protractor
9. An angle that is greater than 90°. — I. Segment Addition Postulate
10. The intersection of two planes is a ________________. — J. Obtuse
11. $AB + BC = AC$ — K. Point
12. An exact location in space. — L. 180°
13. A sunbeam, for example. — M. Isosceles
14. Every angle has exactly one. — N. Pentagon
15. A closed figure with 5 sides. — O. Hexagon

— P. Bisector

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9686http://www.ck12.org/flexr/chapter/9686.