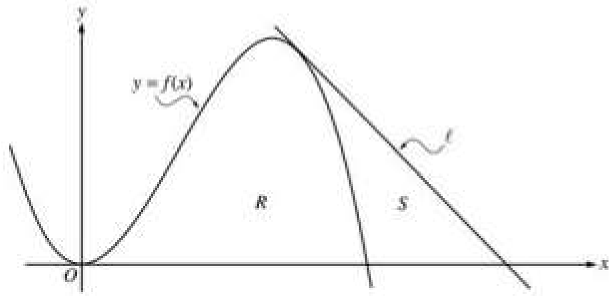
AP Calc BC Spring Break Exam FRQ Solutions

Question 1



Let f be the function given by $f(x) = 4x^2 - x^3$, and let ℓ be the line $y = 18 - 3x$, where ℓ is tangent to the graph of f. Let R be the region bounded by the graph of f and the x-axis, and let S be the region bounded by the graph of f, the line ℓ, and the x-axis, as shown above.

(a) Show that ℓ is tangent to the graph of $y = f(x)$ at the point $x = 3$.

(b) Find the area of S.

(c) Find the volume of the solid generated when R is revolved about the x-axis.

(a) $f'(x) = 8x - 3x^2$; $f'(3) = 24 - 27 = -3$

$f(3) = 36 - 27 = 9$

Tangent line at $x = 3$ is

$y = -3(x-3) + 9 = -3x + 18$,

which is the equation of line ℓ.

2 :
- 1 : finds $f'(3)$ and $f(3)$
- 1 : finds equation of tangent line or shows $(3,9)$ is on both the graph of f and line ℓ

(b) $f(x) = 0$ at $x = 4$

The line intersects the x-axis at $x = 6$.

$$\text{Area} = \frac{1}{2}(3)(9) - \int_3^4 \left(4x^2 - x^3\right) dx = 7.916 \text{ or } 7.917$$

OR

$$\text{Area} = \int_3^4 \left((18 - 3x) - \left(4x^2 - x^3\right)\right) dx + \frac{1}{2}(2)(18 - 12) = 7.916 \text{ or } 7.917$$

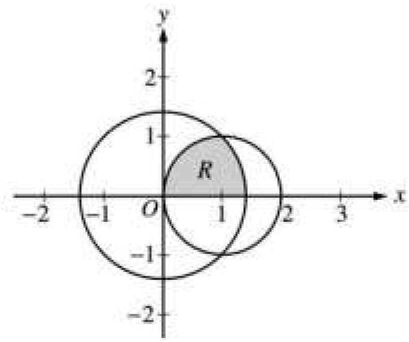
4 :
- 2 : integral for non-triangular region
  - 1 : limits
  - 1 : integrand
- 1 : area of triangular region
- 1 : answer

(c) $$\text{Volume} = \pi \int_0^4 \left(4x^2 - x^3\right)^2 dx = 156.038\pi \text{ or } 490.208$$

3 :
- 1 : limits and constant
- 1 : integrand
- 1 : answer

Question 2

The figure above shows the graphs of the circles $x^2 + y^2 = 2$ and $(x-1)^2 + y^2 = 1$. The graphs intersect at the points $(1, 1)$ and $(1, -1)$. Let R be the shaded region in the first quadrant bounded by the two circles and the x-axis.



(a) Set up an expression involving one or more integrals with respect to x that represents the area of R.

(b) Set up an expression involving one or more integrals with respect to y that represents the area of R.

(c) The polar equations of the circles are $r = \sqrt{2}$ and $r = 2\cos\theta$, respectively. Set up an expression involving one or more integrals with respect to the polar angle θ that represents the area of R.

(a) $\text{Area} = \int_0^1 \sqrt{1-(x-1)^2}\,dx + \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx$

OR

$\text{Area} = \frac{1}{4}(\pi \cdot 1^2) + \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx$

3 :
- 1 : integrand for larger circle
- 1 : integrand or geometric area for smaller circle
- 1 : limits on integral(s)

Note: < −1 > if no addition of terms

(b) $\text{Area} = \int_0^1 \left(\sqrt{2-y^2} - \left(1-\sqrt{1-y^2}\right)\right)dy$

3 :
- 1 : limits
- 2 : integrand
  - < −1 > reversal
  - < −1 > algebra error in solving for x
  - < −1 > add rather than subtract
  - < −2 > other errors

(c) $\text{Area} = \int_0^{\pi/4} \frac{1}{2}(\sqrt{2})^2\,d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\,d\theta$

OR

$\text{Area} = \frac{1}{8}\pi(\sqrt{2})^2 + \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\,d\theta$

3 :
- 1 : integrand or geometric area for larger circle
- 1 : integrand for smaller circle
- 1 : limits on integral(s)

Note: < −1 > if no addition of terms

Question 3

Distance x (mm)	0	60	120	180	240	300	360
Diameter $B(x)$ (mm)	24	30	28	30	26	24	26

A blood vessel is 360 millimeters (mm) long with circular cross sections of varying diameter. The table above gives the measurements of the diameter of the blood vessel at selected points along the length of the blood vessel, where x represents the distance from one end of the blood vessel and $B(x)$ is a twice-differentiable function that represents the diameter at that point.

(a) Write an integral expression in terms of $B(x)$ that represents the average radius, in mm, of the blood vessel between $x = 0$ and $x = 360$.

(b) Approximate the value of your answer from part (a) using the data from the table and a midpoint Riemann sum with three subintervals of equal length. Show the computations that lead to your answer.

(c) Using correct units, explain the meaning of $\pi\int_{125}^{275}\left(\frac{B(x)}{2}\right)^2 dx$ in terms of the blood vessel.

(d) Explain why there must be at least one value x, for $0 < x < 360$, such that $B''(x) = 0$.

(a) $\frac{1}{360}\int_0^{360}\frac{B(x)}{2}\,dx$

$2: \begin{cases} 1: \text{limits and constant} \\ 1: \text{integrand} \end{cases}$

(b) $\frac{1}{360}\left[120\left(\frac{B(60)}{2}+\frac{B(180)}{2}+\frac{B(300)}{2}\right)\right] =$

$\frac{1}{360}[60(30+30+24)] = 14$

$2: \begin{cases} 1: B(60)+B(180)+B(300) \\ 1: \text{answer} \end{cases}$

(c) $\frac{B(x)}{2}$ is the radius, so $\pi\left(\frac{B(x)}{2}\right)^2$ is the area of the cross section at x. The expression is the volume in mm^3 of the blood vessel between 125 mm and 275 mm from the end of the vessel.

$2: \begin{cases} 1: \text{volume in mm}^3 \\ 1: \text{between } x = 125 \text{ and } x = 275 \end{cases}$

(d) By the MVT, $B'(c_1) = 0$ for some c_1 in $(60, 180)$ and $B'(c_2) = 0$ for some c_2 in $(240, 360)$. The MVT applied to $B'(x)$ shows that $B''(x) = 0$ for some x in the interval (c_1, c_2).

$3: \begin{cases} 2: \text{explains why there are two values of } x \text{ where } B'(x) \text{ has the same value} \\ 1: \text{explains why that means } B''(x) = 0 \text{ for } 0 < x < 360 \end{cases}$

Note: max 1/3 if only explains why $B'(x) = 0$ at some x in $(0, 360)$.

Question 4

A particle moves in the xy-plane so that the position of the particle at any time t is given by

$$x(t) = 2e^{3t} + e^{-7t} \quad \text{and} \quad y(t) = 3e^{3t} - e^{-2t}.$$

(a) Find the velocity vector for the particle in terms of t, and find the speed of the particle at time $t = 0$.

(b) Find $\frac{dy}{dx}$ in terms of t, and find $\lim_{t \to \infty} \frac{dy}{dx}$.

(c) Find each value t at which the line tangent to the path of the particle is horizontal, or explain why none exists.

(d) Find each value t at which the line tangent to the path of the particle is vertical, or explain why none exists.

(a) $x'(t) = 6e^{3t} - 7e^{-7t}$

$y'(t) = 9e^{3t} + 2e^{-2t}$

Velocity vector is $< 6e^{3t} - 7e^{-7t}, 9e^{3t} + 2e^{-2t} >$

$$\text{Speed} = \sqrt{x'(0)^2 + y'(0)^2} = \sqrt{(-1)^2 + 11^2} = \sqrt{122}$$

$$3 : \begin{cases} 1 : x'(t) \\ 1 : y'(t) \\ 1 : \text{speed} \end{cases}$$

(b) $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{9e^{3t} + 2e^{-2t}}{6e^{3t} - 7e^{-7t}}$$

$$\lim_{t \to \infty} \frac{dy}{dx} = \lim_{t \to \infty} \frac{9e^{3t} + 2e^{-2t}}{6e^{3t} - 7e^{-7t}} = \frac{9}{6} = \frac{3}{2}$$

$$2 : \begin{cases} 1 : \frac{dy}{dx} \text{ in terms of } t \\ 1 : \text{limit} \end{cases}$$

(c) Need $y'(t) = 0$, but $9e^{3t} + 2e^{-2t} > 0$ for all t, so none exists.

$$2 : \begin{cases} 1 : \text{considers } y'(t) = 0 \\ 1 : \text{explains why none exists} \end{cases}$$

(d) Need $x'(t) = 0$ and $y'(t) \neq 0$.

$$6e^{3t} = 7e^{-7t}$$
$$e^{10t} = \frac{7}{6}$$
$$t = \frac{1}{10}\ln\left(\frac{7}{6}\right)$$

$$2 : \begin{cases} 1 : \text{considers } x'(t) = 0 \\ 1 : \text{solution} \end{cases}$$

Question 5

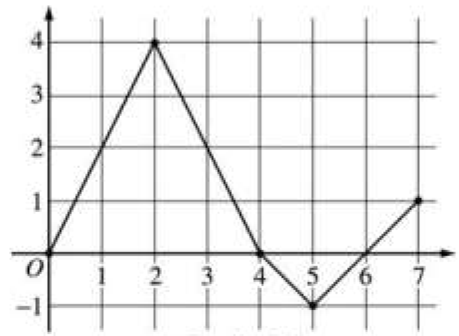
Let f be a function defined on the closed interval $[0, 7]$. The graph of f, consisting of four line segments, is shown above. Let g be the function given by $g(x) = \int_2^x f(t)\,dt$.

(a) Find $g(3)$, $g'(3)$, and $g''(3)$.

(b) Find the average rate of change of g on the interval $0 \le x \le 3$.

(c) For how many values c, where $0 < c < 3$, is $g'(c)$ equal to the average rate found in part (b)? Explain your reasoning.

(d) Find the x-coordinate of each point of inflection of the graph of g on the interval $0 < x < 7$. Justify your answer.



Graph of f

(a) $g(3) = \int_2^3 f(t)\,dt = \frac{1}{2}(4+2) = 3$

$g'(3) = f(3) = 2$

$g''(3) = f'(3) = \frac{0-4}{4-2} = -2$

$3 : \begin{cases} 1 : g(3) \\ 1 : g'(3) \\ 1 : g''(3) \end{cases}$

(b) $\frac{g(3)-g(0)}{3} = \frac{1}{3}\int_0^3 f(t)\,dt$

$= \frac{1}{3}\left(\frac{1}{2}(2)(4) + \frac{1}{2}(4+2)\right) = \frac{7}{3}$

$2 : \begin{cases} 1 : g(3) - g(0) = \int_0^3 f(t)\,dt \\ 1 : \text{answer} \end{cases}$

(c) There are two values of c.

We need $\frac{7}{3} = g'(c) = f(c)$

The graph of f intersects the line $y = \frac{7}{3}$ at two places between 0 and 3.

$2 : \begin{cases} 1 : \text{answer of 2} \\ 1 : \text{reason} \end{cases}$

Note: 1/2 if answer is 1 by MVT

(d) $x = 2$ and $x = 5$

because $g' = f$ changes from increasing to decreasing at $x = 2$, and from decreasing to increasing at $x = 5$.

$2 : \begin{cases} 1 : x = 2 \text{ and } x = 5 \text{ only} \\ 1 : \text{justification} \\ \quad (\text{ignore discussion at } x = 4) \end{cases}$

Question 6

The function f has a Taylor series about $x = 2$ that converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n+1)!}{3^n}$ for $n \geq 1$, and $f(2) = 1$.

(a) Write the first four terms and the general term of the Taylor series for f about $x = 2$.

(b) Find the radius of convergence for the Taylor series for f about $x = 2$. Show the work that leads to your answer.

(c) Let g be a function satisfying $g(2) = 3$ and $g'(x) = f(x)$ for all x. Write the first four terms and the general term of the Taylor series for g about $x = 2$.

(d) Does the Taylor series for g as defined in part (c) converge at $x = -2$? Give a reason for your answer.

(a) $f(2) = 1;\ f'(2) = \dfrac{2!}{3};\ f''(2) = \dfrac{3!}{3^2};\ f'''(2) = \dfrac{4!}{3^3}$

$$f(x) = 1 + \frac{2}{3}(x-2) + \frac{3!}{2!3^2}(x-2)^2 + \frac{4!}{3!3^3}(x-2)^3 + \cdots + \frac{(n+1)!}{n!3^n}(x-2)^n + \cdots$$

$$= 1 + \frac{2}{3}(x-2) + \frac{3}{3^2}(x-2)^2 + \frac{4}{3^3}(x-2)^3 + \cdots + \frac{n+1}{3^n}(x-2)^n + \cdots$$

3 :
- 1 : coefficients $\dfrac{f^{(n)}(2)}{n!}$ in first four terms
- 1 : powers of $(x-2)$ in first four terms
- 1 : general term

(b) $$\lim_{n\to\infty}\left|\frac{\frac{n+2}{3^{n+1}}(x-2)^{n+1}}{\frac{n+1}{3^n}(x-2)^n}\right| = \lim_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{1}{3}|x-2|$$

$= \dfrac{1}{3}|x-2| < 1$ when $|x-2| < 3$

The radius of convergence is 3.

3 :
- 1 : sets up ratio
- 1 : limit
- 1 : applies ratio test to conclude radius of convergence is 3

(c) $g(2) = 3;\ g'(2) = f(2);\ g''(2) = f'(2);\ g'''(2) = f''(2)$

$$g(x) = 3 + (x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{3^2}(x-2)^3 + \cdots + \frac{1}{3^n}(x-2)^{n+1} + \cdots$$

2 :
- 1 : first four terms
- 1 : general term

(d) No, the Taylor series does not converge at $x = -2$ because the geometric series only converges on the interval $|x-2| < 3$.

1 : answer with reason