Section 9.7

Infinite Series:

“Maclaurin and Taylor Polynomials”

All graphics are attributed to:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

Introduction

- In a local linear approximation, the tangent line to the graph of a function is used to obtain a linear approximation of the function near the point of tangency.
- In this section, we will consider how one might improve on the accuracy of local linear approximations by using higher-order polynomials as approximating functions.
- We will also investigate the error associated with such approximations.

Local Linear Approximations

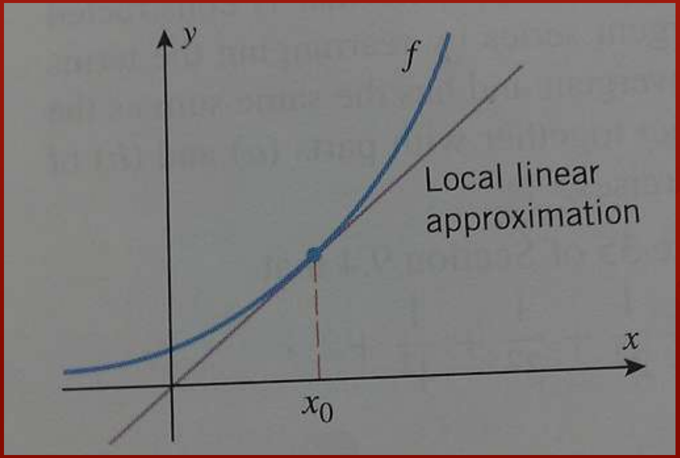
- Remember from Section 3.5 that the local linear approximation of a function f at $_0$ is

$$( ) \approx (_0) + '(_0)(- _0)$$

or more simply $- _1 \approx (- _0)$ and move $_1$.

- This is a polynomial with degree 1 since ^{1}.

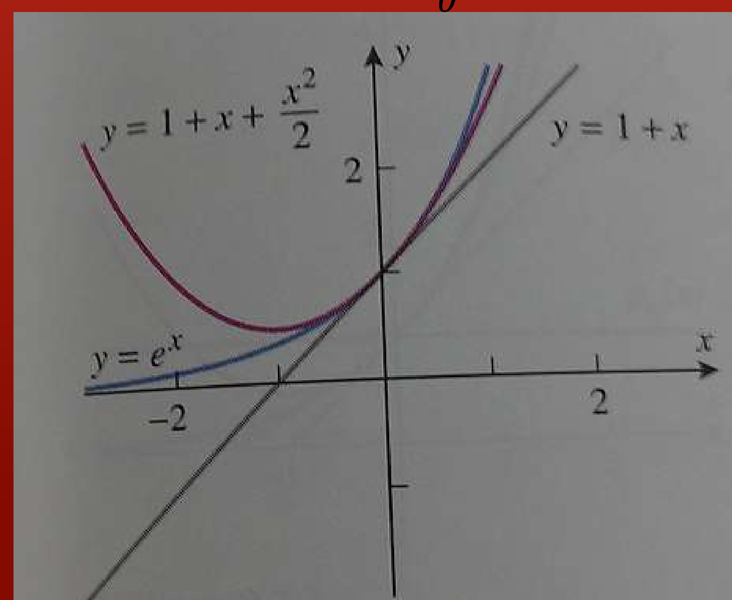
- If the graph of a function f has a pronounced "bend" at $_0$, then we can expect that the accuracy of the local linear approximation of f at $_0$ will decrease rapidly as we progress away from $_0$.



Local Quadratic Approximations

- One way to deal with this problem is to approximate the function f at $_0$ by a polynomial p of degree 2.

- We want to find a polynomial so that the value of the function p at $_0$ (point) and the values of its first two derivatives (slope and concavity) at $_0$ match those of the original function f at $_0$ to make it a good "match" for making approximations since it will remain close to the graph of f over a larger interval around $_0$ than the linear approximation.



Substitution for Local Quadratic Approximation

- A general formula for a local quadratic approximation f at x = 0 comes from $y=ax^2+bx+c$:

  $(\ ) \approx \ _0 + \ _1 + \ _2 \ ^2$ p = $\ _0 + \ _1 + \ _2 \ ^2$

- Remembering the requirements from the previous slide will help perform the substitutions necessary to find this approximation.
  - value of the function p at $_0$ (point) must match the original function f at $_0$: p(0) = f(0)
  - values of its first two derivatives (slope and concavity) at $_0$ must match those of the original function f at $_0$ to make it a good fit: p′(0) = f′(0) and p′′(0) = f′′(0)
- Substitution
  - p(0) = $\ _0 + \ _1 0 + \ _2 0^2 = \ _0$ means p(0) = f(0) = $\ _0$
  - p′(0) = $\ _1 + 2 \ _2 0 = \ _1$ means p′(0) = f′(0) = $\ _1$
  - p′′′′(0) = 2 $\ _2$ means p′′(0) = f′′(0) = 2 $\ _2$

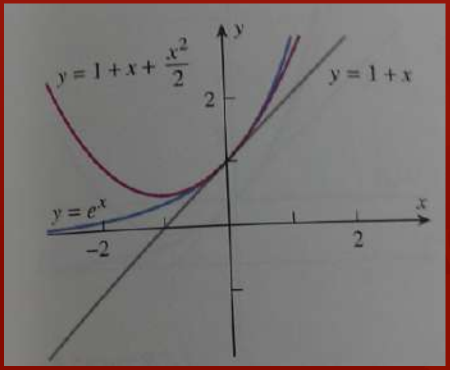
    and gives $\ = \frac{f''(0)}{}$
- Therefore, $(\ ) \approx$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

Example

- Find the local linear and quadratic approximations of at x = 0 and graph y= along with the two approximations.
- Solution
- f′(x) = and f″(x) = so f(0)=f′(0)=f″(0)= ^{0}=1
- Linear approximation: y = mx + b = 1x + 1 = x + 1 ≈
- Quadratic approximation: use y = $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$

  y = $1 + 1 \quad + \frac{\quad}{2}$ ≈
- As expected, the quadratic approximation is more accurate than the local linear approximation (see graph).



Maclaurin Polynomials

- Since the quadratic approximation was better than the local linear approximation, might a cubic or quartic (degree 4) approximation be better yet?
- To find out, we must extend our work on quadratics to a more general idea for higher degree polynomial approximations.

9.7.2 DEFINITION If f can be differentiated n times at 0, then we define the ***nth Maclaurin polynomial for f*** to be

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \qquad (7)$$

- See substitution work similar to that we did for quadratics on page 650 for higher degree polynomials.

Colin Maclaurin (1698-1746)

- Maclaurin polynomials are named after the Scottish mathematician Colin Maclaurin who received his Master's degree and started teaching college math at the age of 17.
- He worked to defend Isaac Newton's methods and ideas and create some of his own.
- He also contributed to astronomy, actuarial sciences, mapping, etc.
- See more info on page 649
- NOTE: The Maclaurin polynomials are the special cases of the Taylor polynomials (see later slides) in which $_0 = 0$.

Example

- Find the Maclaurin polynomials $_{0'}$ $_{1'}$ $_{2'}$ $_{3'}$ for .
- <u>Solution</u>
- All derivatives of are

  so f(0)=f′(0)=f′′(0)=f′′′(0)=…= $^{(\ )}(0) = {}^{0}$=1

- $_0$ = f(0) = 1
- We already found $_1$ & $_2$ earlier (linear and quadratic approx.)
  - $_1$ = x + 1 and $_2$ = $1 + 1 \quad + \frac{2}{2}$
- <u>Cubic approximation</u>: use $_3$ = $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

  $1 + 1 \quad {}^{2} \quad {}^{3}$

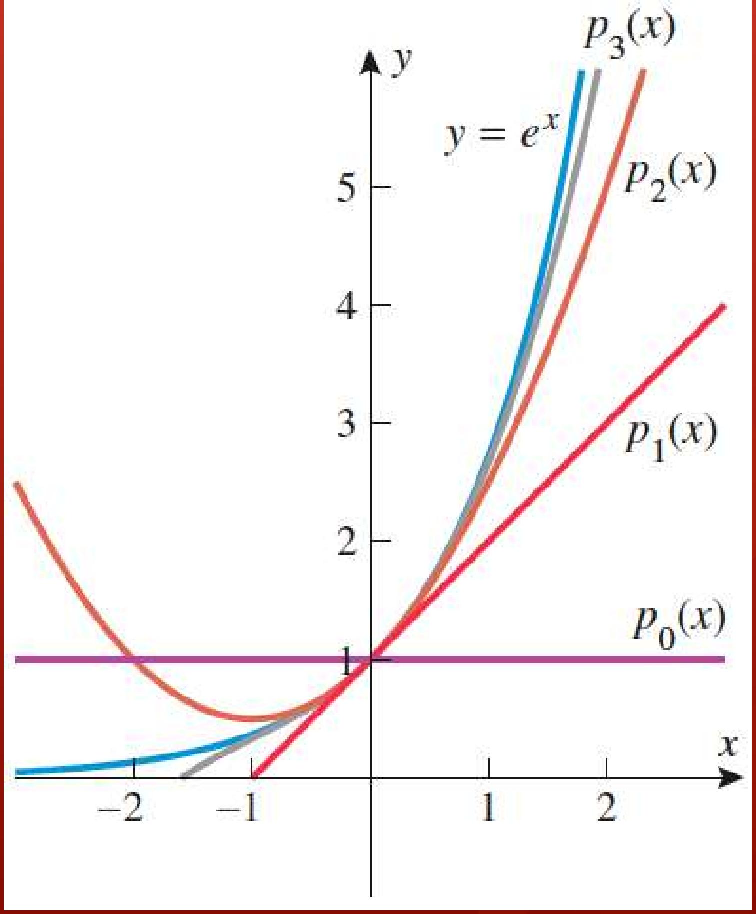
  $$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

- <u>General</u>: use

  $$= 1 + 1 \quad + \frac{2}{2} + \frac{3}{6} + \ldots + \frac{}{!}$$

Analysis of Example Results

- The graphs of ${}_1(\ ),\ {}_2(\ ),\ {}_3(\ )$ are all very good "matches" for near x=0 so they are good approximations near 0.
- The farther x is from 0, the less accurate these approximations become.
- Usually, the higher the degree the Maclaurin polynomial, the larger the interval on which is provides a specified accuracy.



Example

- Find the nth Maclaurin polynomials for sin x.
- Solution:
- Start by finding several derivatives of sin x.
  - $f(x) = \sin x$ $\quad f(0) = \sin 0 = 0$
  - $f'(x) = \cos x$ $\quad f'(0) = \cos 0 = 1$
  - $f''(x) = -\sin x$ $\quad f''(0) = -\sin 0 = 0$
  - $f'''(x) = -\cos x$ $\quad f'''(0) = -\cos 0 = -1$
  - $f''''(x) = \sin x$ $\quad f''''(0) = \sin 0 = 0$
  - and the pattern (0,1,0,-1) continues to repeat for further derivatives at 0.

Example continued

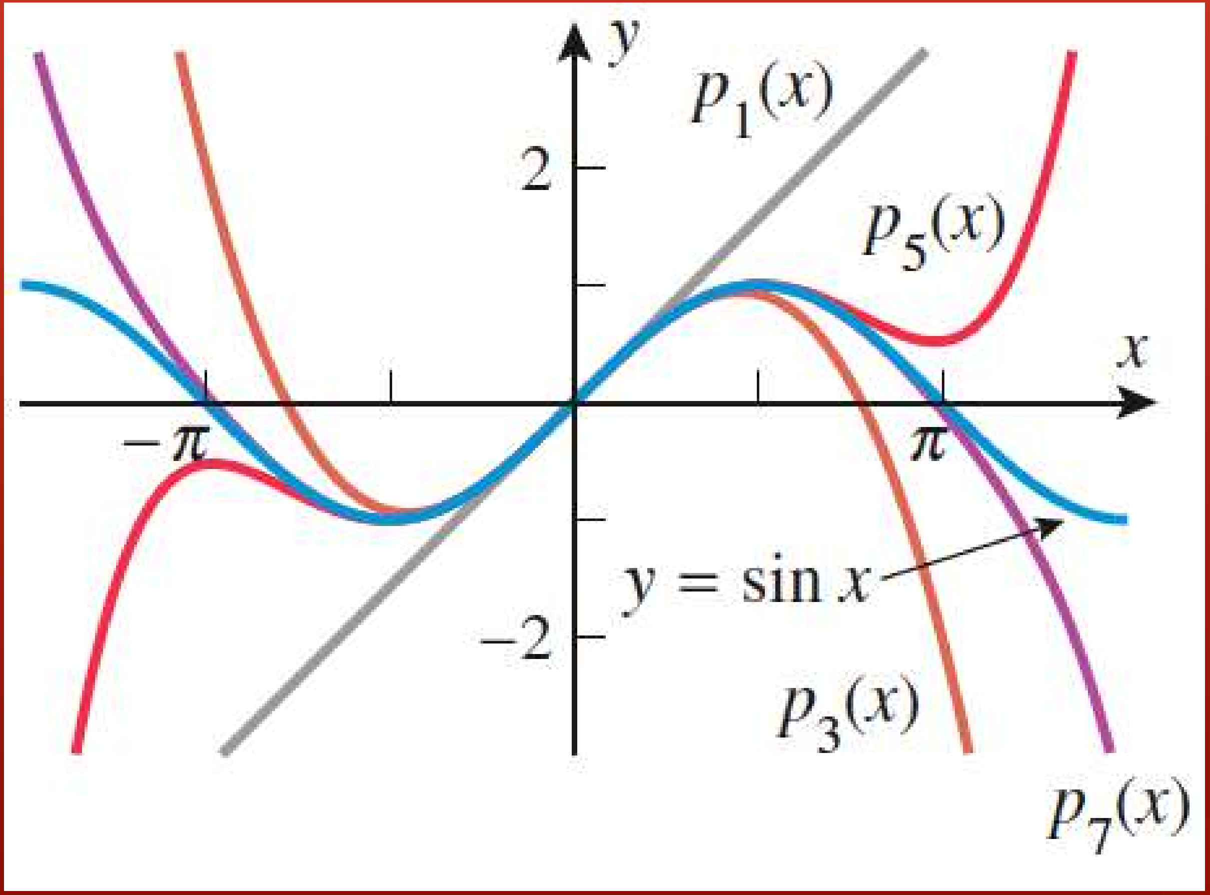
- Use $p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$
- The successive Maclaurin polynomials for sin x are

$$p_0(x) = 0$$
$$p_1(x) = 0 + x$$
$$p_2(x) = 0 + x + 0$$
$$p_3(x) = 0 + x + 0 - \frac{x^3}{3!}$$
$$p_4(x) = 0 + x + 0 - \frac{x^3}{3!} + 0$$

$$p_5(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!}$$
$$p_6(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0$$
$$p_7(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!}$$

- Because every even result is zero, each even-order Maclaurin polynomial after $_0$(x) is the same as the preceding odd-order Maclaurin polynomial and we can write a general nth polynomial accordingly.
- $_{2\ +1}(\ ) = {}_{2\ +2}(\ ) = \ - \frac{3}{3!} + \frac{5}{5!} - \frac{7}{7!} + \dots + (-1)\ * \frac{2\ +1}{(2\ +1)!}$
(k=0,1,2,...)

Graph of Example Results



- If you are interested, see the nth Maclaurin polynomials for cos x on page 652.

Taylor Polynomials

- Until now, we have focused on approximating a function f in the vicinity of x = 0.
- Now we will consider the more general case of approximating f in the vicinity of an arbitrary value of $_0$.
- The basic idea is the same as before; we want to find an nth-degree polynomial p such that its value and the values of its first n derivatives match those of f at $_0$.
- The substitution computations are much like those on slide #6 and they result in:

9.7.3 DEFINITION If f can be differentiated n times at x_0, then we define the *nth Taylor polynomial for f about* $x = x_0$ to be

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \qquad (9)$$

Brook Taylor (1685-1731)

- Taylor polynomials are named after the English mathematician Brook Taylor who claims to have worked/conversed with Isaac Newton on planetary motion and Halley's comet regarding roots of polynomials.
- Supposedly, his writing style was hard to understand and did not receive credit for many of his innovations on a wide range of subjects – magnetism, capillary action, thermometers, perspective, and calculus.
- See more information on page 653.
- Remember, Maclaurin series came later and they are a more specific case of Taylor series.

Example

- Find the first four Taylor polynomials for ln x about x = 2.
- <u>Solution</u>:
- Let $f(x) = \ln x$ ⟶ $f(2) = \ln 2$
- Find the first three derivatives.
  - $f'(x) = \frac{1}{\ }$ $f'(2) = \frac{1}{2}$
  - $f''(x) = -\frac{1}{2}$ $f''(2) = -\frac{1}{4}$
  - $f'''(x) = \frac{2}{3}$ $f'''(2) = \frac{1}{4}$

Example continued

- Use
$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$
combined with the results from the previous slide and $_0 = 2$ to get

$$p_0(x) = f(2) = \ln 2$$

$$p_1(x) = f(2) + f'(2)(x-2) = \ln 2 + \tfrac{1}{2}(x-2)$$

$$p_2(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 = \ln 2 + \tfrac{1}{2}(x-2) - \tfrac{1}{8}(x-2$$

$$p_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$

$$= \ln 2 + \tfrac{1}{2}(x-2) - \tfrac{1}{8}(x-2)^2 + \tfrac{1}{24}(x-2)^3$$

Sigma Notation for Taylor and Maclaurin Polynomials

- We may need to express in sigma notation.

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

- To do this, we use the no derivative of f at x = $_0$.
- Hence, $^{(0)}(\ _0)$ "no derivative" = original function at $_0$ = f($_0$).
- This gives the Taylor polynomial $\sum_{=0} \frac{(\ )(\ _0)}{!}(\ -\ _0)$

$$= (\ _0) + f'(\ _0)(x - \ _0) + \frac{''(\ _0)}{2!}(\ -\ _0)^2 + \ldots + \frac{(\ )(\ _0)}{!}(\ -\ _0)$$

- In particular, we can get the Maclaurin polynomial for f(x) as

$$\sum_{=0} \frac{(\ )(0)}{!}(\ -\ _0) = (0) + f'(0)x + \frac{''(0)}{2!}\ ^2 + \ldots + \frac{(\ )(0)}{!}$$

Example

- Find the nth Maclaurin polynomial for $\frac{1}{1-}$ and express it in sigma notation.
- <u>Solution:</u>
- Let f(x) = $\frac{1}{1-}$ f(0) = 1 = 0!
- Find the first k derivatives at x = 0.
  - f′(x) = $\frac{1}{(1-\ )^2}$ f′(0) = $1 = 1!$
  - f″(x) = $\frac{2}{(1-\ )^3}$ f″(0) = 2 = 2!
  - f‴(x) = $\frac{3*2}{(1-\ )^4}$ f‴(0) = $3!$
  - f⁗(x) = $\frac{4*3*2}{(1-\ )^5}$ f⁗(0) = 4!

    and so on
  - $^{(\ )}$(x) = $\frac{!}{(1-\ )^{\ +1}}$ $^{(\ )}$(0) = k!
- <u>Substitute</u> into $\sum_{=0} \frac{^{(\ )}(0)}{!}(\ -\ _0) = (0) + f'(0)\ ) + \frac{''(0)}{2!}\ ^2 + \ldots + \frac{^{(\ )}(0)}{!}$

  from the previous slide.

Sigma Notation for a Taylor Polynomial

- The computations and substitutions are similar to those in the previous example except you use the more general form

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

.

- See example 6 on page 655

The nTH Remainder

- It will be convenient to have a notation for the error in the approximation ( ) ≈ ( ).
- Therefore, we will let ( ) (the nth remainder) denote the difference between f(x) and its nth Taylor polynomial.
- ( ) = f(x) - ( ) = ( ) - $\sum_{=0} \frac{(\ )(\ _0)}{!}(\ -\ _0)$

original function – Taylor polynomial

- This can be rewritten as

$$f(x) = p_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_n(x)$$

which is called Taylor's formula with remainder.

Accuracy of the Approximation ( ) ≈ ( )

- Finding a bound for ( ) gives an indication of the accuracy of the approximation ( ) ≈ ( ).

9.7.4 THEOREM (*The Remainder Estimation Theorem*) *If the function f can be differentiated $n+1$ times on an interval containing the number x_0, and if M is an upper bound for $|f^{(n+1)}(x)|$ on the interval, that is, $|f^{(n+1)}(x)| \leq M$ for all x in the interval, then*

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1} \qquad (14)$$

for all x in the interval.

- If you are interested, there is a proof on pages A41-42.
- This bound | ( )| is called the Lagrange error bound.

Example given accuracy

- Use an nth Maclaurin polynomial for to approximate e to five decimal place accuracy.
- Solution:
- All derivatives of = .
- On slide #10, we found the nth Maclaurin polynomial for .

$$\sum_{=0} \frac{}{!} = 1 + 1 \quad + \frac{2}{2} + \frac{3}{6} + \ldots + \frac{}{!}$$

- This gives $= \; ^{1} \approx \sum_{=0} \frac{1}{!} = 1 + 1 + \frac{1^2}{2} + \frac{1^3}{6} + \ldots + \frac{1}{!}$
- Five decimal place accuracy means $\pm .000005$ or less of an error:

$$| \quad (\;)| \leq .000005$$

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}$$

Example continued

- $|R_n(x)| \le \frac{M}{(n+1)!}|x-x_0|^{n+1}$ gives $\left| \; (\;) \right| \le \frac{\;}{(\; +1)!} * \left|1-0\right|^{\; +1} = \frac{\;}{(\; +1)!}$
- M is an upper bound of the value of $^{(\; +1)}(\;) =$ for x in the interval [0,1].
- is an increasing function, so its maximum value on the interval [0,1] occurs at x = 1: $\le$ on this interval which makes M = e for this problem.

$$\left| \; (\;) \right| \le \frac{\;}{(\; +1)!}$$

- Since e is what we are trying to approximate, it is not very helpful to have e in the problem.
- e<3 which is less accurate but easier to deal with.

$\left| \; (\;) \right| \le \frac{3}{(\; +1)!}$ ⇒ $\frac{3}{(\; +1)!} \le .000005$ ⇒ (n+1)! ≥ 600,000

- 9!=362,880 which is the smallest value of n that gives the required accuracy since 10!=3,628,800
- $\sum_{\; =0} \frac{\;}{\;!} = 1 + 1 \; + \frac{\;^2}{2} + \frac{\;^3}{6} + \ldots + \frac{\;}{\;!}$ gives $1 + 1 + \frac{1^2}{2} + \frac{1^3}{6} + \ldots + \frac{1^9}{9!} \approx$

2.71828

Another Accuracy Example

- Use the Remainder Estimation Theorem to find an interval containing x=0 throughout which f(x)=cos x can be approximated by p(x) = 1 – $\left(\frac{{}^{2}}{2!}\right)$ to three decimal-place accuracy.
- Solution:
  - f must be differentiable n+1 times on an interval containing the number x=0 according to the theorem and cos x is differentiable everywhere.
  - Similar to f(x)=sin x on slides #12-13, p(x) is both the second and third Maclaurin polynomial for cos x.
  - When this happens you want to choose the degree of n of the polynomial to be as large as possible, so we will take n=3.
  - Therefore, we need $\left| {}_3(\ ) \right| \le .0005$

Example continued

- This gives us $|\ _3(\ )| \le \frac{}{(3+1)!} * |\ - 0|^{3+1} =$ $\frac{|\ |^4}{24}$ where M is an upper bound for

$$|\ ^{(4)}(\ )| = |\cos\ |.$$

- Since $|\cos\ | \le 1$ for every real number x, we can take M=1 as that upper bound.

$$|\ _3(\ )| \le \frac{|\ |^4}{24} \rightarrow \frac{|\ |^4}{24} \le .0005$$

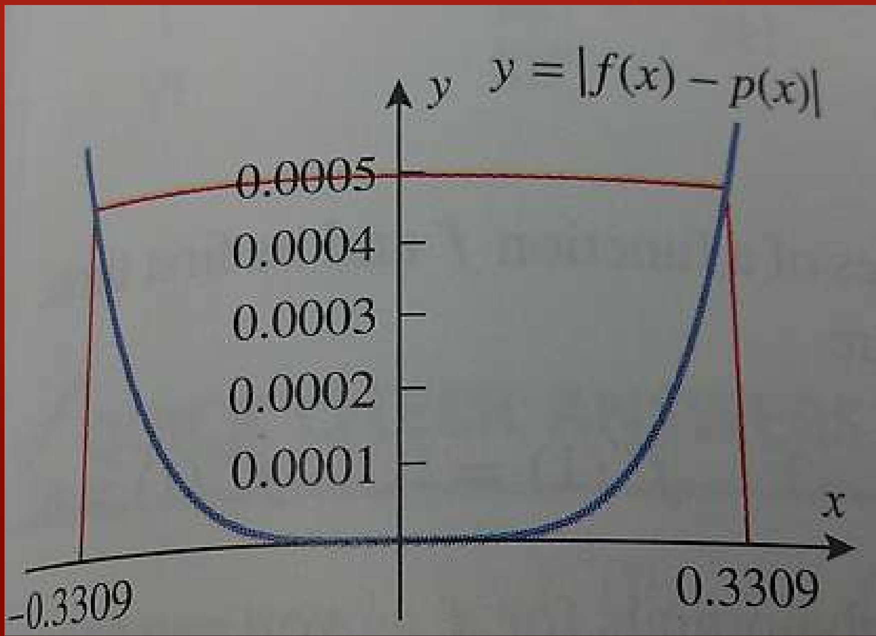
$$|\ | \le .3309$$

- This tells us that one interval is

[-.3309,.3309] which we can check by

graphing $|\ (\ ) - \ (\ )|$

original function – Taylor polynomial



Getting Ready to Race