8.5

Counting Principles

What You Should Learn

- Solve simple counting problems
- Use the Fundamental Counting Principle to solve more complicated counting problems
- Use permutations to solve counting problems
- Use combinations to solve counting problems

Simple Counting Problems

Example 1 – *Selecting Pairs of Numbers at Random*

Eight pieces of paper are numbered from 1 to 8 and placed in a box. One piece of paper is drawn from the box, its number is written down, and the piece of paper is *returned to the box*. Then, a second piece of paper is drawn from the box, and its number is written down.

Finally, the two numbers are added together. In how many different ways can a sum of 12 be obtained?

Example 1 – *Solution*

To solve this problem, count the number of different ways that a sum of 12 can be obtained using two numbers from 1 to 8.

First number	4	5	6	7	8
Second number	8	7	6	5	4

From this list, you can see that a sum of 12 can occur in five different ways.

The Fundamental Counting Principle

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Fundamental Counting Principle

Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways that the two events can occur is $m_1 \cdot m_2$.

Example 3 – *Using the Fundamental Counting Principle*

How many different pairs of letters from the English alphabet are possible?

Solution:

There are two events in this situation. The first event is the choice of the first letter, and the second event is the choice of the second letter.

Because the English alphabet contains 26 letters, it follows that the number of two-letter pairs is

$$26 \cdot 26 = 676.$$

Permutations

Permutations

One important application of the Fundamental Counting Principle is in determining the number of ways that n elements can be arranged (in order).

An ordering of elements is called a **permutation** of the elements.

Definition of Permutation

A **permutation** of n different elements is an ordering of the elements such that one element is first, one is second, one is third, and so on.

Example 5 – *Finding the Number of Permutations of n Elements*

How many permutations of the following letters are possible?

A B C D E F

Solution:

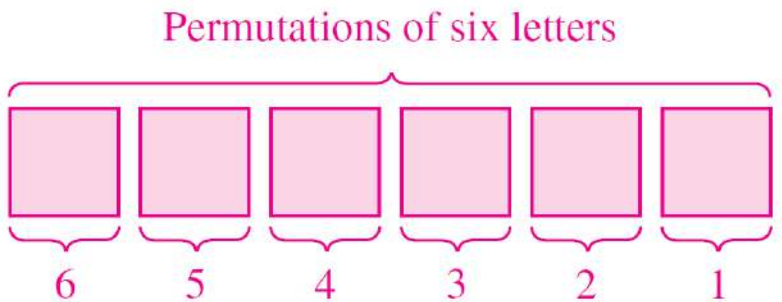
Consider the following reasoning.

First position:	Any of the *six* letters
Second position:	Any of the remaining *five* letters
Third position:	Any of the remaining *four* letters
Fourth position:	Any of the remaining *three* letters
Fifth position:	Either of the remaining *two* letters
Sixth position:	The *one* remaining letter

Example 5 – *Solution*

cont'd

So, the numbers of choices for the six positions are as follows.



The total number of permutations of the six letters is

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$$

Permutations

Number of Permutations of n Elements

The number of permutations of n elements is given by

$$n \cdot (n-1) \cdots 4 \cdot 3 \cdot 2 \cdot 1 = n!.$$

In other words, there are $n!$ different ways that n elements can be ordered.

Permutations of n Elements Taken r at a Time

The number of **permutations of n elements taken r at a time** is given by

$$_{n}P_{r} = \frac{n!}{(n-r)!}$$

$$= n(n-1)(n-2) \cdots (n-r+1).$$

Permutations

Using this formula, find the number of permutations of eight horses taken three at a time is

$$_8P_3 = \frac{8!}{(8-3)!}$$

$$= \frac{8!}{5!}$$

$$= \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}}$$

$$= 336$$

Example 7 – *Distinguishable Permutations*

In how many distinguishable ways can the letters in BANANA be written?

Solution:

This word has six letters, of which three are A's, two are N's, and one is a B. So, the number of distinguishable ways in which the letters can be written is

$$\frac{6!}{3! \cdot 2! \cdot 1!} = \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!} \cdot 2!}$$

$$= 60.$$

When you have repeats, you must divide by the # of repeats! (factorial) for each reapeated item.

Example 7 – *Solution*

cont'd

The 60 different distinguishable permutations are as follows.

AAABNN	AAANBN	AAANNB	AABANN
AABNAN	AABNNA	AANABN	AANANB
AANBAN	AANBNA	AANNAB	AANNBA
ABAANN	ABANAN	ABANNA	ABNAAN
ABNANA	ABNNAA	ANAABN	ANAANB
ANABAN	ANABNA	ANANAB	ANANBA
ANBAAN	ANBANA	ANBNAA	ANNAAB

Example 7 – *Solution*

cont'd

ANNABA	ANNBAA	BAAANN	BAANAN
BAANNA	BANAAN	BANANA	BANNAA
BNAAAN	BNAANA	BNANAA	BNNAAA
NAAABN	NAAANB	NAABAN	NAABNA
NAANAB	NAANBA	NABAAN	NABANA
NABNAA	NANAAB	NANABA	NANBAA
NBAAAN	NBAANA	NBANAA	NBNAAA
NNAAAB	NNAABA	NNABAA	NNBAAA

Combinations

Combinations

When you count the number of possible permutations of a set of elements, order is important.

As a final topic in this section, you will look at a method for selecting subsets of a larger set in which order *is not* important.

Such subsets are called **combinations of *n* elements taken *r* at a time**. For instance, the combinations

{A, B, C} and {B, A, C}

are equivalent because both sets contain the same three elements, and the order in which the elements are listed is not important.

Combinations

So, you would count only one of the two sets. A common example of a combination is a card game in which the player is free to reorder the cards after they have been dealt.

Combinations of n Elements Taken r at a Time

The number of **combinations of n elements taken r at a time** is given by

$$_nC_r = \frac{n!}{(n-r)!r!}.$$

Example 8 – *Combinations of n Elements Taken r at a Time*

a. In how many different ways can three letters be chosen from the letters A, B, C, D, and E? (The order of the three letters is not important.)

b. A standard poker hand consists of five cards dealt from a deck of 52. How many different poker hands are possible? (After the cards are dealt, the player may reorder them, so order is not important.)

Example 8 – *Solution*

a. You can find the number of different ways in which the letters can be chosen by using the formula for the number of combinations of five elements taken three at a time, as follows.

$$ {}_5C_3 = \frac{5!}{2!3!} = \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3!}}{\cancel{2} \cdot 1 \cdot \cancel{3!}} = 10 $$

b. You can find the number of different poker hands by using the formula for the number of combinations of 52 elements taken five at a time, as follows.

$$ {}_{52}C_5 = \frac{52!}{47!5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{47!}} = 2{,}598{,}960 $$