8.1 Sequences and Series

What You Should Learn

- Use sequence notation to write the terms of sequences.
- Use factorial notation.
- Use summation notation to write sums.
- Find sums of infinite series.
- Use sequences and series to model and solve real-life problems.

Sequences

Sequences

Please read this slide but do not copy it or the next two slides down.

In mathematics, the word *sequence* is used in much the same way as in ordinary English.

Saying that a collection is listed *in sequence* means that it is ordered so that it has a first member, a second member, a third member, and so on.

Mathematically, you can think of a sequence as a *function* whose domain is the set of positive integers.

Sequences

Instead of using function notation, sequences are usually written using subscript notation, as shown in the following definition.

Definition of Sequence

An **infinite sequence** is a function whose domain is the set of positive integers. The function values

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

are the **terms** of the sequence. If the domain of a function consists of the first n positive integers only, then the sequence is a **finite sequence.**

Sequences

On occasion, it is convenient to begin subscripting a sequence with 0 instead of 1 so that the terms of the sequence become

$a_0, a_1, a_2, a_3 \ldots$

The domain of the function is the set of nonnegative integers.

Example 1 – *Writing the terms of a Sequence*

Write the first four terms of each sequence.

a. $a_n = 3_n - 2$ **b.** $a_n = 3 + (-1)^n$

USE SUBSTITUTUON

Solution:

a. The first four terms of the sequence given by

$a_n = 3_n - 2$

are

$a_1 = 3(1) - 2 = 1$ 1st term

$a_2 = 3(2) - 2 = 4$ 2nd term

$a_3 = 3(3) - 2 = 7$ 3rd term

$a_4 = 3(4) - 2 = 10.$ 4th term

Example 1 – *Solution*

cont'd

b. The first four terms of the sequence given by

$a_n = 3 + (-1)^n$

are

$a_1 = 3 + (-1)^1 = 3 - 1 = 2$ 1st term

$a_2 = 3 + (-1)^2 = 3 - 1 = 4$ 2nd term

$a_3 = 3 + (-1)^3 = 3 - 1 = 2$ 3rd term

$a_4 = 3 + (-1)^4 = 3 - 1 = 4.$ 4th term

Factorial Notation

Factorial Notation

Some very important sequences in mathematics involve terms that are defined with special types of products called **factorials.**

Copy the definition below:

Definition of Factorial

If n is a positive integer, then ***n* factorial** is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot (n - 1) \cdot n.$$

As a special case, zero factorial is defined as $0! = 1$.

Example 6 – *Writing the Terms of a Sequence Involving Factorials*

Write the first five terms of the sequence given by $a_n = \dfrac{2^n}{n!}$.
Begin with *n* = 0.

Solution:

$$a_0 = \frac{2^0}{0!} = \frac{1}{1} = 1$$ 0th term

$$a_1 = \frac{2^1}{1!} = \frac{2}{1} = 2$$ 1st term

$$a_2 = \frac{2^2}{2!} = \frac{4}{2} = 2$$ 2nd term

$$a_3 = \frac{2^3}{3!} = \frac{8}{6} = \frac{4}{3}$$ 3rd term

$$a_4 = \frac{2^4}{4!} = \frac{16}{24} = \frac{2}{3}$$ 4th term

Summation Notation

Summation Notation

There is a convenient notation for the sum of the terms of a finite sequence. It is called **summation notation** or **sigma notation** because it involves the use of the uppercase Greek letter sigma, written as Σ.

Copy the definition below:

Definition of Summation Notation

The sum of the first n terms of a sequence is represented by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdot \cdot \cdot + a_n$$

where i is called the **index of summation,** n is the **upper limit of summation,** and 1 is the **lower limit of summation.**

Example 8 – *Sigma Notation for Sums*

a. $\sum_{i=1}^{5} 3i = 3(1) + 3(2) + 3(3) + 3(4) + 3(5)$

$= 3(1 + 2 + 3 + 4 + 5)$

$= 3(15)$

$= 45$

b. $\sum_{k=3}^{6} (1 + k^2) = (1 + 3^2) + (1 + 4^2) + (1 + 5^2) + (1 + 6^2)$

$= 10 + 17 + 26 + 37$

$= 90$

Example 8 – *Sigma Notation for Sums* cont'd

c. $$\sum_{n=0}^{8} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40{,}320}$$

$$\approx 2.71828$$

For the summation in part (c), note that the sum is very close to the irrational number $e \approx 2.718281828$.

It can be shown that as more terms of the sequence whose nth term is $1/n!$ are added, the sum becomes closer and closer to e.

Summation Notation

Properties of Sums

1. $\sum_{i=1}^{n} c = cn$, c is a constant.

2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$, c is a constant.

3. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

4. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

Series

Series

Many applications involve the sum of the terms of a finite or an infinite sequence. Such a sum is called a **series.**

Definition of a Series

Consider the infinite sequence $a_1, a_2, a_3, \ldots, a_i, \ldots$

1. The sum of the first n terms of the sequence is called a **finite series** or the **partial sum** of the sequence and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

2. The sum of all the terms of the infinite sequence is called an **infinite series** and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_i + \cdots = \sum_{i=1}^{\infty} a_i.$$

Example 9 – *Finding the Sum of a Series*

For the series

$$\sum_{i=1}^{\infty} \frac{3}{10^i}$$

find (a) the third partial sum and (b) the sum.

Solution:

a. The third partial sum is

$$\sum_{i=1}^{3} \frac{3}{10^i} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3}$$

$$= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000}$$

Example 1 – *Solution*

cont'd

= 0.3 + 0.03 + 0.003

= 0.333…..

b. The sum of the series is

$$\sum_{i=1}^{\infty} \frac{3}{10^i} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \cdots$$

$$= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10{,}000} + \frac{3}{100{,}000} + \cdots$$

= 0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 . . .

= 0.33333 . . .

$$= \frac{1}{3}.$$

Application

Application

Sequences have many applications in situations that involve recognizable patterns. One such model is illustrated in Example 10.

Example 10 – *Population of the United States*

From 1980 through 2008, the resident population of the United States can be approximated by the model

$$a_n = 226.4 + 2.41n + 0.016n^2, \quad n = 0, 1, . . . , 28$$

where a_n is the population (in millions) and n represents the year, with $n = 0$ corresponding to 1980. Find the last five terms of this finite sequence.

Example 10 – *Solution*

The last five terms of this finite sequence are as follows.

$a_{24} = 226.4 + 2.41(24) + 0.016(24)^2$

≈ 293.5 2004 population

$a_{25} = 226.4 + 2.41(25) + 0.016(25)^2$

≈ 296.7 2005 population

$a_{26} = 226.4 + 2.41(26) + 0.016(26)^2$

≈ 299.9 2006 population

Example 10 – *Solution*

cont'd

$a_{27} = 226.4 + 2.41(27) + 0.016(27)^2$

≈ 303.1 2007 population

$a_{28} = 226.4 + 2.41(28) + 0.016(28)^2$

≈ 306.4 2008 population