Section 5.10

Integration: "Logarithmic and Other Functions Defined by Integrals"

All graphics are attributed to:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

Introduction

- Remember that the natural logarithm function ln x is the inverse of e^x.

- In this section we will show how ln x can be defined as an integral and work to recognize when integrals that appear in solutions of problems can be expressed as natural logarithms.

Connection Between Natural Logarithms and Integrals

$$t_1 = e, \quad t_2 = e^2, \quad t_3 = e^3, \ldots, \quad t_n = e^n, \ldots$$

each of the areas would be 1 (Figure 5.10.1*b*). Thus, in modern integral notation

$$\int_1^{e^n} \frac{1}{t}\,dt = n$$

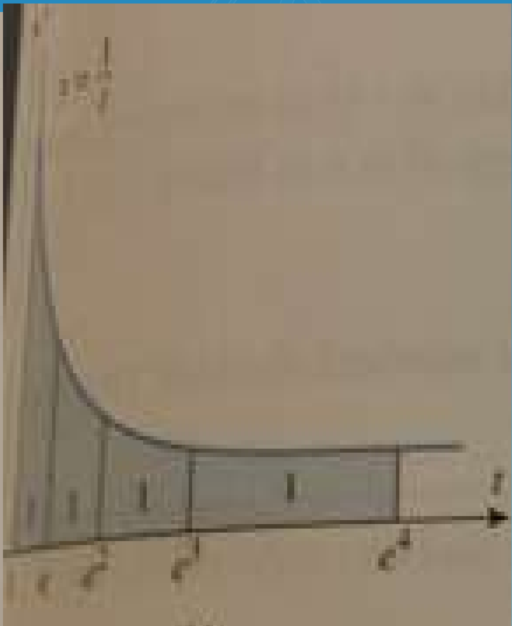
which can be expressed as

$$\int_1^{e^n} \frac{1}{t}\,dt = \ln(e^n)$$

By comparing the upper limit of the integral and the expression inside the logarithm, it is a natural leap to the more general result

$$\int_1^{x} \frac{1}{t}\,dt = \ln x$$

which today we take as the formal definition of the natural logarithm.



(*b*)

5.10.1 DEFINITION The *natural logarithm* of x is denoted by $\ln x$ and is defined by the integral

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0 \tag{1}$$

Continuity and Differentiability Natural Logarithm Properties

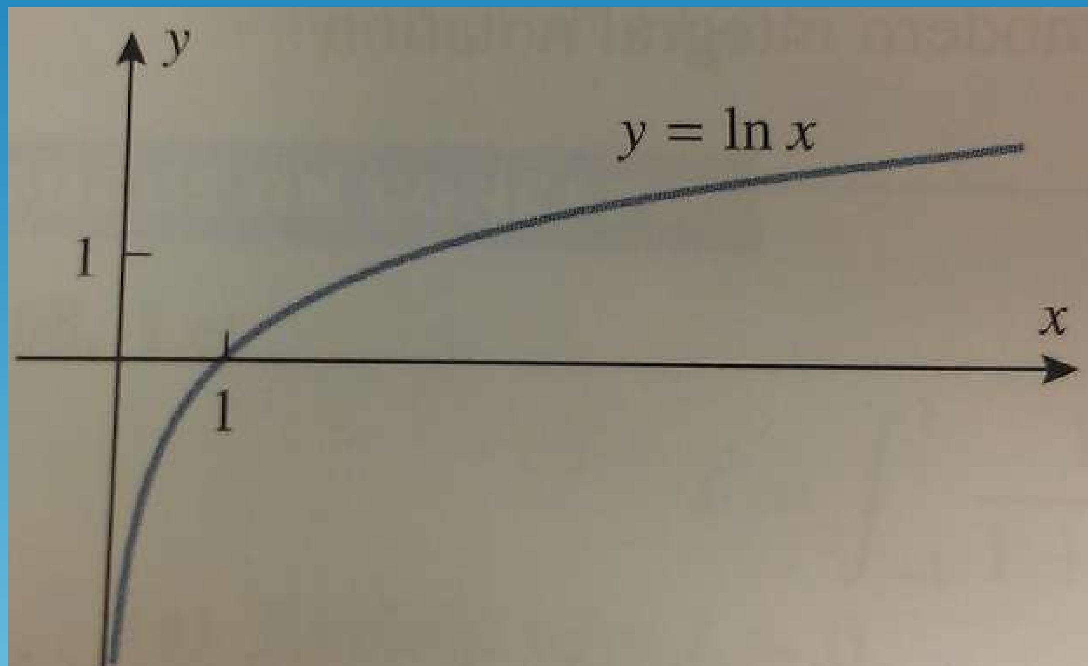
- Please read the bottom half of page 397 and the top part of page 398 regarding the our previous assumption that e^x is continuous and the resulting properties of logarithms.
- Below are listed the product, quotient, and power properties from Algebra II and Pre-Calculus/Math Analysis in terms of the natural logarithm (proofs on pages 398-399):

5.10.2 THEOREM *For any positive numbers a and c and any rational number r:*

(a) $\ln ac = \ln a + \ln c$ (b) $\ln \frac{1}{c} = -\ln c$

(c) $\ln \frac{a}{c} = \ln a - \ln c$ (d) $\ln a^r = r \ln a$

Graph, Domain, Range, and End Behavior of ln x



5.10.3 THEOREM

(*a*) *The domain of* $\ln x$ *is* $(0, +\infty)$.

(*b*) $\lim_{x \to 0^+} \ln x = -\infty$ *and* $\lim_{x \to +\infty} \ln x = +\infty$

(*c*) *The range of* $\ln x$ *is* $(-\infty, +\infty)$.

Definition and Derivative of e^x

5.10.4 DEFINITION The inverse of the natural logarithm function $\ln x$ is denoted by e^x and is called the ***natural exponential function***.

5.10.5 THEOREM *The natural exponential function e^x is differentiable, and hence continuous, on $(-\infty, +\infty)$, and its derivative is*

$$\frac{d}{dx}[e^x] = e^x$$

Irrational Exponents

- The logarithm power property states that if a>0 and r is a rational number, then $\ln a^r = r \ln a$
- If you raise e to the power of each side, then you get $e^{\ln a^r} = e^{r \ln a}$.
- When the property $e^{\ln x} = x$ is applied, $e^{\ln a^r} = a^r$.
- By substitution: $a^r = e^{\ln a^r} = e^{r \ln a}$

5.10.6 DEFINITION If $a > 0$ and r is a real number, a^r is defined by

$$a^r = e^{r \ln a} \tag{7}$$

Applications of Definition 5.10.6

- When that definition and algebra I exponent rules regarding adding, subtracting, and multiplying exponents are applied, we can prove (more formally on page 402) some of the derivatives and limits we have been using (such as the power rule and the derivative of b^x).

5.10.7 THEOREM

(*a*) *For any real number r, the power function x^r is differentiable on $(0, +\infty)$ and its derivative is*

$$\frac{d}{dx}[x^r] = rx^{r-1}$$

(*b*) *For $b > 0$ and $b \neq 1$, the base b exponential function b^x is differentiable on $(-\infty, +\infty)$ and its derivative is*

$$\frac{d}{dx}[b^x] = b^x \ln b$$

General Logarithms

- When b>0 and b does not equal 1, b^x is a one to one function (meaning it passes the horizontal and vertical line tests) and so it has an inverse that is a function.

$y = b^x = e^{\ln b^x}$	by $e^{\ln x} = x$
$y = b^x = e^{\ln b^x} = e^{x \ln b}$	by the power prop.
$\ln y = \ln(e^{x \ln b})$	take natural log. both sides
$\ln y = \ln(e^{x \ln b}) = x \ln b$	by $\ln e^x = x$
$\ln y / \ln b = x$	divide both sides by ln b
$\ln x / \ln b = y$	interchange x and y

Therefore, the inverse function for b^x is $\ln x / \ln b$.

Change of Base Formula = Inverse Function for b^x

- The inverse function for b^x is ln x/ln b which you may remember is the change of base formula:

5.10.9 DEFINITION For $b > 0$ and $b \neq 1$, the *base b logarithm* function, denoted $\log_b x$, is defined by

$$\log_b x = \frac{\ln x}{\ln b} \tag{9}$$

- Therefore, $\log_b x$ is the inverse function for b^x by substitution.

Functions Defined by Integrals

- The functions we have dealt with so far this year are called elementary functions: they include polynomial, rational, power, exponential, logarithmic, and trigonometric and all other functions that can be obtained from these by addition, subtraction, multiplication, division, root extraction, and composition.
- There are many important functions that do not fall into this "elementary function" category which often result from solving initial value/condition problems of this form:

$$dy/dx = f(x) \qquad \text{given } y(x_0) = y_0$$

Solving Functions Defined by Integrals

- The basic method for solving functions of this type is to integrate f(x), and then use the initial condition to determine the constant of integration C as we did in Section 5.2.
- Instead, we could find a general formula (indefinite integral) for the solution and then apply that formula to solve specific problems.

$$y(x) = y_0 + \int_{x_0}^{x} f(t)\,dt$$

Integrals with Functions as Limits of Integration

- Various applications can lead to integrals in which at least one of the limits of integration is a function of x.
- For now, we will differentiate (take the derivative of) these integrals only.
- To do so, we will apply the chain rule:

$$\frac{d}{dx}\left[\int_a^{g(x)} f(t)\,dt\right] \rightarrow \frac{d}{dx}[F(g(x))] = F'(g(x))g'(x) = f(g(x))g'(x)$$

Theorem 5.6.3

To differentiate an integral with a constant lower limit and a function as the upper limit, substitute the upper limit into the integrand, and multiply by the derivative of the upper limit.

Example

$$\frac{d}{dx}\left[\int_{1}^{\sin x}(1-t^2)\,dt\right] = (1-\sin^2 x)\cos x = \cos^3 x$$

Guess who this is…