1 Functions and Their Graphs

1.6 Inverse Functions

What You Should Learn

- Find inverse functions informally and verify that two functions are inverse functions of each other.
- Use graphs of functions to decide whether functions have inverse functions.
- Determine whether functions are one–to–one.
- Find inverse functions algebraically.

Inverse Functions

Inverse Functions

We have know that a function can be represented by a set of ordered pairs.

For instance, the function $f(x) = x + 4$ from the set $A = \{1, 2, 3, 4\}$ to the set $B = \{5, 6, 7, 8\}$ can be written as follows.

$f(x) = x + 4$: $\{(1, 5), (2, 6), (3, 7), (4, 8)\}$

All I did was plug in the numbers in set A into the function f(x)=x + 4 to get the associated y values which are set B.

Inverse Functions

In this case, by interchanging the first and second coordinates of each of these ordered pairs, you can form the **inverse function** of which is denoted by f^{-1} It is a function from the set *B* to the set *A* and can be written as follows.

$$f^{-1}(x) = x - 4: \{(5, 1), (6, 2), (7, 3), (8, 4)\}$$

Note that the domain of is equal to the range of f^{-1} and vice versa, as shown in Figure 1.56.

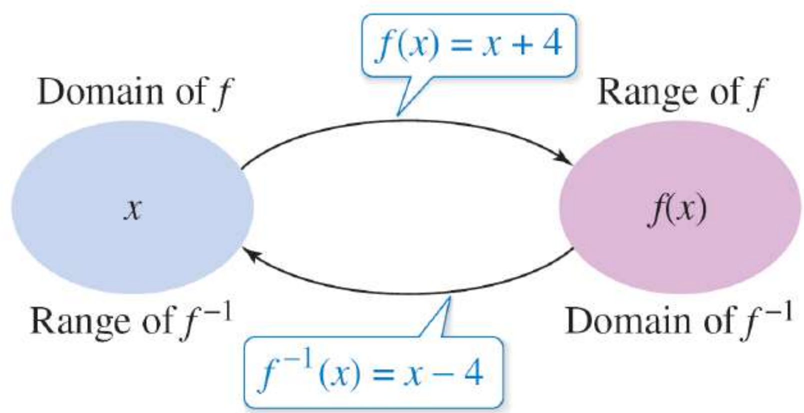


Figure 1.56

Inverse Functions

You may remember from last year: when you are looking for the inverse, interchange x and y and solve for y.

<u>Example</u>

If f(x) = x – 4, you may want to rewrite it y = x – 4.

Then interchange x and y to get x = y – 4.

Solve for y by adding 4 to both sides gives you y = x + 4 which is the inverse function: $f^{-1}(x) = x + 4$

Inverse Functions

Also note that the functions f and f^{-1} have the effect of "undoing" each other. In other words, when you form the composition of f with f^{-1} or the composition of with you obtain the identity function.

$$f(f^{-1}(x)) = f(x - 4) = (x - 4) + 4 = x$$

$$f^{-1}(f(x)) = f^{-1}(x + 4) = (x + 4) - 4 = x$$

Example 1 – *Finding Inverse Functions Informally*

Find the inverse function of f(x) = 4x. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

Solution:

The function *multiplies* each input by 4. To "undo" this function, you need to *divide* each input by 4. So, the inverse function of $f(x) = 4$ is given by

$$f^{-1}(x) = \frac{x}{4}.$$

Example 1 – *Solution*

cont'd

You can verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function as follows.

$$f(f^{-1}(x)) = f\left(\frac{x}{4}\right)$$

$$= 4\left(\frac{x}{4}\right)$$

$$= x$$

$$f^{-1}(f(x)) = f^{-1}(4x)$$

$$= \frac{4x}{4}$$

$$= x$$

Inverse Functions

Definition of Inverse Function

Let f and g be two functions such that

$$f(g(x)) = x \qquad \text{for every } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \qquad \text{for every } x \text{ in the domain of } f.$$

Under these conditions, the function g is the **inverse function** of the function f. The function g is denoted by f^{-1} (read "f-inverse"). So,

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

The domain of f must be equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.

If the function *g* is the inverse function of the function *f* then it must also be true that the function *f* is the inverse function of the function *g*. For this reason, you can say that the functions *f* and *g* are *inverse functions of each other*.

The Graph of an Inverse Function

The Graph of an Inverse Function

The graphs of a function and its inverse function f^{-1} are related to each other in the following way. If the point (a, b) lies on the graph of then the point (b, a) must lie on the graph of f^{-1} and vice versa.

This means that the graph of f^{-1} is a reflection of the graph of f in the line $y = x$.

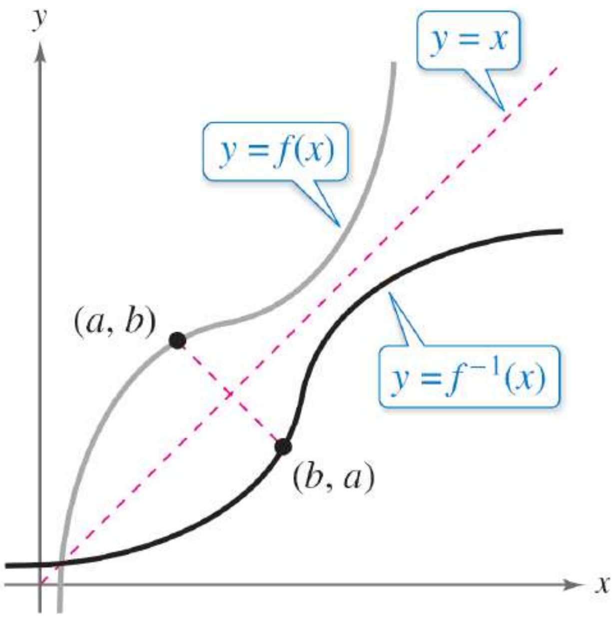


Figure 1.57

Example 5 – *Verifying Inverse Functions Graphically*

Verify that the functions *f* and *g* are inverse functions of each other graphically.

$$f(x) = 2x^3 - 1 \qquad \text{and} \qquad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

Solution:

From Figure 1.58, you can conclude that *f* and *g* are inverse functions of each other.

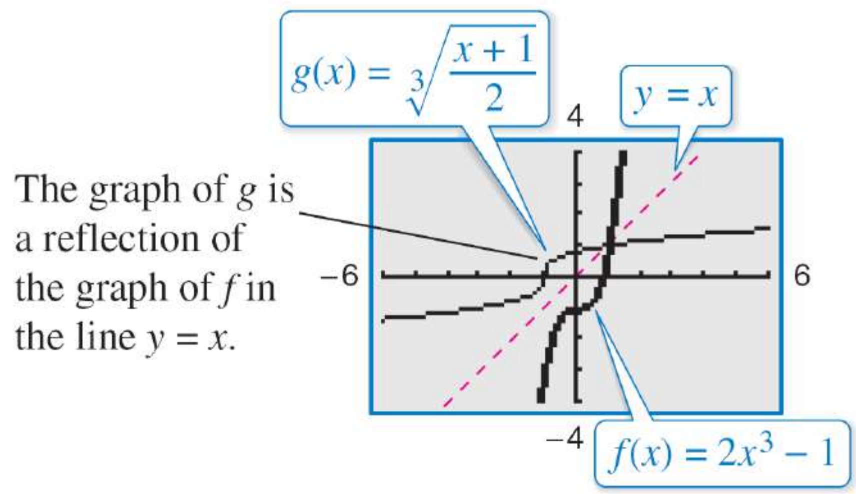


Figure 1.58

The Existence of an Inverse Function

The Existence of an Inverse Function

To have an inverse function, a function must be **one-to-one,** which means that each x-value has only one y-value ***and*** each y-value has only one x-value.

Definition of a One-to-One Function

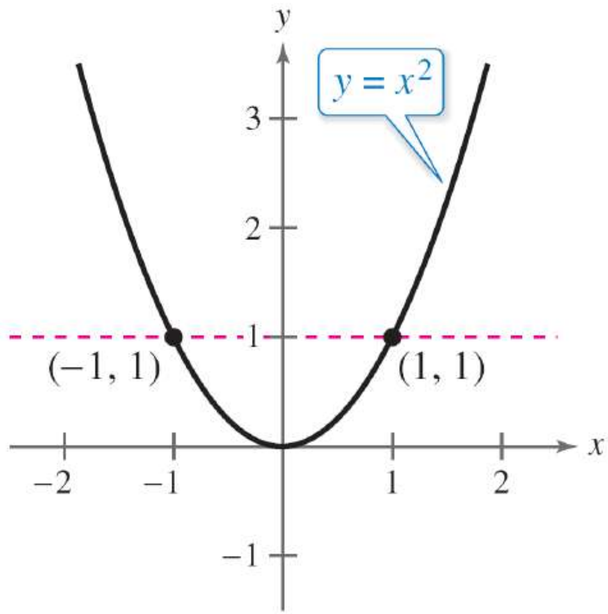
A function f is **one-to-one** when, for a and b in its domain, $f(a) = f(b)$ implies that $a = b$.

Existence of an Inverse Function

A function f has an inverse function f^{-1} if and only if f is one-to-one.

The Existence of an Inverse Function

From figure 1.61, it is easy to tell whether a function of x is one-to-one. Simply check to see that every horizontal line intersects the graph of the function at most once. This is called the **Horizontal Line Test.**



$f(x) = x^2$ is not one-to-one.

Figure 1.61

The Existence of an Inverse Function

Two special types of functions that pass the Horizontal Line Test are those that are increasing or decreasing on their entire domains.

1. If *f* is *increasing* on its entire domain, then *f* is one-to-one.

2. If *f* is *decreasing* on its entire domain, then *f* is one-to-one.

Example 7 – *Testing for One-to-One Functions*

Is the function $f(x) = \sqrt{x} + 1$ one-to-one?

Solution:

Let *a* and *b* be nonnegative real numbers with *f*(*a*) = *f*(*b*).

$$\sqrt{a} + 1 = \sqrt{b} + 1$$ Set *f* (*a*) = *f* (*b*).

$$\sqrt{a} = \sqrt{b}$$

$$a = b$$

So, *f*(*a*) = *f*(*b*) implies that *a* = *b.* You can conclude that is one-to-one and *does* have an inverse function.

Example 7 – *Testing for One-to-One Functions*

You may also want to test by graphing.

If your graph passes the vertical line test ***and*** the horizontal line test then it is a function which has an inverse. Therefore, it is a one-to-one function.

Finding Inverse Functions Algebraically

Finding Inverse Functions Algebraically

For simple functions, you can find inverse functions by inspection. For more complicated functions, however, it is best to use the following guidelines.

Finding an Inverse Function

1. Use the Horizontal Line Test to decide whether f has an inverse function.
2. In the equation for $f(x)$, replace $f(x)$ by y.
3. Interchange the roles of x and y, and solve for y.
4. Replace y by $f^{-1}(x)$ in the new equation.
5. Verify that f and f^{-1} are inverse functions of each other by showing that the domain of f is equal to the range of f^{-1}, the range of f is equal to the domain of f^{-1}, and $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Example 8 – *Finding an Inverse Function Algebraically*

Find the inverse function of $f(x) = \dfrac{5 - 3x}{2}$.

Solution:

The graph of *f* in Figure 1.63 passes the Horizontal Line Test. So, you know that *f* is one-to-one and has an inverse function.

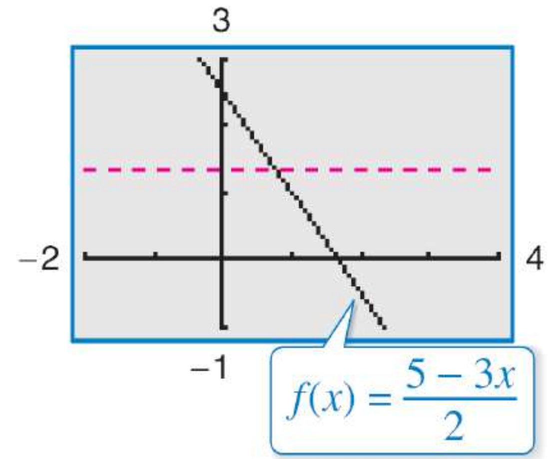


Figure 1.63

$$f(x) = \frac{5 - 3x}{2}$$ Write original function.

$$y = \frac{5 - 3x}{2}$$ Replace *f* (*x*) by *y*.

$$x = \frac{5 - 3y}{2}$$ Interchange *x* and *y*.

$$2x = 5 - 3y$$ Multiply each side by 2.

Example 8 – *Solution*

cont'd

$$3y = 5 - 2x$$

Isolate the *y*-term.

$$y = \frac{5 - 2x}{3}$$

Solve for *y*.

$$f^{-1}(x) = \frac{5 - 2x}{3}$$

Replace *y* by $f^{-1}(x)$

The domains and ranges of *f* and f^{-1} consist of all real numbers. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.