Section 9.1

Infinite Series: “Sequences”

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Introduction

- In this chapter, we will be dealing with infinite series which are sums that involve infinitely many terms.
- We can use them for many things such as approximating trig functions and logarithms, solving differential equations, evaluating difficult integrals, constructing mathematical models of physical laws, and more.
- Since it is impossible to add up infinitely many numbers directly, we must define exactly what we mean by the sum of an infinite series.
- Also, it is important to realize that not all infinite series actually have a sum so we will need to find which series do have sums and which do not.

"Sequence" – everyday vs. mathematical applications

- In everyday language, the term "sequence" means a succession of things in a definite order – chronological order, size order, or logical order.

- In mathematics, the term "sequence" is commonly used to denote a succession of numbers whose order is determined by a rule or a function.

Definitions and Examples

- Infinite sequence – an unending succession of numbers called terms.
- Terms – numbers that have a definite order; first term a_1, second term a_2, third term a_3, fourth term a_4, and so on. They are often written a_1, a_2, a_3, a_4, …, where the … indicates that the sequence continues indefinitely.
- Easy examples / pattern / general term

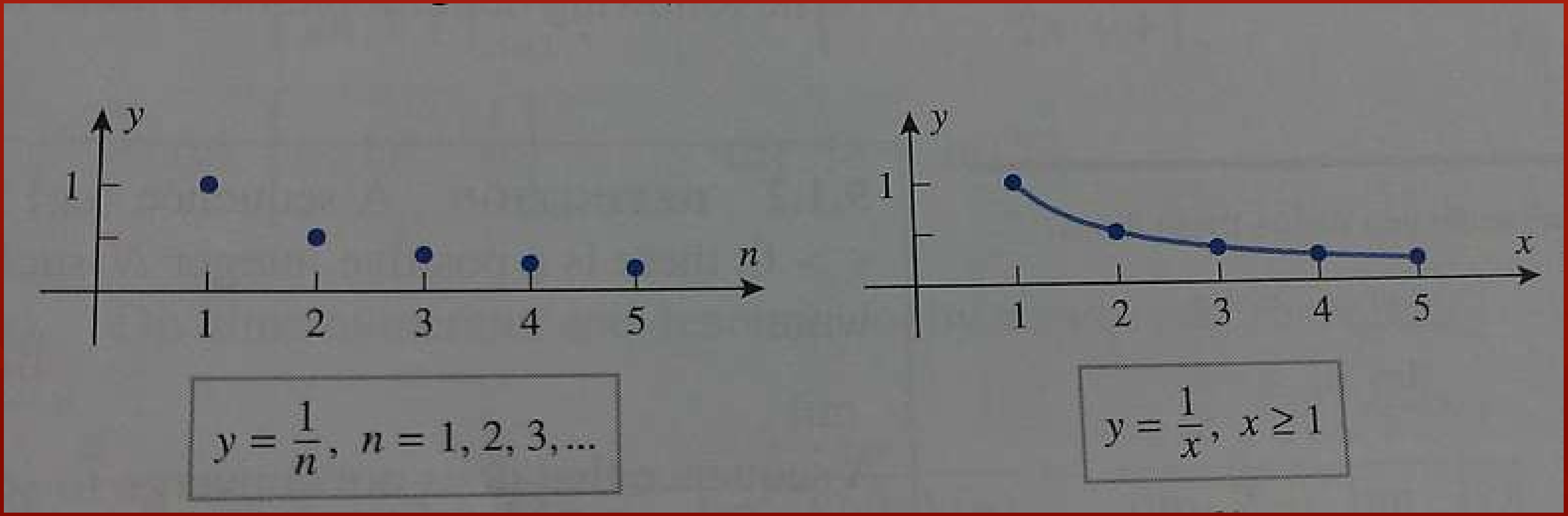
Easy examples	pattern	general term
1,2,3,4,…	add one	n
2,4,6,8,…	multiply by 2	$2n$
1,-1,1,-1,…	alternating +/-	$(-1)^{n-1}$

Example with Brace Notation

9.1.1 DEFINITION A *sequence* is a function whose domain is a set of integers.

Graphs of Sequences

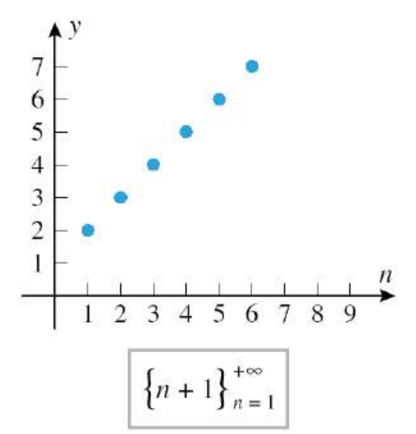
- Since sequences are functions, we will sometimes graph the sequence.
- Because the term numbers in the sequence below y = 1/n are integers, the graph consists of a succession of isolated points instead of a continuous curve like y = 1/x would be if you were not graphing a sequence.



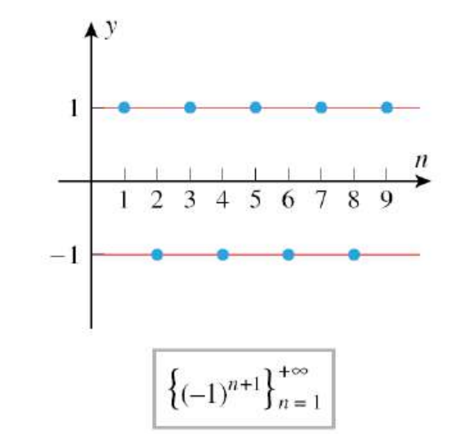
Limits of a Sequence

- Since sequences are functions, we can discuss their limits.
- However, because a sequence is only defined for integer values, the only limit that makes sense is the limit of a_n as n approaches infinity.
- Examples:

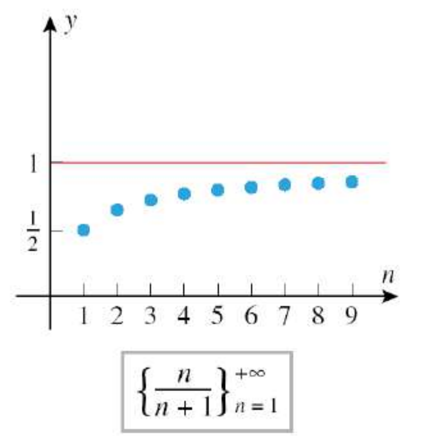
Increases without bound (DIVERGE)



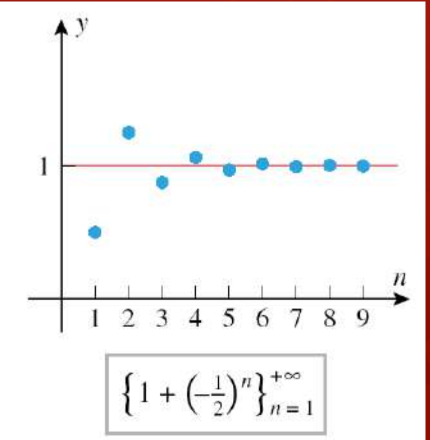
Oscillates between -1 and 1 (DIVERGE)



Increases toward a “limiting value” of 1 (CONVERGE)

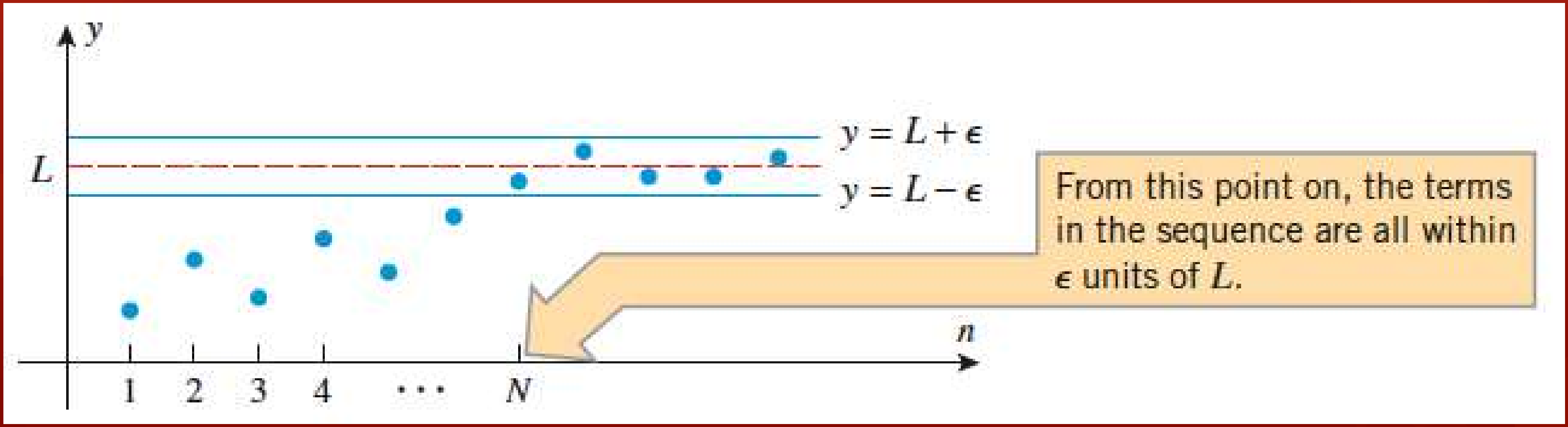


Tends toward a “limiting value” of 1 in an oscillatory fashion (CONVERGE)



Limits of a Sequence con't

- Informally speaking, the limit of a sequence $\{a_n\}$ is intended to describe how a_n behaves as n approaches infinity.
- More specifically, we will say that a sequence $\{a_n\}$ approaches a limit L if the terms in the sequence eventually become arbitrarily close to L (see picture below and definition on following slide).



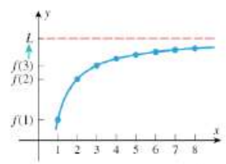
Converge and Diverge

9.1.2 DEFINITION A sequence $\{a_n\}$ is said to *converge* to the *limit* L if given any $\epsilon > 0$, there is a positive integer N such that $|a_n - L| < \epsilon$ for $n \geq N$. In this case we write

$$\lim_{n \to +\infty} a_n = L$$

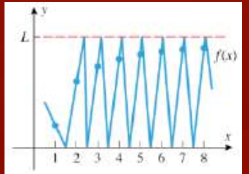
A sequence that does not converge to some finite limit is said to *diverge*.

- Look back at the examples on slide #8 to see which converge, which diverge, and why.
- If the general term of a sequence is f(n), where f(x) is a function defined on the entire interval $[1, +\infty)$, then the values of f(n) can be viewed as "sample values" of f(x) taken at the positive integers. Thus,
  - if f(x) $\to$ L as x $\to + \infty$, then f(n) $\to$ L as n $\to + \infty$
- However, the converse is not true. You cannot infer the same about f(x) given f(n).



If $f(x) \to L$ as $x \to +\infty$, then $f(n) \to L$ as $n \to +\infty$.

(a)



$f(n) \to L$ as $n \to +\infty$, but $f(x)$ diverges by oscillation as $x \to +\infty$.

(b)

Familiar Properties of Limits Apply to Sequences ☺

9.1.3 THEOREM *Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ converge to limits L_1 and L_2, respectively, and c is a constant. Then:*

(a) $\lim_{n\to+\infty} c = c$

(b) $\lim_{n\to+\infty} ca_n = c\lim_{n\to+\infty} a_n = cL_1$

(c) $\lim_{n\to+\infty} (a_n + b_n) = \lim_{n\to+\infty} a_n + \lim_{n\to+\infty} b_n = L_1 + L_2$

(d) $\lim_{n\to+\infty} (a_n - b_n) = \lim_{n\to+\infty} a_n - \lim_{n\to+\infty} b_n = L_1 - L_2$

(e) $\lim_{n\to+\infty} (a_n b_n) = \lim_{n\to+\infty} a_n \cdot \lim_{n\to+\infty} b_n = L_1 L_2$

(f) $\lim_{n\to+\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to+\infty} a_n}{\lim_{n\to+\infty} b_n} = \frac{L_1}{L_2}$ (*if* $L_2 \neq 0$)

Converge or Diverge Examples

- To determine whether the following sequences converge or diverge, we will examine the limit as n approaches ∞.
- a) $\left\{\frac{}{2\quad+1}\right\}_{n=1}^{+\infty}$
  - Solution: Divide the numerator and denominator by the highest power of n in the denominator (like chapter 1).
  - $\lim_{\to+\infty}\frac{}{2\quad+1} = \lim_{\to+\infty}\frac{/}{(2\quad+1)/} = \frac{\lim_{\to+\infty} 1}{\lim_{\to+\infty}(2+\frac{1}{})} = \frac{1}{2+0} = \frac{1}{2} \to$

    converges
- b) $\left\{(-1)^{n+1}\frac{}{2\quad+1}\right\}_{n=1}^{+\infty}$
  - Solution: This is the same as a) except the $(-1)^{n+1}$ makes it oscillate between -1 and 1 times the answer we got in a). Therefore, the odd-numbered terms approach $\frac{1}{2}$ and the even-numbered terms approach - $\frac{1}{2}$ and the sequence diverges.

Converge or Diverge Examples con't

- c) $\left\{(-1)^{n+1}\frac{1}{n}\right\}_{n=1}^{+\infty}$
  - Solution: Like in b), the $(-1)^{n+1}$ makes this oscillate between positives and negatives. However, the 1/n approaches 0. Therefore, the terms approach 0 from both sides, making the limit 0. $\rightarrow$ converges to 0
- d) $\{8 - 2n\}_{n=1}^{+\infty}$
  - Solution: $\lim_{n \to +\infty} (8 - 2n) = -\infty$ by end behavior or graphing. Therefore, the sequence diverges.
- See example #4 on page 601 for more sequences.

Limit Example Using L'Hopital's Rule

- Find the limit of the sequence $\{ — \}_{n=1}^{+\infty}$

- Solution: $\lim_{\to +\infty} — = \frac{\infty}{\infty}$ which is indeterminate form.

- Therefore, we will use L'Hopital's rule where you take the derivative of the top and bottom separately and then evaluate the limit.

- $\lim_{\to +\infty} — = \lim_{\to +\infty} — = \lim_{\to +\infty} \frac{1}{} = \frac{1}{+\infty} = 0 \to \lim_{\to +\infty} —$
  $= 0$

Another Limit Example Using L'Hopital's Rule

- Show that the $\lim_{\to +\infty} \sqrt{}$
- Solution: $\lim_{\to +\infty} \sqrt{} = \infty^{\frac{1}{\infty}} = \infty^{0}$ which is indeterminate form
- Therefore, we will use L'Hopital's rule.
- The problem is that we do not currently have a fraction so we have some work to do first to get a fraction.
- $\lim_{\to +\infty} \sqrt{} = \lim_{\to +\infty} {}^{\frac{1}{}} = \lim_{\to +\infty} {}^{\frac{1}{}}$ by the inverse property
- $= \lim_{\to +\infty} {}^{\frac{1}{}}$ by the power property
- $= \lim_{\to +\infty} \frac{}{} = \lim_{\to +\infty} \frac{}{} = \lim_{\to +\infty} \frac{\frac{1}{}}{1} = \frac{\frac{1}{+\infty}}{1} = {}^{0} = 1$

Example Looking at Even-Numbered Terms and Odd-Numbered Terms Separately

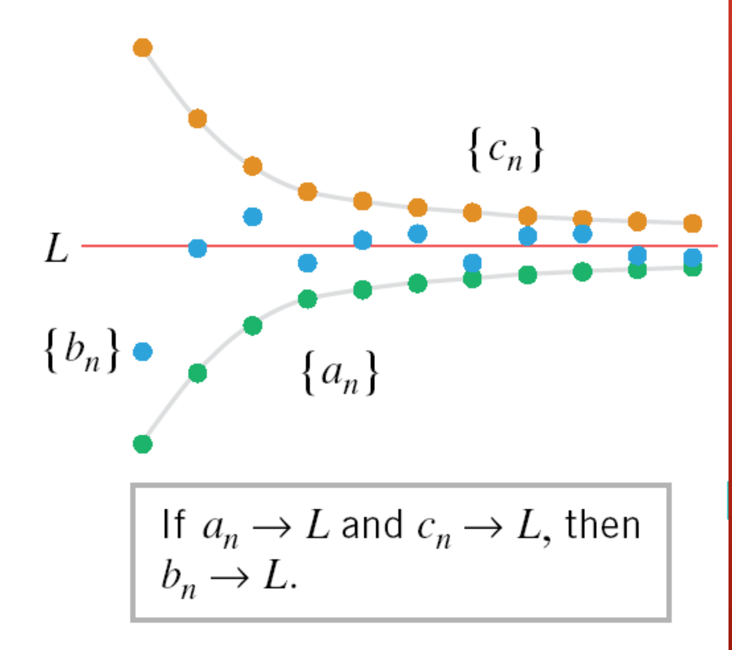
- Sometimes the even-numbered and odd-numbered terms of a sequence behave sufficiently differently that it is desirable to investigate their convergence separately.

- Example: The sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{2^3}, \frac{1}{3^3}, \ldots$ converges to 0 since the odd-numbered terms $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ converge to 0 & the even-numbered terms $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ also converge to 0.

- Example 2: The sequence $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \ldots$ diverges since the odd-numbered terms 1,1,1,… converge to 1 and the even-numbered terms $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ converge to 0, which is different than 1. That is why they diverge.

The Squeezing Theorem for Sequences

- This idea is very similar to the squeezing theorem we learned in chapter 1 for trigonometry limits.
- We will use it to find limits of sequences that cannot be directly obtained and must therefore be compared to other sequences and "squeezed" between them.



9.1.5 THEOREM (*The Squeezing Theorem for Sequences*) *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences such that*

$$a_n \leq b_n \leq c_n \quad (\textit{for all values of } n \textit{ beyond some index } N)$$

If the sequences $\{a_n\}$ and $\{c_n\}$ have a common limit L as $n \to +\infty$, then $\{b_n\}$ also has the limit L as $n \to +\infty$.

9.1.6 THEOREM *If* $\lim_{n \to +\infty} |a_n| = 0$, *then* $\lim_{n \to +\infty} a_n = 0$.

Rocksanna at Mono Lake