Section 5.5

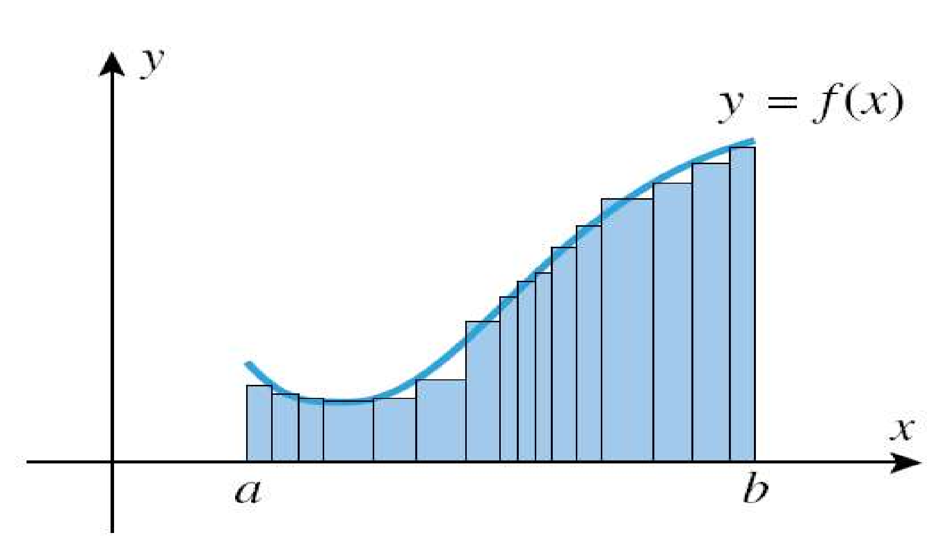
Integration: "The Definite Integral"

All graphics are attributed to:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

Introduction

- In this section we will introduce the concept of a "definite integral," which will link the concept of area to other important concepts such as length, volume, density, probability, and work (most of those in Chapter Six).
- In the last section, we divided the interval [a,b] into n subintervals of equal length to form n rectangles whose area we found.
- For some functions, it may be more convenient to use rectangles with different widths.



Formula Adjustment

- To allow for the possibility of different widths, we must change the formula in section 5.4. Instead of the width being the same for each rectangle (Δx), we need to make that Δx_k.

5.5.1 DEFINITION A function f is said to be *integrable* on a finite closed interval $[a, b]$ if the limit

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

exists and does not depend on the choice of partitions or on the choice of the points x_k^* in the subintervals. When this is the case we denote the limit by the symbol

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

which is called the *definite integral* of f from a to b. The numbers a and b are called the *lower limit of integration* and the *upper limit of integration*, respectively, and $f(x)$ is called the *integrand*.

Riemann sum

- The sum on the previous slide is called a Riemann sum and the definite integral is sometimes called the Riemann integral in honor of the German mathematician Bernhard Riemann (1826-1866).
- He formulated many of the basic concepts of integral calculus and we will use Riemann sums to help us find volume, surface area, and arc length in Chapter Six.
- NOTE: Although a function does not have to be continuous on an interval to be integrable (able to be integrated) on that interval, we will generally be taking the definite integrals of continuous functions.

The Definite Integral

- This is the formula you have been waiting for…the short way. ☺

5.5.2 THEOREM *If a function f is continuous on an interval $[a, b]$, then f is integrable on $[a, b]$, and the net signed area A between the graph of f and the interval $[a, b]$ is*

$$A = \int_a^b f(x)\,dx \tag{1}$$

Examples

▶ **Example 1** Sketch the region whose area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry.

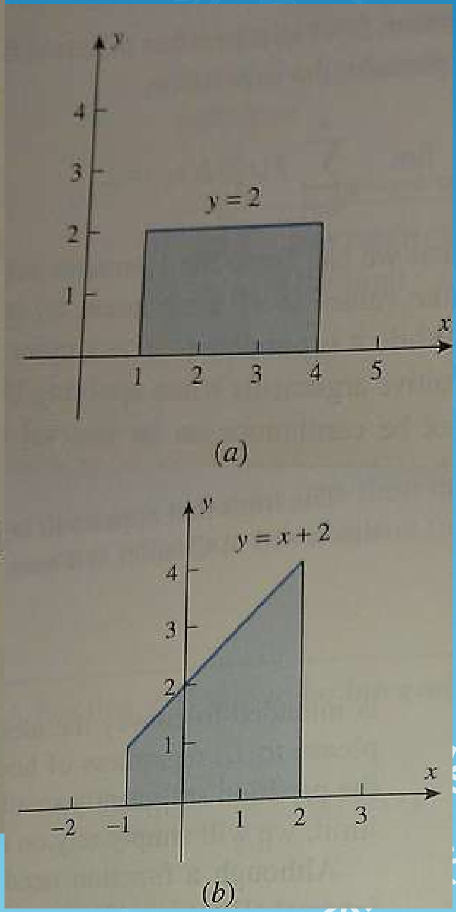
(a) $\int_{1}^{4} 2\,dx$ (b) $\int_{-1}^{2} (x+2)\,dx$ (c) $\int_{0}^{1} \sqrt{1-x^2}\,dx$

Solution (a). The graph of the integrand is the horizontal line $y = 2$, so the region is a rectangle of height 2 extending over the interval from 1 to 4 (Figure 5.5.4*a*). Thus,

$$\int_{1}^{4} 2\,dx = (\text{area of rectangle}) = 2(3) = 6$$

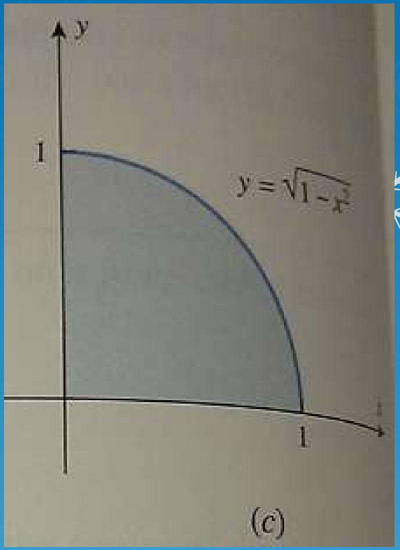
Solution (b). The graph of the integrand is the line $y = x + 2$, so the region is a trapezoid whose base extends from $x = -1$ to $x = 2$ (Figure 5.5.4*b*). Thus,

$$\int_{-1}^{2} (x+2)\,dx = (\text{area of trapezoid}) = \frac{1}{2}(1+4)(3) = \frac{15}{2}$$



Partial Circle Example

(c) $\int_0^1 \sqrt{1 - x^2}\, dx$



- Many people do not recognize this as a circle since it is not in graphing form. You may want to square both sides and add x^2 to both sides.

Solution (c). The graph of $y = \sqrt{1 - x^2}$ is the upper semicircle of radius 1, centered at the origin, so the region is the right quarter-circle extending from $x = 0$ to $x = 1$ (Figure 5.5.4c). Thus,

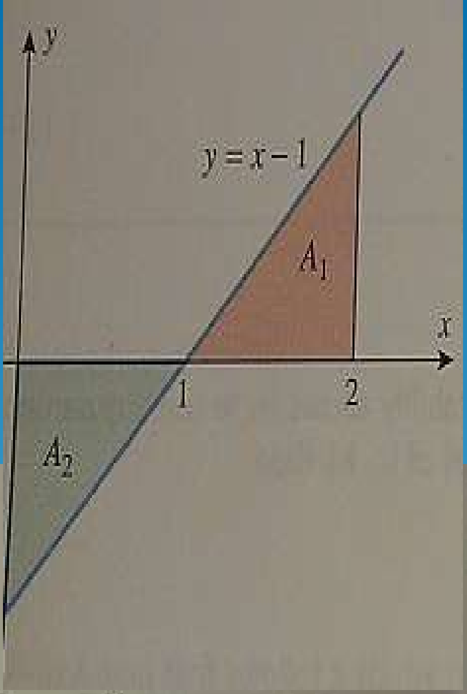
$$\int_0^1 \sqrt{1 - x^2}\, dx = \text{(area of quarter-circle)} = \frac{1}{4}\pi(1^2) = \frac{\pi}{4}$$

Net Signed Area – Another Example

- We briefly discussed net signed area in Section 5.4. Here is another example:

▶ Example 2 Evaluate

$$\text{(a)} \int_0^2 (x-1)\,dx \qquad \text{(b)} \int_0^1 (x-1)\,dx$$



Solution. The graph of $y = x - 1$ is shown in Figure 5.5.5, and we leave it for you to verify that the shaded triangular regions both have area $\frac{1}{2}$. Over the interval $[0, 2]$ the net signed area is $A_1 - A_2 = \frac{1}{2} - \frac{1}{2} = 0$, and over the interval $[0, 1]$ the net signed area is $-A_2 = -\frac{1}{2}$. Thus,

$$\int_0^2 (x-1)\,dx = 0 \quad \text{and} \quad \int_0^1 (x-1)\,dx = -\tfrac{1}{2}$$

- The area of the pink shaded region is 1/2 since $Area_1 = A_1 = (1/2)b*h = (1/2)(1)(1) = 1/2$
- The area of the green shaded region is -½ since $Area_2 = A_2 = (1/2)b*h = (1/2)(1)(1) = 1/2$ and it is below the x-axis so it is negative.

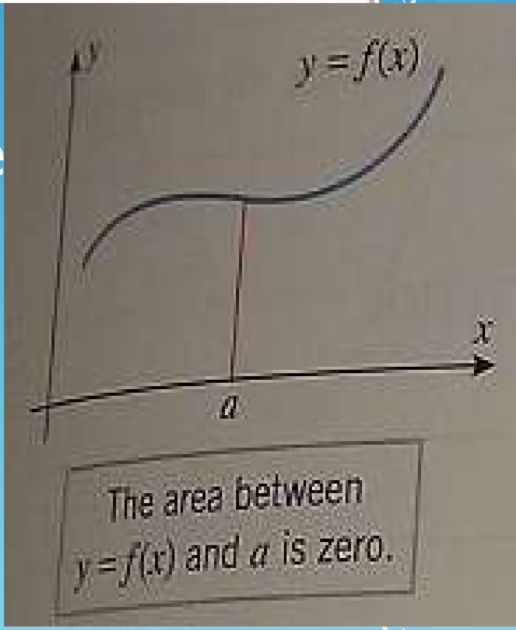
Properties of Definite Integrals

5.5.3 DEFINITION

(a) If a is in the domain of f, we define

$$\int_a^a f(x)\,dx = 0$$

- This property makes sense if you look at the figure to the right. Between the limits of integration (from a to a) there is no area under the curve y=f(x). Therefore, it equals zero.



More Properties of Definite Integrals

(b) If f is integrable on $[a, b]$, then we define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

- This property makes sense when you think of the earlier circle example and reverse the limits of integration (a and b):

$$\int_1^0 \sqrt{1-x^2}\,dx = -\int_0^1 \sqrt{1-x^2}\,dx = -\frac{\pi}{4}$$

Example 1(c)

More Properties of Definite Integrals

5.5.4 THEOREM *If f and g are integrable on $[a, b]$ and if c is a constant, then cf, $f + g$, and $f - g$ are integrable on $[a, b]$ and*

(a) $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

(b) $\displaystyle\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(c) $\displaystyle\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

5.5.5 THEOREM *If f is integrable on a closed interval containing the three points a, b, and c, then*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

no matter how the points are ordered.

Example Using Properties

▸ **Example 4** Evaluate

$$\int_0^1 (5 - 3\sqrt{1 - x^2})\, dx$$

Solution. From parts (*a*) and (*c*) of Theorem 5.5.4 we can write

$$\int_0^1 (5 - 3\sqrt{1 - x^2})\, dx = \int_0^1 5\, dx - \int_0^1 3\sqrt{1 - x^2}\, dx = \int_0^1 5\, dx - 3\int_0^1 \sqrt{1 - x^2}$$

The first integral in this difference can be interpreted as the area of a rectangle of hei[illegible] and base 1, so its value is 5, and from Example 1 the value of the second integral is [illegible] Thus,

$$\int_0^1 (5 - 3\sqrt{1 - x^2})\, dx = 5 - 3\left(\frac{\pi}{4}\right) = 5 - \frac{3\pi}{4}$$ ◂

Discontinuities and Integrability

- This year we will only learn basic results about whether and when functions are integrable.
- There are many other conditions and exceptions to discuss, but it is more than we are ready for at this point.

5.5.6 THEOREM

(*a*) *If f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$, then*

$$\int_a^b f(x)\,dx \geq 0$$

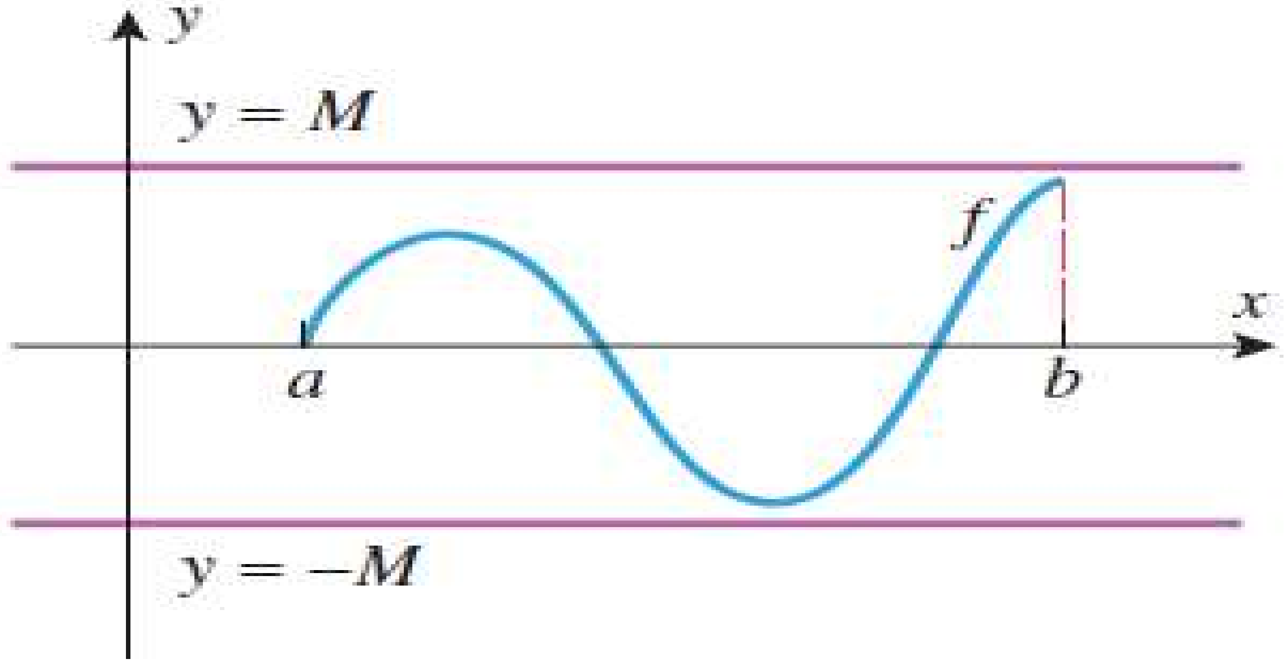
(*b*) *If f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$, then*

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

5.5.7 DEFINITION A function f that is defined on an interval is said to be ***bounded*** on the interval if there is a positive number M such that

$$-M \leq f(x) \leq M$$

for all x in the interval. Geometrically, this means that the graph of f over the interval lies between the lines $y = -M$ and $y = M$.

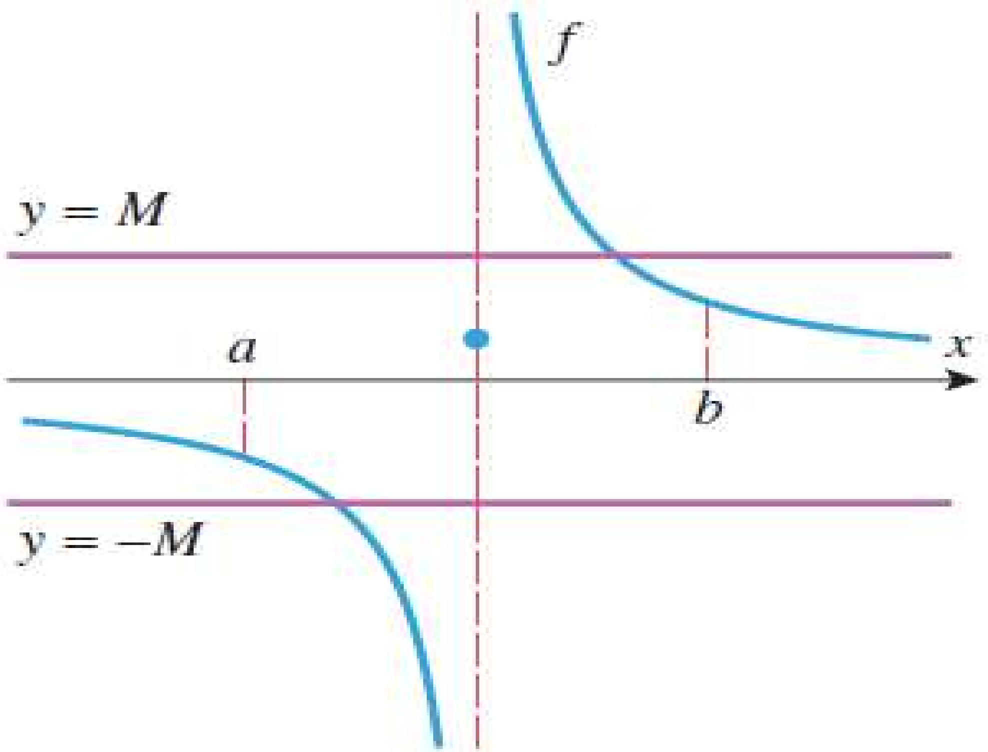


f is bounded on $[a, b]$.

5.5.8 THEOREM *Let f be a function that is defined on the finite closed interval $[a, b]$.*

(a) *If f has finitely many discontinuities in $[a, b]$ but is bounded on $[a, b]$, then f is integrable on $[a, b]$.*

(b) *If f is not bounded on $[a, b]$, then f is not integrable on $[a, b]$.*



f is not bounded on $[a, b]$.

Hiking Trail around Cinque Terre in Northwestern Italy